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On the exact analytical formula for dimensionless injection rate in CO_2 storage based on Special Trans Functions Theory

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Abstract. In this article, Special Trans Functions Theory (STFT) is used to obtain exact solutions of the transcendental equation that describes Dimensionless Injection Rate (DIR) for CO_2 storage. The obtained results were compared with DIR values calculated by a sequential asymptotic formula (SAF) based on the Lambert W function. The proposed exact analytical approach to determining DIR solutions implies considerable qualitative improvement compared to the conventional analytical and numerical methods. Determining STFT DIR solution for CO_2 storage is novel and precise, while the computations are simple, practical and do not require significant computation time. Note that the formulas derived within STFT in applied physics and engineering domains are characterized by structural robustness and complexity. It is a kind of "genetic characteristic" of STFT formulas, which does not affect the efficiency and simplicity of their applications. This paper additionally analyzes the importance of choosing an exact analytical solution for transcendent functional equations that describe the phenomenon of CO_2 storage, since these above-mentioned nonlinear forms describe many other significantly different physical phenomena. This implies the generalization of STF Theory.

Keywords. STFT; CO_2 storage; DIR; Lambert W function; SAF.

1. Introduction

The subject of the theoretical (mathematical and computational) analysis presented in this article is the qualitative improvement of the conventional solution (so-called Sequential Approximate Formula based on the Lambert W function) to the transcendental functional equation for Dimensionless Injection Rate), Ref. [1], with the usage of the Perovich's Special Trans Functions Theory - STFT.

Let us note that the aforementioned transcendental functional equation is obtained within CO_2 storage phenomenon studies as a logical consequence of the research line presented by references [2–5] and also by references [6–11].

Namely, a transcendental equation for DIR of CO_2 storage (marked as α) appears in the article Ref. [1]. This equation takes the form

$$P - P_0 = \frac{\mu_c \alpha(\beta - \varepsilon\, ln\alpha)}{\mu_b(c_r + c_b)} \tag{1}$$

where P [$ML^{-1}\ T^{-2}$] and P_0 [$ML^{-1}\ T^{-2}$] are the current and initial fluid pressures, μ_c [$ML^{-1}\ T^{-1}$] is the dynamic viscosity of pure CO_2, μ_b [$ML^{-1}\ T^{-1}$] is the dynamic viscosity of brine (free of CO_2), c_r [$M^{-1}\ L\ T^2$] is the compressibility of the rock, c_b [$M^{-1}\ L\ T^2$] is the compressibility of the brine.

After simple structural modification formula (1) takes the form

$$C_\varepsilon e^{-\ln\alpha} = \beta_\varepsilon - \ln\alpha \tag{2}$$

where

$$C_\varepsilon = \frac{(P - P_0)\mu_b(c_r + c_b)}{\mu_c \varepsilon}, \quad \beta_\varepsilon = \frac{\beta}{\varepsilon}. \tag{3}$$

If

$$\Psi = \beta_\varepsilon - \ln\alpha, \quad \ln\alpha = \beta_\varepsilon - \Psi \tag{4}$$

is replaced into Eq. (1), then $\Psi = C_\varepsilon e^{-\beta_\varepsilon} e^{\Psi}$ is obtained. This equation can be rewritten in the transcendental functional equation form

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Table 1. Values of the parameters which characterize the analyzed problem of CO_2 storage [1].

Parameter	Symbol	Value
Injection rate	M_0	15 kg/s
Well radius	r_w	0.2 m
Radial extent	r_E	20 km
Porosity	ϕ	0.2
Rock compressibility	c_r	$4.5 \cdot 10^{-10}\ P_a^{-1}$
Initial pressure	P_0	$10\ MP_a$
Permeability reduction factor due to salt precipitation	k_{IS}	1
Formation thickness	H	30 m
Permeability	k	100 mD
CO_2 density	ρ_c	797 kg/m^3
CO_2 viscosity	μ_c	$7.1 \cdot 10^{-5}\ P_a s$
Brine viscosity	μ_b	$9.63 \cdot 10^{-4}\ P_a s$
Brine compressibility	c_b	$3.54 \cdot 10^{-10}\ P_a^{-1}$

$$\Psi(x) = xe^{\Psi(x)} \tag{5}$$

where

$$x = C_\varepsilon e^{-\beta_\varepsilon} \tag{6}$$

From Eq. (4) we get

$$\alpha = e^{\beta_\varepsilon - \Psi}. \tag{7}$$

or,

$$\alpha = \frac{C_\varepsilon}{\Psi} \tag{8}$$

having in mind that $e^{-\Psi} = \frac{x}{\Psi}$. Additionally, $\alpha[-]$ is defined in Ref. [1] as

