Sādhanā (2022)47:280
https://doi.org/10.1007/s12046-022-01998-w

Fuzzy tree covering number for fuzzy graphs with its real-life application in electricity distribution system

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MS received 7 July 2022; revised 12 August 2022; accepted 20 August 2022

Abstract. Tree covering number ($T(G)$) for a crisp graph G, which is the minimum number of simple trees those are not have common vertices, appear as induced subgraphs for covering up all vertices of the crisp graph G. The main objective of the theoretical portion of this article is to define a fuzzy tree covering number for fuzzy graphs using edge membership and vertex membership functions. The newly defined parameter tread as an effective fuzzy mathematical tool, which is more fruitful and valuable for solving real-life problems with uncertainties. Also, some bounds together with specific characterizations for fuzzy tree covering number are established in this article. In the last part of this article, the electricity distribution system is taken to reflect the realistic applicability of fuzzy tree covering number for fuzzy graphs along with many related results.

Keywords. Fuzzy graph; fuzzy tree; tree cover number; fuzzy tree covering number.

1. Introduction

In graph theory, a collection $F_G = \{G_1, G_2, ..., G_n\}$ of sub graphs for a crisp graph G is called a tree cover of G if G_i is a tree for all indices $i = 1, 2, ..., n$. In the case of each edge $e \in E(G)$, it should be satisfied that there exists $G_i \in F_G$ such that $e \in E(G_i)$. All the sub-graphs G_i's are acyclic and connected. Here, for a crisp graph G, $T(G)$ represents tree covering number and defined by the following.

$$T(G) = min\{|F_G| : F_G \text{ is a tree cover for the crisp graph } G\}.$$

In 2011, for the research on maximum positive semi-definite nullity of a crisp graph, the earlier said graph parameter was first introduced, and the existing literature on this topic is much less. It is also seen in some studies that the upper limit of the value for the tree cover number of a crisp graph is considered as the maximum positive semi-definite nullity of that graph.

The primary motivation for developing this article comes from the following "Teacher station location" modelling real-life problem. A university wanted to locate k teachers in its coverage area. Each teacher would be assigned a specific set of students with who they would take classes in day shifts. The objective is to determine where to locate the teacher stations and how to assign students to teachers so that the last completion time is minimized. Also, fuzzy graphs are now an efficient mathematical tool for modelling realistic problems. Therefore, the fuzzy tree covering number of fuzzy graphs will play an essential role in showing the impact of the fuzzy tree cover on a system. Rosenfield [24] had introduced the concept of fuzzy graphs.

The definition of a fuzzy tree covering number of fuzzy graphs is made by using membership values of edges and vertices to handle real-life vagueness and impreciseness. In this paper, some characterizations of fuzzy tree covering number are also obtained by relating the type of fuzzy graph and vertex/edge membership functions. In [2] and [10], outer planar and tree-width definitions are introduced. Rosalio *et al*. [23] had worked on tree covers for crisp graphs.

Some tree-covering problems are discussed in the work of Arkin, Hassin, and Levin [1]. The vertices of a graph are partitioned into clusters. Different types covering problems and models to solve real-life problems are discussed by Bhattacharya and Pal [4] [5, 6, 7]. Mordeson and Peng [16] have given the idea of various operations on fuzzy graphs. Some new ideas about fuzzy planar graphs are presented in Samanta and Pal's research study [25] [29]. There are many studies on the fuzzy colouring of fuzzy graphs and fuzzy tolerance graphs provided by Samanta and Pal [28]. Also, Ghorai and Pal [12] have given some fabulous research on m-polar fuzzy planar graphs in detail.

Financial support of the first author offered by the Department of Science and Technology (DST), New Delhi, India, for research through the letter No. DST/INSPIRE/03/2019/001407 and Reg. no.-IF190612 is thankfully acknowledged.

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Published online: 19 December 2022

Barioli *et al*. [2] have studied on minimum semi-definite rank of outer-planar graphs and cover number for trees. Ekstrand*et al*. [10] have handled partial 2-trees and their particular parameter, semi-definite zero forcing number, and semi-definite maximum nullity, which are positive. On the other hand, Fallat and Hogban [11] have a detailed discussion on some critical parameters of graphs like zero forcing number, minimum rank and maximum nullity. Vague graphs with their strengths are described by Samanta *et al*. [26].

On network optimization with maximal covering problem, there is a nice work of Bergman *et al*. [3]. Applicability of covering games on facility location problems are shown in the study of Cardinal and Hoefer [8]. Chang and Zadeh [9] had a research work on fuzzy controls and mappings. Hakimi [13] has shown a graph theoretic approach on optimum distribution. Fuzzy graphs are used in the work of Koczy [14] for evaluation and network optimization. Involving distance constraint, network location problems are analyzed by Moon and Chaudhry [15]. Nayeem and Pal [17] have characterized shortest path problems. Fuzzy edge covering problem with minimum weight is solved by Ni. [18]. Real-life applications on modern trend of fuzzy graph theory are discussed by Pal *et al*. [19]. Pramanik *et al*. [20] have worked on interval-valued fuzzy planar graphs. Conditional covering problems are handled in the study of Rana *et al*. [21]. Rashmanlou and Pal [22] have worked on isometry of interval-valued fuzzy graphs. Also, Samanta *et al*. [27] have given the idea of fuzzy colouring for fuzzy graphs.

1.1 *Novelty*

In existing literature of graph theory, there is a concept of tree covering number for crisp graph only. But, in this article, new definition of fuzzy tree covering number and semi-tree covering number for fuzzy graphs are introduced. The summarized form of the entire work done on fuzzy tree covering number for fuzzy graphs as follows.

(i) Defining new parameters for fuzzy graphs i.e., notch of fuzzy tree, fuzzy tree covering number, semi-tree covering number.
(ii) Also, we have provided many efficient results on fuzzy tree covering number for fuzzy graphs with details proof.
(iii) In some cases, we have proved specific bounds for fuzzy tree covering numbers which are very strong aspects to be used in modeling real-life problems.

