

## HEILBRONN'S CLASS-NUMBER THEOREM.

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1. LET  $h(d)$  denote the number of non-equivalent forms of the type  
(1)  $Q(X, Y) = aX^2 + bXY + cY^2$ ,  $(a, b, c) = 1$ ,  $a > 0$   
of the discriminant

$$(2) \quad d = b^2 - 4ac.$$

Heilbronn<sup>1</sup> has recently proved the beautiful result (which includes a classical conjecture of Gauss) :

*Theorem.*—If  $d$  runs over all negative fundamental discriminants, then

$$\begin{aligned} h(d) &\rightarrow \infty \\ \text{for } d &\rightarrow -\infty. \end{aligned}$$

2. I shall now introduce the problem discussed in this paper.

There are examples of negative discriminants (e.g.,<sup>2</sup>  $d = -232$  and  $d = -1848$ ) with the following property: *all non-equivalent reduced primitive forms of negative discriminant  $-d$  have their middle coefficients equal to zero.* In such cases, we shall say, briefly, that " $d$  (or  $-d$ ) has the property P". It seems natural to inquire whether the number of negative discriminants  $d$  with the property P is finite or not. It will be proved here by the methods developed by Heilbronn in K that :

*Theorem 1.*—*There is only a finite number of negative discriminants  $d$ , with  $\frac{1}{4}d$  quadratfrei, having the property P.*

It is clear that if  $d$  has the property P, then

$$(3) \quad h(d) = H = 2^{t-1}$$

where  $t$  is the number of different prime factors contained in  $\frac{d}{4}$ .

In what follows we shall assume that  $d$  has the property P, and then prove that this assumption leads to a contradiction for all large  $-d$ .

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<sup>1</sup> "On the class-number in imaginary quadratic fields", hereafter referred to as K. The success of Heilbronn's argument rests essentially on—

(i) a theorem of Hecke (see Landau, *Gött. Nachr.*, 1918);

(ii) a generalization of some recent work of Deuring (*Math. Ztschr.*, 1933).

I am indebted to Dr. Heilbronn for a copy of his manuscript.

<sup>2</sup> The numbers 232 and 1848 are "idoneal". See Dickson's *History of the Theory of Numbers*, I, 361-65.

Notation.

The following notation is almost identical with that used in K.

All roman letters except  $o$ ,  $O$ ,  $s$ ,  $L$ ,  $Q$  ( $P$  and  $K$ ) denote rational integers.  $\chi(n)$  denotes a real character mod  $m$  ( $m > 0$ ) such that

$$L_0(s) = L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

vanishes<sup>3</sup> for at least one  $\rho$  in the halfplane  $\sigma > \frac{1}{2}$ ;  $m$ ,  $\chi$  and  $\rho$  are fixed throughout the paper;

$$(4) \quad \rho = \theta + i\phi \quad (\theta > \frac{1}{2}, \phi \geq 0).$$

We introduce the following Dirichlet series:

$$(5) \quad \zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

$$(6) \quad L_0(s) = \sum_{n=1}^{\infty} \chi(n) n^{-s},$$

$$(7) \quad L_1(s) = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) n^{-s},$$

$$(8) \quad L_2(s) = \sum_{n=1}^{\infty} \chi(n) \left(\frac{d}{n}\right) n^{-s},$$

where  $\left(\frac{d}{n}\right)$  is the Kronecker symbol.

The constants implied in the signs  $O$  and  $o$  depend only on  $m$ ,  $s$ ,  $\rho$ , but they are independent of  $d$ ,  $Q$  and (here we differ from K)  $H$ .

*Proof of Theorem 1.*

*Lemma 1:* If  $1 \leq l_2 \leq m$  and if for  $\sigma > 1$

$$\psi(s) = \psi(s, m, l_1, l_2, Q) = \sum_{\substack{Y=1 \\ Y \equiv l_2(m)}}^{\infty} \sum_{\substack{X=-\infty \\ X \equiv l_1(m)}}^{\infty} Q^{-s(X, Y)},$$

then

$\psi(s)$  is regular, and

$$(9) \quad \psi(s) = O(|d|^{\frac{1}{2} - \frac{1}{2}\sigma} + |d|^{-\frac{1}{2}\sigma})$$

for  $d \rightarrow -\infty$  if  $\sigma > \frac{1}{2}$ ,  $s \neq 1$ .