$$\alpha = \frac{M_o \mu_b (c_r + c_b)}{4\pi H \rho_c k}. \tag{9}$$

where M_0 [MT^{-1}] is the mass injection rate of CO_2, H[L] is the formation thickness, ρ_c [ML^{-3}] is the density of pure CO_2, and k[L^2] is permeability. A more detailed definition of variables β and ε can be found in article Ref. [1]. The values of the parameters that appear in Eqs. (1) and (9), as well as in the derived formulas for α (Eqs. (7) and (8)) are given in table 1.

The transcendental functional equation of the form

$$\Psi(x) = xe^{\Psi(x)}, \quad 0 \le x \le \frac{1}{e}, \tag{10}$$

is presented in the previous derivation of formulas (7) and (8). Formula (10) describes physical phenomena within the CO_2 storage, as well as many very different physical processes in applied physics and the engineering domain ([12–17] and many other articles). A graphical presentation of the Eq. (10) is given in figure 1. It is evident that Eq. (10) has two solutions in the defined domain $x \in [0, e^{-1}]$. The larger solution is of interest to this work.

Eq. (1) is solved in Ref. [1] by using the Sequential Approximate Formula (SAF) based on the Lambert W

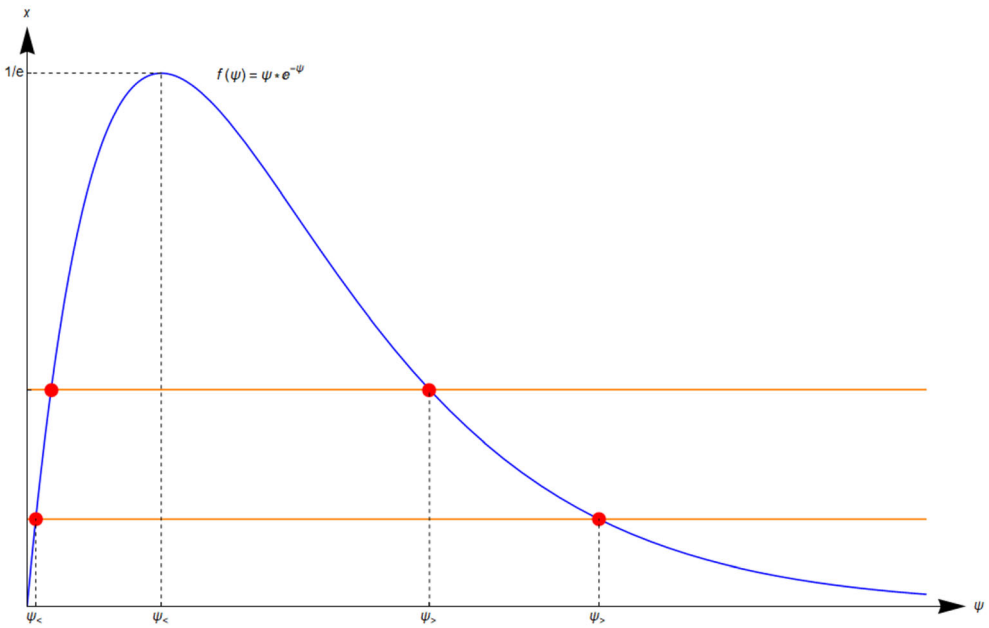

Figure 1. Two solutions to the transcendental equation (10).