1.2 *Organization of paper*

The introductory part of this article is described in section 1. Section 2 is arranged with some basic definitions and new-defined parameters of fuzzy graphs. Section 3 establishes some relevant results on bounds of fuzzy tree covering number of fuzzy graphs with their essential attributes. The applicability of fuzzy tree covering number and related results are shown by solving a problem in the electricity distribution system in Section 4. In Section 5, significance of the proposed ideas are provided. Section 6 reflects an overall conclusion for the entire work of this article.

1.3 *Highlights of contributions*

The main contributions in this article are highlighted in this portion.

(a) Existing definition of tree covering number for crisp graphs have its particular limitations as it does not include any type of uncertainties. But, in our introduced definition of fuzzy tree covering number of fuzzy graphs are more well planned idea to handle impreciseness and vagueness in any system.
(b) More significantly, fuzzy tree covering number is calculated in fuzzy sense which is efficiently relatable to our real-life application purpose.
(c) We are the first to introduce such new concept for fuzzy graphs and give some important results for various types of fuzzy graphs.
(d) Also, we have given bounds for fuzzy tree covering number of fuzzy graphs which are very fruitful in some particular real-life situations. In developing any model with programming problems with uncertainties, the results can be directly used for the ease of calculation process.

2. Preliminaries

Some necessary definitions and concepts are described in this part of the article.

Definition 1 [16] A non empty set $G = (V, \sigma, \mu)$ together with one function $\sigma : V \rightarrow [0,1]$ called membership function of vertices and another one $\mu : V \times V \rightarrow [0,1]$ known as membership function of edges; is said to be fuzzy graph which satisfies the following $\mu(x,y) \leq min\{\sigma(x), \sigma(y)\}$.

Definition 2 [23] Consider a crisp graph G and $F_G = \{G_1, G_2, ..., G_n\}$ be a set of some sub-graphs of given crisp graph; where for all arbitrary indices $i = 1, 2, ..., n$, G_i is a tree. For each edge $e \in E(G)$, if there exists $G_i \in F_G$ such that $e \in E(G_i)$, then for G, F_G is said to be a tree cover.

Definition 3 For a crisp graph G the tree covering number, $T(G)$, is given by,

$$T(G) = min\{|F_G| : F_G \text{ is a tree cover of } G\}.$$

Definition 4 For a fuzzy graph $G = (V, \sigma, \mu)$ let, $\tau(G) = \{T_1, T_2, ..., T_n\}$ be a set of sub-graphs of G and for all $i = 1, 2, ..., n$, T_i is a fuzzy tree in G. If for every edge uv with $\mu(uv) \neq 0$ in G, there exists $T_i \in \tau(G)$ with the condition that $\mu(uv) \neq 0$ in T_i; then $\tau(G)$ is a fuzzy tree cover of G.

Definition 5 For an arbitrary fuzzy graph $H = (V, \sigma, \mu)$, let a fuzzy tree denoted by T. Then, the notch of a fuzzy tree is presented by the symbol $N_T(H)$ which is defined in the following form,

$$N_T(H) = \sum_{u,v \in V(H)} \frac{\mu(uv)}{max\{\sigma(u), \sigma(v)\}}.$$

Example 1 Considering a fuzzy graph $G = (V, \sigma, \mu)$ shown in figure 1. Then, the notch of the fuzzy tree $T = \{v_1, v_6, v_5, v_4, v_3, v_2\}$ in that fuzzy graph is calculated by the following process.

$$N_T(G) = \frac{0.2}{max\{0.3, 0.5\}} + \frac{0.4}{max\{0.5, 0.7\}} + \frac{0.2}{max\{0.4, 0.5\}} + \frac{0.2}{max\{0.3, 0.7\}} + \frac{0.1}{max\{0.1, 0.5\}} = 0.4 + 0.57 + 0.4 + 0.29 + 0.2 = 1.86$$

Definition 6 For a fuzzy graph $G = (V, \sigma, \mu)$, if $\tau(G) = \{T_1, T_2, ..., T_n\}$ be taken as a fuzzy tree cover; then $T_c(G)$ represents fuzzy tree covering number and it is defined as,

$$T_c(G) = min\{N_{T_i}(G) : T_i \in \tau(G)\}.$$

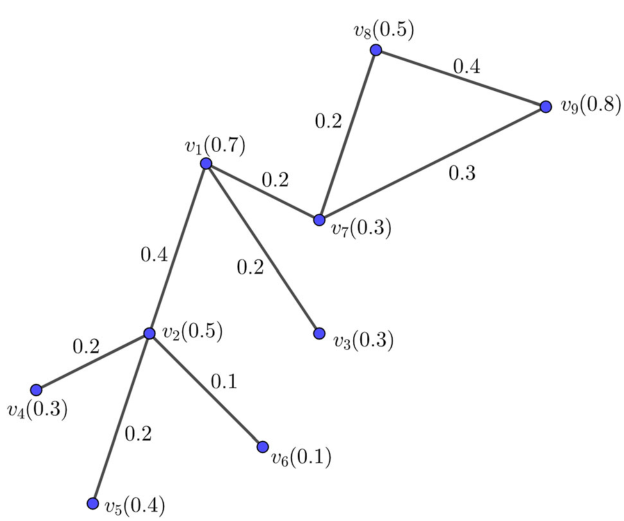

Figure 1. Considered fuzzy graph G.

Example 2 In G of figure 1, fuzzy tree cover set is, $\tau(G) = \{T_1, T_2\}$.

Here, $T_1 = \{v_1, v_2, v_3\}$ and $T_2 = \{v_2, v_4, v_5\}$.

Now, $N_{T_1}(G) = 0.57 + 0.29 = 0.86$ and $N_{T_2}(G) = 0.4 + 0.4 + 0.2 = 1$.

So, $T_c(G) = min\{0.86, 1\} = 0.86$.