*Proof:* This result is contained in the proof of lemma 9 of K.

<sup>3</sup> If this assumption were not true for any  $m > 0$  it would follow, from a theorem of Hecke (in a paper by Landau previously cited), that

$$h(d) > \frac{c\sqrt{-d}}{\log(-d)},$$

so that (3) would be false for sufficiently large  $-d$ .

*Lemma 2:* For<sup>4</sup>  $\sigma > \frac{1}{2}$ ,  $s \neq 1$ ,

$$(10) \quad L_0(s) L_2(s) = \zeta(2s) \prod_{p/m} (1 - p^{-2s}) \sum_a \chi(a) a^{-s} + \\ O(H |d|^{\frac{1}{2} - \frac{1}{2}\sigma}) + O(H |d|^{-\frac{1}{2}\sigma}).$$

*Proof:* This follows from (9) above and the proof of lemma 10 in K.

*Lemma 3:* If  $\sigma \geq \frac{1}{2}$ ,

$$(11) \quad \left| \sum_a \chi(a) a^{-s} \right| \geq \frac{1}{4} H^{-2} + O(H |d|^{-\frac{1}{2}\sigma}).$$

*Proof:* Since  $d$  has the property P,

$$(12)^5 \quad d = -4ac, \quad a/\frac{d}{4}, \quad 0 < a \leq \frac{1}{2} |\sqrt{d}|.$$

Hence

$$(13) \quad \sum_a \chi(a) a^{-s} = \prod_{p/\frac{1}{4}d} (1 + \chi(p) p^{-s}) - \sum_{\substack{r > \frac{|\sqrt{d}|}{2} \\ r/\frac{1}{4}d, \mu(r) \neq 0}} \chi(r) r^{-s}$$

As in K it is easy to see that

$$(14) \quad \left| \prod_{p/\frac{1}{4}d} (1 + \chi(p) p^{-s}) \right| \geq \frac{1}{4} H^{-2}$$

Further, from (12),

$$(15) \quad \sum_{\substack{r > \frac{1}{2} |\sqrt{d}| \\ r/\frac{1}{4}d, \mu(r) \neq 0}} \chi(r) r^{-s} = O(2^t |d|^{-\frac{1}{2}\sigma}).$$

(11) follows from (13), (14), (15) and (3).

*Proof of Theorem 1:* We put  $s = \rho = \theta + i\phi$  in lemma 2 and we let  $d$  tend to  $-\infty$ . Then we get

$$(16) \quad 0 = \lim_{d \rightarrow -\infty} \zeta(2\rho) \prod_{p/m} (1 - p^{-2\rho}) \sum_a \chi(a) a^{-\rho} + \\ O(H |d|^{\frac{1}{2} - \frac{1}{2}\theta}) + O(H |d|^{-\frac{1}{2}\theta}),$$

and from lemma 3,

$$(17) \quad \left| \sum_a \chi(a) a^{-\rho} \right| \geq \frac{1}{4} H^{-2} + O(H |d|^{-\frac{1}{2}\theta}).$$

since  $\theta > \frac{1}{2}$  we see that, if

$$(18) \quad H < |d|^\epsilon$$

for every  $\epsilon = \epsilon(\theta) > 0$ , then (16) and (17) contradict each other for  $-d > d_0(\theta)$ . Hence there is an  $\epsilon = \epsilon(\theta) > 0$  such that

$$(19) \quad H \geq |d|^\epsilon \text{ for } -d > d_0(\theta).$$

Now (19) contradicts (3) since  $2^t < |d|^{\frac{1}{2}\epsilon}$  for all  $-d > d_0(\epsilon)$ .

Hence (3) is false for large  $-d$  and our theorem is proved.

<sup>4</sup>  $a$  runs through the minima of the  $H$  quadratic forms belonging to  $d$ .

<sup>5</sup>  $p/q$  means " $p$  is a divisor of  $q$ ",