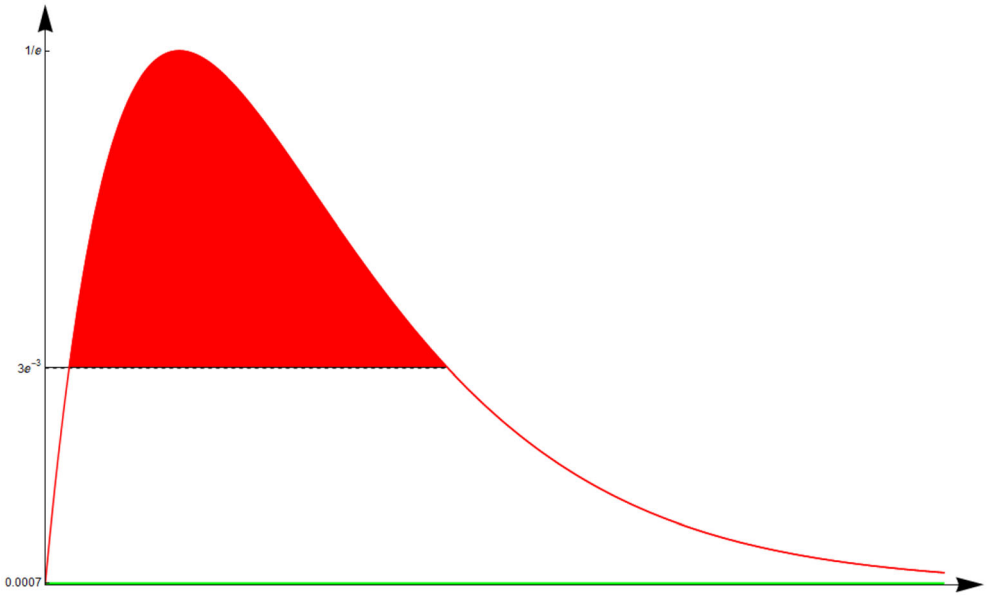

Figure 2. The constraint of the functional parameter x in the application of SAF based on the Lambert W function.

function. Note that this approximate approach is very complicated, not comprehensive and has low precision. Additionally, it depends a lot on the parameters given in table 1. Namely, SAF implies the condition $x<3/e^3$ (i.e., $x<0.15$). But, in the Ref. [1], that constraint is even more restricted, i.e. $x<0.0007$. So, it can simply be said that the parameters are strictly defined for the SAF method, based on the Lambert W function, to be applicable at all.

Figure 2 shows the subset of parameters in table 1 for which it is possible to apply the Sequential Approximate Formula based on the Lambert W function to determine the solution of Eq.(10). concerning the set of all parameter values. We see that it is less than 0.2% (notice the green area in figure 2). A brief description of SAF method is given in subsection 2.1.

It is important to point out that in this article, the transcendental functional equation for the analysis of the dimensionless injection rate (Eq. (10)) is presented in detail. An analytical closed solution was obtained for this equation using the Special Trans Function Theory (STFT). Let us note that STFT is a very useful tool for solving broad families of transcendental functional equations and obtaining exact analytical solutions [12–15, 17–20] and other publications. Since STFT ensures reaching extreme precisions in numerical results the highest precision in defining the Dimensionless Injection Rate has been achieved so far in this paper.

Also, the STFT exact analytical solution is completely invariant to the parameter values given in table 1. Of course, this is a significant advantage over the Sequential Asymptotic Formula based on the W function. To underline that in the SAF approach less than 0.2%, (figure 2) while in the STFT approach 100% of the set of parameters is the domain of validity of the methods.

Based on the above, it is more than clear that the STF Theory qualitatively improves the solutions of the DIR transcendental equation (10) obtained by applying the Sequential Asymptotic Formula based on Lambert's W function:

- In an analytical sense: namely, STFT solutions are exact analytical in closed form. Theoretically, this leads to the fact that for these families of transcendental equations we consider the problem closed in the real domain. On the other hand, we have Sequentially Asymptotic Formulas;
- In terms of accuracy: the precision of STFT formulas is very high, practically unlimited, while the precision of SAF is extremely low;
- In the degree of dependence on the parameters given in table 1: STFT formulas are invariant to the values of the parameters given in table 1, while the SAF is highly dependent on those parameters. SAF is a valid method with limited accuracy for only 0.2% of parameter set values.

The logical consequence of the above is that in the analysis of Eq. (10) STFT is the standard method. Consequently, the application of SAF based on the W function under all defined constraints would be possible if STFT did not exist.

This article is structured as follows. Analytical approaches to solving DIR are presented in section 2. Subsection

2.1 gives a summary of SAF methods for solving DIR. In subsection 2.2 Special Trans Functions are presented as exact analytical solutions of Eq.(10). In subsection 2.3 the formula for DIR obtained using STFT is presented. In section 3 there are numerical and graphical simulations of the formulas obtained in section 2. The conclusion of the paper is given in section 4.

2. Concerning the analytical solutions obtained within the DIR transcendental equation analysis

In this section, the subject matter is determining analytical solutions to the transcendental equation (10). Firstly, by using the SAF based on the W function as an asymptotic analytical method, and, secondly, by obtaining exact analytical STF solutions as special trans functions, $tran_<(x)$ and $tran_>(x)$. Also, the exact analytical formula for DIR is presented.