Definition 7 Suppose U_1 and U_2 are two sets of vertices such that $U_1 \cap U_2 = \phi$. Then the definition of join (sum) of $H_1 = (U_1, \sigma_1, \mu_1)$ and $H_2 = (U_2, \sigma_2, \mu_2)$ is given as a fuzzy graph $H = H_1 + H_2 : (\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ on $H* : (U, P)$ where $U = (U_1 \cup U_2)$ and, $P = P_1 \cup P_2 \cup P'$, where P' is the set of all edges joining vertices of U_1 with vertices of U_2, with

$$(\sigma_1 + \sigma_2)(s) = \begin{cases} (\sigma_1 \cup \sigma_2)(s) & \text{if } s \in U_1 \cup U_2 \\ \sigma_1(s) & \text{if } s \in U_1 \\ \sigma_2(s) & \text{if } s \in U_2 \end{cases}$$

and,

$$(\mu_1 + \mu_2)(st) = \begin{cases} (\mu_1 \cup \mu_2)(st) & \text{if} st \in P_1 \cup P_2 \\ \sigma_1(s) \wedge \sigma_2(t) & \text{if } st \in P' \\ \mu_1(st) & \text{if } st \in P_1 \\ \mu_2(st) & \text{if } st \in P_2 \end{cases}$$

Definition 8 Consider $H = (V, \sigma, \mu)$ as a fuzzy graph. Then, an arbitrary edge $st \in P(H)$ is called a strong edge if and only if $\mu(st) = min\{\sigma(s), \sigma(t)\}$.

Definition 9 Let us take an arbitrary fuzzy graph $H = (V, \sigma, \mu)$, $O(E)$ reflects the order of edges and defined as the cardinality of $E(H)$; where, $E(H) = \{uv : \mu(uv) \neq 0\}$.

Definition 10 Let us take a fuzzy graph $H = (U, \sigma, \mu)$, $O(U)$ represents the order of vertices and defined as the cardinality of $U(H)$, here, $U(H) = \{s : \sigma(s) \neq 0\}$.

Definition 11 If $H = (U, \sigma, \mu)$ is a fuzzy graph, $\delta_T(H)$ is giving edge-notch for the fuzzy tree T which is defined below. For all arbitrary edges, $st \in P(H)$,

$$\delta_T(H) = \sum_{\mu(st) \neq 0} \mu(st).$$

Definition 12 Given $G = (V, \sigma, \mu)$ be a fuzzy graph, $T_{sc}(G)$ is standing for semi-tree covering number which is defined as minimum edge-notch for a tree cover in G i.e.,

$$T_{sc}(G) = min\{\delta_T(G) : T \in \tau(G)\}.$$

Definition 13 Two fuzzy vertices of a fuzzy graph are called fuzzy independent if there is no strong edge between them. A fuzzy subset S of σ is a fuzzy vertex independent set of G if any two vertices of S are fuzzy independent.

Definition 14 A fuzzy vertex independent set S is a maximal fuzzy independent set if no super-set of S is a fuzzy vertex independent set.

Definition 15 The cardinality of any maximum fuzzy independent set of a fuzzy graph $G = (V, \sigma, \mu)$ is called its fuzzy vertex independent number, denoted by $\beta(G)$.

3. Some characterizations

In this portion, we provide some essential characterizations of fuzzy tree covering number of fuzzy graphs and pertinent properties of fuzzy graphs considering the results of fuzzy tree covering number equal to zero, fuzzy tree covering numbers equal to one, and other cases for bounds of fuzzy tree covering numbers.

Clearly, for any fuzzy graph $H = (V, \sigma, \mu)$, $T_c(H) \geq 0$.

The ferocity of upper bounds for fuzzy tree covering numbers is found, and some relationships are established, and relationships are established in the following theorems.

Theorem 1 *Let, a fuzzy graph be* $H = (P, \sigma, \mu)$. *Then* $T_c(H) = 0$ *if and only if* $H \cong \overline{K_n}$.

Proof Assume that, $T_c(H) = 0$

Then, $\sum_{s,t \in U(H)} \frac{\mu(st)}{max\{\sigma(s),\sigma(t)\}} = 0$

That is, $\mu(st) = 0$.

Also, for existence of $T_c(H)$, $max\{\sigma(s), \sigma(t)\} \neq 0 \Rightarrow$ either $\sigma(s) \neq 0$ or, $\sigma(t) \neq 0$.

Thus, $\mu(st) = 0 \forall st \in E(H)$ and, either $\sigma(s) \neq 0$ or, $\sigma(t) \neq 0$.

$\Rightarrow H \cong \overline{K_n}$.

Conversely, assume that $H \cong \overline{K_n}$.

By definition of $\overline{K_n}$, $T_c(H) \leq 0$.

Combining this with the inequality, $T_c(H) \geq 0$, we get $T_c(H) = 0$. □

Theorem 2 *Let, a fuzzy graph be* $H = (U, \sigma, \mu)$ *and all edges of that fuzzy graph are strong edges. Then,* $T_c(H) = 1$ *if and only if for all edges,* $st \in P(H), \sigma(s) = \sigma(t)$.

Proof Let, $T_c(H) = 1$.

Now, $min\{N_{T_i}(H) : T_i \in \tau(H)\} = 1$

Then, $N_{T_i}(H) = 1$ for some $T_i \in \tau(H)$

So, $\sum_{s,t \in U(H)} \frac{\mu(st)}{max\{\sigma(s),\sigma(t)\}} = 1$

Implies, $\mu(st) = max\{\sigma(s), \sigma(t)\}$

Finally, $min\{\sigma(s), \sigma(t)\} = max\{\sigma(s), \sigma(t)\}$, as all edges are strong edge in H

So we get, $\sigma(s) = \sigma(t) \forall s, t \in U(H)$.

The converse case is similarly proved.

Hence, the proof of the theorem.

Theorem 3 *Let us take a fuzzy graph,* $H = (U, \sigma, \mu)$ *with* $O(U) = n$. *Then,* $T_c(H) \leq O(U)$ *if and only if H is a non-trivial fuzzy graph which is connected.*

Proof Assume, $T_c(H) \leq O(U) = n$.

Then, there exists a fuzzy tree cover $\tau(H)$ of H with the condition that $min\{N_{T_i}(H) : T_i \in \tau(H)\} \leq n$

Implies, $N_{T_i}(H) \leq n$ for some T_i in $\tau(H)$

So, $\sum_{s,t \in U(H)} \frac{\mu(st)}{max\{\sigma(s),\sigma(t)\}} \leq n$

Then, at least one of $\sigma(s)$ and $\sigma(t)$ is non-zero and, $\mu(st) \neq 0 \forall$ arbitrary edges $st \in P(H)$

Finally, we have H is a non-trivial fuzzy graph which is connected.

The converse case is similarly proved.

Theorem 4 *Let,* $H = (U, \sigma, \mu)$ *be a fuzzy graph. Then* $T_c(H) = 1$ *if and only if* σ *and* μ *are constant functions.*

Proof Assume firstly, $T_c(H) = 1$.

Then, $\frac{\mu(st)}{max\{\sigma(s),\sigma(t)\}} = 1$

That is, $\mu(st) = max\{\sigma(s), \sigma(t)\}$

This implies that vertex membership, σ and edge-membership function μ are constant.

Theorem 5 *Let,* $H = (U, \sigma, \mu)$ *be a non-trivial connected fuzzy graph with* $O(U) = n$ *and* $|\tau(H)| = 1$. *Then,* $T_c(H) \leq 1$ *if and only if, H is a fuzzy tree.*

Proof Let us first assume, $T_c(H) \leq 1$.

That is, $T_c(H) \geq 0$

or, $\sum_{s,t \in U(H)} \frac{\mu(st)}{max\{\sigma(s),\sigma(t)\}} \geq 0$

For all edges, $st \in P(H)$, $\mu(st) \neq 0$.

Also, a fuzzy tree covers $\tau(H)$ exists for a fuzzy graph H such that $|\tau(H)| = 1$.

Let, $\tau(H) = \{T\}$ where T is a fuzzy tree.

Since H is a non-trivial graph which is connected.

So, $H = T$, which implies H is a fuzzy tree.

Conversely, let H be a fuzzy tree.

Let, $\tau(H) = \{H\}$. That means that $\tau(H)$ is a fuzzy tree cover of H.

By definition, we have, $T_c(H) = min\{N_{T_i}(H) : T_i \in \tau(H)\} = N_T(H) < 1$, for a fuzzy tree.

These imply, $T_c(H) < 1$.

Hence, the proof of the theorem. □

Theorem 6 *For a non-empty fuzzy graph* $H = (U, \sigma, \mu)$, $O(U) = n$ *and,* $O(P) = m$; *P represents edge-set. Then, notch of H is always greater than or, equal to* $\frac{O(P)}{O(U)}$ *i.e.,* $N_T(H) \geq \frac{m}{n}$.

Proof Let, $H = (U, \sigma, \mu)$ be a fuzzy graph.

So, we have $O(U) = \{|U(H)| : \sigma(s) \neq 0 \text{ for } s \in U(H)\}$.

But, all $\sigma(s) \in [0, 1]$ and, $\mu(s) \in [0, 1]$ for a fuzzy graph.

Therefore,

$$\sum \sigma(s) \leq O(U) \tag{1}$$

and,

$$\sum \mu(st) \leq O(P) \tag{2}$$

From (1), we have,

$$\frac{1}{\sum \sigma(s)} \geq \frac{1}{O(U)} \quad (3)$$

From (2) and (3), we get,

$$\frac{O(P)}{O(U)} \leq \frac{\mu(st)}{max\{\sigma(s), \sigma(t)\}} = N_T(H)$$

$$N_T(H) \geq \frac{O(P)}{O(U)} = \frac{m}{n}.$$

□

Theorem 7 *For any fuzzy cycle* C_n, $1 \leq T_c(C_n) \leq 2$.

Proof The contra-positive of the implication in Theorem 1 asserts that, $T_c(C_n) \geq 1$.

Now, consider a fuzzy cycle, $C_n = [u_1, u_2, u_3, ..., u_n, u_1]$.

Let, $\tau = \{T_1, T_2\}$ where, $T_1 = [u_1, u_2, ..., u_n]$ and, $T_2 = [u_n, u_1]$.

Then, T_1 and T_2 are fuzzy trees in C_n.

Hence, τ is a fuzzy tree cover of C_n.

Thus, $T_c(C_n) \leq 2 \times 1 = 2 \Rightarrow 1 \leq T_c(C_n) \leq 2$.

Hence, the proof of the theorem is completed.

Theorem 8 *For a fuzzy fan* F_n, $1 \leq T_c(F_n) \leq \sum_{u \in V(F_n)} \sigma(u)$.

Proof By Theorem 5, we have, $T_c(F_n) \geq 1$.

Note that by definition, $F_n = P_n + K_1$.

Let, $P_n = \{u_1, u_2, ..., u_n\}$ such that $\sigma(u_i) \neq 0$ and $V(K_1) = \{v\}$ for $\sigma(v) \neq 0$.

Then, $\mu(vu_i) \neq 0$ for $vu_i \in E(F_n)$; $i = 1, 2, ..., n$.

Consider, $\tau = \{T_1, T_2\}$ where, $T_1 = P_n$ and, $T_2 =$ the star, $K_{1,n} = \{v\} + \overline{K_n}$ with,

$$V(\overline{K_n}) = \{u_1, u_2, ..., u_n\} = \{u_i : \sigma(u_i) \neq 0\}.$$

Then, T_1 and T_2 are fuzzy trees.

Also, it is noted that τ is a fuzzy tree cover of F_n.

Therefore, $T_c(F_n) \leq \sum_{u \in V(F_n)} \sigma(u)$

That is, $1 \leq T_c(F_n) \leq \sum_{u \in V(F_n)} \sigma(u)$.

Hence, the theorem. □

Theorem 9 *Let,* $G = (V, \sigma, \mu)$ *be a simple fuzzy graph which is connected, and let an independent set be* $S \subseteq V(G)$. *Then,* $T_{sc}(G) \leq O(V) - \beta(G)$, *where* $\beta(G)$ *is the independence number of the fuzzy graph* G.

Proof Assume that, $V(G) = \{v_i : \sigma(v_i) \neq 0 \text{ for } i = 1, 2, ..., n\}$; suppose that,

$$S = \{v_i : \mu(v_i v_j) = 0 \text{ for all } v_i, v_j \in V(G) \\ \text{and } i = 1, 2, ..., k\}$$

is an independent set for that fuzzy graph G.

We are interested in developing a fuzzy tree cover of the size $(n - k)$ using the following iterative process.

For the value, $i = k + 1$, let T_{v_i} be the fuzzy tree which is induced by the set of vertices $\{v_{k+1}\} \cup \{N(v_{k+1}) \cap S\}$.

For $i = k + 2$ to n, let us consider that T_{v_i} be the fuzzy tree which is induced by the set of vertices in $\{v_i\} \cup \{N(v_i) \cap S\}$ such that $\sigma(v_j) = 0$ for all $k + 1 \leq j < i$.

Since, $\mu(uv) \neq 0$ for all $uv \in E(G)$, each $s \in S$ has at least one neighbor in $\{v_{k+1}, ..., v_n\}$.

That is, $\mu(s, v_i) \neq 0$ for at least one i and, $i = k + 1, ..., n$.

So, this described process will provide a fuzzy tree cover of G of size $(n - k)$.

Thus, $T_{sc}(G) \leq O(V) - \beta(G)$, where $O(V) = n$ and, $\beta(G) = k$.

Hence, the proof of the theorem. □

Theorem 10 *For a fuzzy graph* $H = (U, \sigma, \mu)$ *and,* $\mu(st) \neq 0$ *for* $st \in E(H)$,

$$T_{sc}(H) - \mu(st) \leq T_{sc}(H - st) \leq T_{sc}(H) + \mu(st).$$

Proof Let, $\sigma(s) \neq 0$ and $\sigma(t) \neq 0$ for $s, t \in U(H)$.

Also, $\mu(st) \neq 0$ for $st \in E(H)$.

Now, we have deleted the edge st and get the fuzzy graph, $(H - st)$ (here, st could be a multi-edge).

Let $\tau(H)$ represent a fuzzy tree cover of $(H - st)$ with minimum size.

If s and t are such that $s \in T_i$ and, $t \in T_j$ with $\mu(st) = 0$ in $\tau(H)$; then, $\tau(H)$ is a fuzzy tree cover of H.

So, $T_{sc}(H) \leq T_{sc}(H - st)$.

Let two arbitrary vertices of H are s and t with the conditions that $s, t \in T_i$ and $\mu(st) \neq 0$, i.e. they are belonging to the same fuzzy tree, T_{st}; then, the graph which is induced by the vertices of T_{st} contains a fuzzy cycle in that fuzzy graph.

Therefore, $\tau(H)$ is not a fuzzy tree cover of the fuzzy graph H.

Although, the vertices of T_{st} may be partitioned into two sets P and Q, such that the vertex s is contained in the fuzzy tree, which is induced by the vertices of P and the vertex t is in the fuzzy tree which is induced by vertices of Q.

These two fuzzy trees are represented by T_P and T_Q, respectively.

So, $\{(\tau(H) \backslash T_{st}) \cup T_P \cup T_Q\}$ is a fuzzy tree cover of that fuzzy graph of size, $T_{sc}(H - st) + \mu(st)$.

This shows that, $T_{sc}(H) - \mu(st) \leq T_{sc}(H - st)$.

Now, we have to show, $T_{sc}(H - st) \leq T_{sc}(H) + \mu(st)$.

Let us assume that, there is a minimum fuzzy tree cover, $\tau(H)$ of H such that s and t are in separate trees, i.e. $\mu(st) = 0$ for $s \in T_i$, $t \in T_j$ for $i \neq j$.

Then, $\tau(H)$ is a fuzzy tree cover of the fuzzy graph $(H - st)$.

So, $T_{sc}(H - st) \leq T_{sc}(H)$.

Otherwise, let $\tau(H)$ be a minimum fuzzy tree cover of H that involves the edge st and let T_{st} be the fuzzy tree that contains st.

Now, we delete the edge st from T_{st} and a fuzzy tree cover of $(H - st)$ of size $\{T_{sc}(H) + \mu(st)\}$ is produced.

This proves that, $T_{sc}(H - st) \leq T_{sc}(H) + \mu(st)$, which completes the proof. □

Theorem 11 *Let, $H = (U, \sigma, \mu)$ be a fuzzy graph such that for edges $st \in E(H)$ with $\mu(st) \neq 0$ is a fuzzy bridge. Then, st is in a fuzzy tree in each minimum fuzzy tree cover of H.*

Proof It will be noted that there is no path which does not include the fuzzy edge st from s to t. That for all possible path P in the said fuzzy graph, we have, $\sum_{s,t \in P} \mu(st) \neq 0$.

Thus, for any fuzzy tree cover that does not include the fuzzy edge st, it is the situation that $s \in T_i$ and $t \in T_j$ such that $\mu(st) = 0$, i.e. the fuzzy vertices s and t are in separate fuzzy trees of H.

The two fuzzy trees can be combined to form one fuzzy tree by imposing a non-zero membership value to the fuzzy edge st. That means, assume, $\mu(st) \neq 0$ in H.

Hence, the proof is completed. □

4. An application

According to the annual report, 2020–2021 of West Bengal State Electricity Distribution Company Limited (WBSEDCL), the detailed management discussion and analysis are described in this section.

Describing the Company's objectives, estimates and expectations, management discussion and analysis may constitute 'forward-looking statements' according to the statements of the Annual Report with the applicability of laws and regulations. On the basis of reasonable assumptions, the expectations and the actual real-life results might differ.

Therefore, there are always uncertainty and impreciseness in the data on electric supply, demand and power stock amount. These real-life uncertain environments in the electric distribution process have allowed us to use the fuzzy graph concept.

In this article, fuzzy graphs describe the fuzzy system of electric distribution. Also, the concept of fuzzy tree covering of fuzzy graphs and fuzzy tree covering numbers are used to solve a real-life problem as follows.

Find an arrangement or strategy for a fuzzy graph (fuzzy system) to get the most prominent coverage of the electric supply area. Also, compare two cases of fuzzy tree covering number and semi tree covering number in the fuzzy system of the electric distribution process for a better strategy.

This portion reflects that the applicability of introduced definitions and developed theorems for solving such problems is the practical proof for large problem sizes in real-life.

In this article, MU stands for million units, and it is a unit of energy. The structure of the zones of WBSEDCL is as follows. There are mainly five major zones, namely

(i) Kolkata Zone,
(ii) Burdwan Zone,
(iii) Midnapore Zone,
(iv) Berhampore Zone,
(v) Siliguri Zone.

Also, the main zones are sub-divided for the distribution of electricity throughout different districts of West Bengal, India, in 21 sub-zones. Figure 2 provides the structure in detailed.

4.1 *Fuzzy graph construction*

For transforming the fuzzy environment in the distribution scenario of WBSEDCL, we must first clarify the vertices and edges of the fuzzy graph and their membership values.

In this application part, the Regional or, Zonal offices are taken as vertices, which are parts of different main zones. At the same time, the central zone sections are directly connected to the office of WBSEDCL. Therefore, we can construct a tree-like structure for the flow of distribution of energy all over the state of West Bengal. Also, the main zones are connected to each other.

All distribution points are treated as vertices, and there exists an edge between two vertices if there is an exchange of official data related to the distribution of energy of WBSEDCL.

4.1.1 *Membership values* In the energy distribution system of WBSEDCL, due to the demand of De-centralized Bulk Consumers, the amount of demand energy and the total capacity of any zone is not fixed; it is dependent on real-life situations. Therefore, the approximated demand amounts in MU as per WBSEDCL Annual Report 2020-2021 are represented as triangular fuzzy numbers to assign the membership values of the vertices. The edge-membership values between the regional offices and the central zones are the approximated percentage of difference in demand amount in the main zone. On the other side, the edge-membership values of the edges between main zones are the approximated per cent of the difference in demand amount for 2020-2021. The parameter that helps assign edge-membership values between main zones is directly related to the capacity addition for De-centralized Bulk Consumers for 2020-2021.

4.1.2 *Scaling technique* From tables 1 and 2, we can see that the upper limit of the approximated demand in MU is 623 for Paschim Burdwan Regional Office, and the lower limit of the approximated demand in MU is 36 for Dakshin Dinajpur Regional Office. In this section, we use a scaling technique to convert all the values between 0 and 1 to quickly verify all our developed theorems for a fuzzy graph in this application.

For finding the membership values, we fixed the minimum approximated demand in 30 MU in each regional

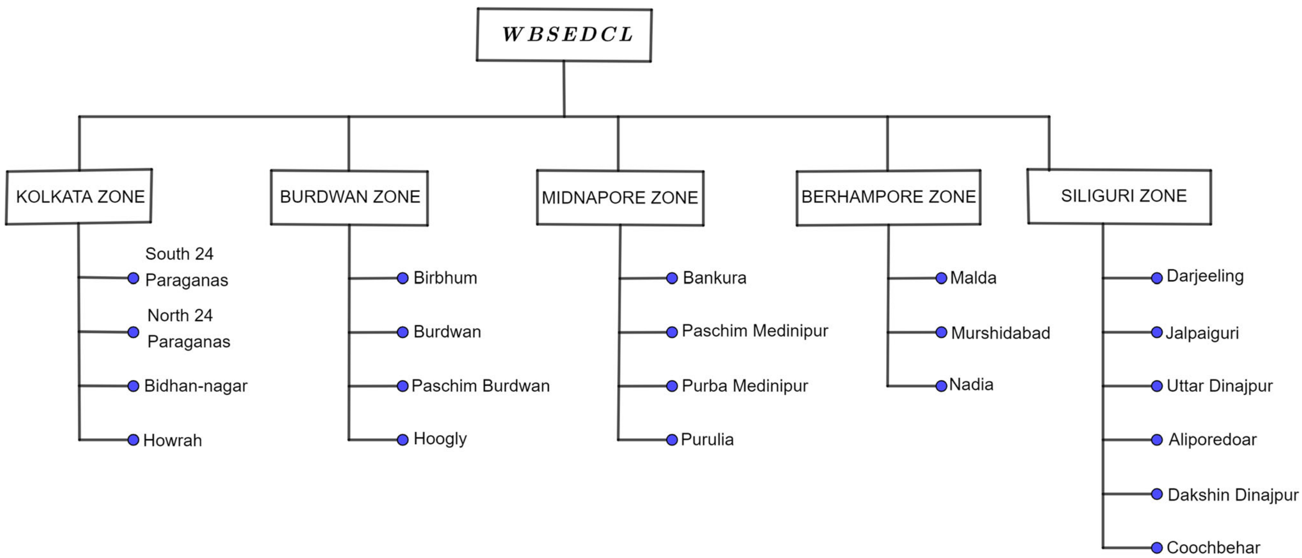

Figure 2. Structure of different zones of WBSEDCL.

Table 1. Details of vertex and edge-membership values for this application.

Name of Zone	Regional office	v_i	Demand (in Mu)	Approx. demand	$\mu(v_iv_j)$	$\sigma(v_i)$
Kolkata Zone	South 24 paraganas	v_7	151.110	151	$\mu(v_2v_7) = 0.036$	0.03
Kolkata Zone	North 24 paraganas	v_8	209.800	209	$\mu(v_2v_8) = 0.049$	0.089
Kolkata Zone	Bidhan-nagar	v_9	212.030	212	$\mu(v_2v_9) = 0.05$	0.08
Kolkata Zone	Howrah	v_{10}	428.340	428	$\mu(v_2v_{10}) = 0.1$	0.1
Burdwan Zone	Birbhum	v_{11}	152.040	152	$\mu(v_3v_{11}) = 0.04$	0.098
Burdwan Zone	Burdwan	v_{12}	485.540	485	$\mu(v_3v_{12}) = 0.12$	0.048
Burdwan Zone	Paschim Burdwan	v_{13}	623.500	623	$\mu(v_3v_{13}) = 0.15$	0.04
Burdwan Zone	Hoogly	v_{14}	377.320	377	$\mu(v_3v_{14}) = 0.09$	0.048
Midnapore Zone	Bankura	v_{15}	126.490	126	$\mu(v_4v_{15}) = 0.03$	0.032
Midnapore Zone	Paschim Midnapore	v_{16}	182.040	182	$\mu(v_4v_{16}) = 0.04$	0.045
Midnapore Zone	Purba Midnapore	v_{17}	112.470	112	$\mu(v_4v_{17}) = 0.03$	0.07
Midnapore Zone	Purulia	v_{18}	58.110	58	$\mu(v_4v_{18}) = 0.01$	0.02
Berhumpore Zone	Malda	v_{19}	71.210	71	$\mu(v_5v_{19}) = 0.02$	0.02
Berhumpore Zone	Murshidabad	v_{20}	83.350	83	$\mu(v_5v_{20}) = 0.02$	0.02
Berhumpore Zone	Nadia	v_{21}	79.620	79	$\mu(v_5v_{21}) = 0.018$	0.04

Table 2. Details of vertex and edge-membership values for this application.

Name of Zone	Regional office	v_i	Demand (in Mu)	Approx. demand	$\mu(v_iv_j)$	$\sigma(v_i)$
Siliguri Zone	Darjeeling	v_{22}	230.000	230	$\mu(v_6v_{22}) = 0.055$	0.037
Siliguri Zone	Jalpaiguri	v_{23}	265.790	265	$\mu(v_6v_{23}) = 0.06$	0.067
Siliguri Zone	Uttar Dinajpur	v_{24}	115.090	115	$\mu(v_6v_{24}) = 0.03$	0.29
Siliguri Zone	Aliporeduar	v_{25}	57.020	57	$\mu(v_6v_{25}) = 0.01$	0.29
Siliguri Zone	Dakshin Dinajpur	v_{26}	36.910	36	$\mu(v_6v_{26}) = 0.008$	0.02
Siliguri Zone	Cooch-behar	v_{27}	59.070	59	$\mu(v_6v_{27}) = 0.01$	0.02

section; which reflects vertex membership value 0 to the corresponding vertex, and the maximum approximated demand is 4200 MU; that will give vertex membership value 1 to the particular vertex. For example, the membership value of the vertex v_6 is calculated as,

$$1 - \left(\frac{4116 - 763}{4200}\right) \cong 0.2.$$

Similarly, other membership values are calculated. For the first level of the constructed fuzzy graph, we must maintain $\mu(v_i v_j) = min\{\sigma(v_i), \sigma(v_j)\}$ for $v_i, v_j \in V(G)$.

Table 2 contains all the information for edge and vertex membership values of the fuzzy graph in this application. Also, table 3 stands for providing vertex membership values corresponding to vertices for main zones in the fuzzy system.

The constructed fuzzy graph is given in figure 3.

From figure 3, we can see that the possible trees which construct a fuzzy tree cover for the fuzzy graph G as follows.

$$\tau(G) = \{T_1, T_2, T_3, T_4, T_5\}.$$

Here, $T_1 = \{v_1, v_2, v_9, v_7, v_8, v_{10}\}$, $T_2 = \{v_1, v_3, v_{12}, v_{13}, v_{14}, v_{11}\}$,

$T_3 = \{v_1, v_4, v_{18}, v_{17}, v_{16}, v_{15}\}$, $T_4 = \{v_1, v_5, v_{21}, v_{20}, v_{19}\}$, $T_5 = \{v_1, v_6, v_{27}, v_{23}, v_{26}, v_{25}, V_{24}, v_{22}\}$.

Now, notches for these fuzzy trees are calculated.

$N_{T_1} = 0.25 + 0.4 + 0.2 + 0.196 + 0.144 = 1.19$,
$N_{T_2} = 0.24 + 0.154 + 0.46 + 0.58 + 0.35 = 1.78$,
$N_{T_3} = 0.11 + 0.23 + 0.308 + 0.23 + 0.076 = 1.03$,
$N_{T_4} = 0.06 + 0.25 + 0.25 + 0.225 = 0.785$,
$N_{T_5} = 0.184 + 0.275 + 0.3 + 0.15 + 0.05 + 0.04 + 0.05 = 1.049$.

Therefore,
$T_c(G) = min\{N_{T_1}(G), N_{T_2}(G), N_{T_3}(G), N_{T_4}(G), N_{T_5}(G)\}$,
i.e.,
$T_c(G) = min\{1.19, 1.78, 1.03, 0.785, 1.049\} = 0.785$.

Also, all the edge-notches for all fuzzy trees in $\tau(G)$ are evaluated in the next portion.

$\delta_{T_1}(G) = (0.238 + 0.036 + 0.049 + 0.05 + 0.1 + 0.238 + 0.11 + 0.06 + 0.18) = 1.061$,

$\delta_{T_2}(G) = (0.241 + 0.238 + 0.04 + 0.12 + 0.15 + 0.09 + 0.11 + 0.18) = 1.229$,

$\delta_{T_3}(G) = (0.11 + 0.11 + 0.11 + 0.03 + 0.04 + 0.03 + 0.01 + 0.06 + 0.11) = 0.61$,

$\delta_{T_4}(G) = (0.06 + 0.06 + 0.018 + 0.02 + 0.02 + 0.06 + 0.06 + 0.06 + 0.11) = 0.468$,

$\delta_{T_5}(G) = (0.18 + 0.18 + 0.18 + 0.11 + 0.055 + 0.06 + 0.06 + 0.03 + 0.01 + 0.008 + 0.01) = 0.883$.

Therefore,
$T_{sc}(G) = min\{\delta_{T_1}(G), \delta_{T_2}(G), \delta_{T_3}(G), \delta_{T_4}(G), \delta_{T_5}(G)\}$,

That is, $T_{sc}(G) = min\{1.061, 1.229, 0.61, 0.468, 0.883\} = 0.463$.

Thus, it is observed from the above that, $T_c(G) = 0.785 > T_{sc}(G) = 0.463$.

4.2 *Insightful analysis*

Considering the constructed fuzzy graph G in this application part, we have seen that the fuzzy tree covering number is greater than the semi tree covering number of that fuzzy system. Here, the fuzzy graph represents the fuzzy environment of the electricity distribution system of WBSEDCL. The result proves that the fuzzy relation between the regional zones and distribution system is more essential data for better coverage in the process. So, using the fuzzy relationships and connections between the main zones in the electricity distribution system is always efficient and realistic. That is, the exchange of official data between regional offices will be a practical step in decreasing the imbalance of energy distribution in different zones, which is relevant to arranging a smooth flow of power and reducing electric deficiency in the state of West Bengal in India.

5. Significance of the proposed ideas

The consequences of our proposed ideas with new definitions and results are encapsulated in this section.

(i) In this article, new-define parameter fuzzy tree covering number for fuzzy graphs has the flexibility to handle uncertain situations in any system. The parameter reflects the importance of tree covers in a fuzzy graph, where each tree stands for a particular connected system in a fuzzy network.

(ii) In existing literature, only number of edges are taken into the concept for crisp graphs. On the other hand, vertex and edge membership values are having significance in new-defined term fuzzy tree covering number of fuzzy graphs.

(iii) Also, we have provided the relation between fuzzy tree covering number and independence number for fuzzy graphs. This result is acting a strong role in the field of fuzzy graph theory.

Table 3. Vertex-membership values corresponding to the main zones with WBSEDCL.

Name of Zone	v_i	Demand in MU (approx.)	$\sigma(v_i)$
WBSEDCL	v_1	4116	0.98
Kolkata Zone	v_2	1001	0.25
Burdwan Zone	v_3	10014	0.26
Midnapore Zone	v_4	479	0.13
Berhumpore Zone	v_5	234	0.08
Siliguri Zone	v_6	763	0.2

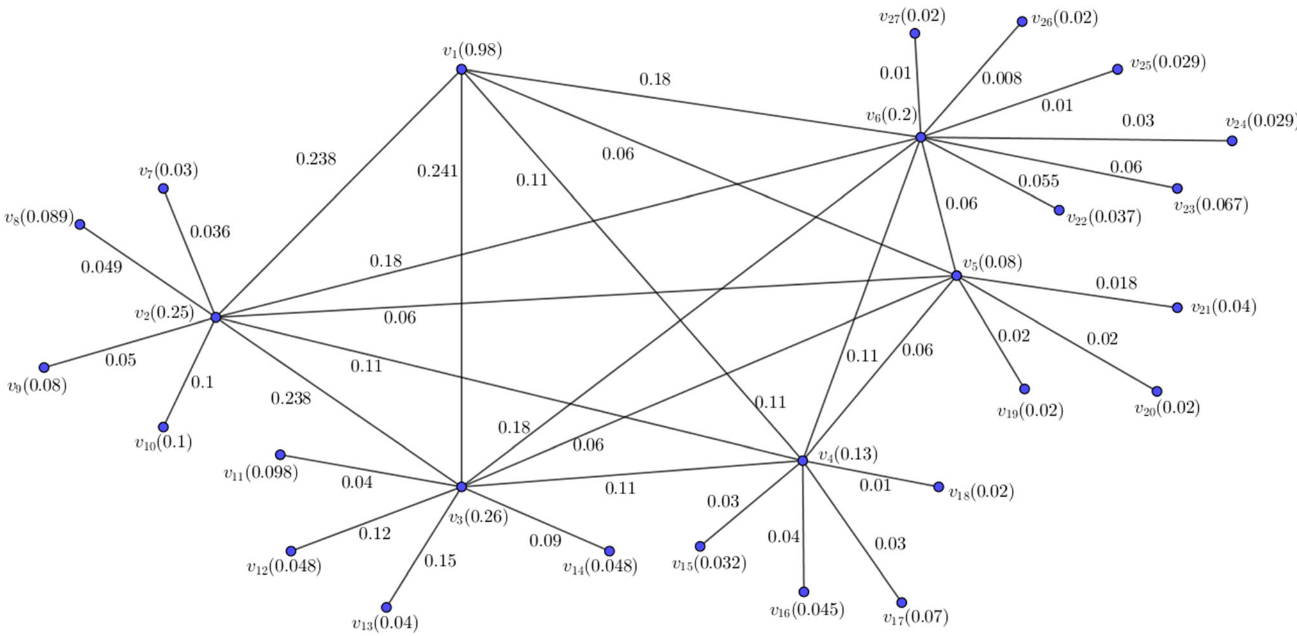

Figure 3. Constructed fuzzy graph.

(iv) Another definition semi-tree covering number is speaking for the case of tree cover taking role of only fuzzy edges in a fuzzy graphs.
(v) In the application part, we characterized an electricity distribution system with impreciseness by the help of the introduced definitions and results.
(vi) Lastly, we have shown that the strength of fuzzy tree covering number is more than semi-tree covering number of a fuzzy graph for real-life applicability.

These all are representing high-powered significance of our proposed ideas, definitions and results with proper proof.

6. Conclusion

This article introduces new definitions of fuzzy tree covering number and semi tree covering number of fuzzy graphs. Also, bounds of fuzzy tree covering numbers for the fuzzy tree, fuzzy cycle, and fuzzy fans are characterized. The relation between the semi-tree covering number and fuzzy vertex independent number for a fuzzy graph is constructed. The defined parameters are used to solve the problem of the electricity distribution system in the application part, which is very closely connected to any such covering problem of fuzzy graphs.

In the future, we will try to define other types of fuzzy covering numbers for fuzzy graphs and evaluate some bounds for the same. Also, we are interested in working on fuzzy graphoidal covering number of fuzzy graphs and showing real-life applicability of those covering numbers for better coverage in a fuzzy system.

Data availability All the data for the application part are taken from the Annual Report of Reserve Bank of India 2020-2021, published on May 24, 2021.

Declarations

Conflict of interest As per the declaration, there is no conflict of interest between them.

Ethical approval There is no harmful performance with human or animal participants by any of the authors during the construction of this article.

Informed consent Before submitting the work, all co-authors and the responsible authorities-tacitly or explicitly-at the institute/ organization where the work is developed have declared their consent to submit this article.

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