2.1 *DIR analytical solutions modelling by using the SAF based on the Lambert W functions presented in Ref. [1]*

In his work from 1758. Lambert studied the equation of the form

$$\Psi^{\alpha} - \Psi^{\beta} = (\alpha - \beta)v\Psi^{\alpha+\beta}. \tag{11}$$

If $\alpha \to \beta$ then equation (11) can be written in the form $\ln \Psi = -v\Psi^{\beta}$. In the special case when β is negative, equation (11) has a solution that can be written using Lambert's W function: $\Psi = \exp\left[-\frac{W(-\beta v)}{\beta}\right]$. For $-\frac{1}{e} \le x < 0$, Lambert's W function takes two values. Lambert's W function has a so-called principal branch, which is denoted by $W_0(x)$ or $W(x)$, and is obtained for $W(x) \ge -1$. The branch of the function obtained for $W(x) \le -1$ is denoted by $W_{-1}(x)$. A more detailed presentation of Lambert's W function can be found in papers [16, 21–24]. In article [15], equation (11) is analytically solved using Perovich's STFT.

There are several approximative and asymptotic formulas for Lambert's W function. The interested reader can find a review of them in the work of Corless [22] and Veberič [16]. In various domains [16], robust and complicated numerical methods for solving modified Lambert transcendental equations have been applied using advanced software tools [16, 21, 23, 24]. In paper Ref. [1], an asymptotic formula that yields reasonably accurate results for $W(z) \ge 3$ is used and has the following form [16]:

$$\begin{aligned} W(z) &= \ln(z) \ - \ln(\ln(z)\ ) \\ &+ \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} c_{km}(\ln(\ln(z)\ )\ )^{m+1}(\ln(z)\ )^{-k-m-1} \\ &= L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(L_2-2)}{2{L_1}^2} + \frac{L_2\left(2{L_2}^2-9L_2+6\right)}{6{L_1}^3} \\ &+ \frac{L_2\left(3{L_2}^3-22{L_2}^2+36L_2-12\right)}{12{L_1}^4} \\ &+ \frac{L_2\left(12{L_2}^4-125{L_2}^3+350{L_2}^2-300L_2+60\right)}{60{L_1}^5} \\ &+ O\left[\left(\frac{L_2}{L_1}\right)^6\right] \end{aligned} \tag{12}$$

where

$$L_1 = \ln(z)\ ,\ L_2 = \ln(\ln(z)\ )$$

for W_0 branch and

$$L_1 = \ln(-z)\ ,\ L_2 = \ln(-\ln(-z)\ ) \tag{13}$$

for $W_{-1}(x)$ branch.

Equation (12) based on the SAF approach to the Lambert W function represents an analytical solution for the transcendental equation (10). In addition, equation (12) is an asymptotic solution that gives reasonably accurate results under the condition $x \le \frac{3}{e^3}$, i.e., for $W(x) > 3$. More specifically, formula (12) can be applied under the above conditions for x to obtain an approximate solution of equation (10). This strict constraint is graphically shown in figure 2 by the dashed black line. Consequently, all solutions of the transcendental equation (12) above this line can not be obtained by the proposed formula. Namely, the application of this formula requires that a large set of parameters be precisely set so that the value of the variable x reaches the set limits. This problem significantly complicates the application of SAF methodology, especially in the field of application of physical phenomena that depend on a large number of different parameters. For most combinations of parameters, the value of the variable x will never reach the target constraint. Or even worst, in the final analysis, it may be necessary to artificially adjust individual parameters to make the formula applicable. Generally speaking, due to the set constraint the formula becomes impractical to use in many scientific areas.

In cases where greater precision is desired, it is necessary to take into account the subintervals of each branch individually. This further complicates research and requires additional knowledge of numerical mathematics and programming skills. For example, if the principal branch is calculated, the domain interval must be divided into four subintervals $[-e^{-1}, -0.32358170806015724)$, $[-0.32358170806015724,$

0.14546954290661823), [0.14546954290661823, 8.706658967856612) and $[8.706658967856612, +\infty)$ a separate algorithm for calculating the Lambert W function is applied to each subinterval. Despite such complicated modelling, the approach is generally inefficient because it guarantees only 3 exact decimals on the entire domain. A similar situation occurs when modelling the $W_{-1}(x)$ branch. Here, the domain is divided into three subintervals, over which fundamentally different algorithms are applied and 5 exact decimal places are guaranteed [16].

Such a complex approach to solving transcendental equations is simply not acceptable, especially given the existence of a universal tool developed within STF theory. There is no need to expect the researcher to follow different algorithms and predict in advance the subintervals in which the observed parameter values (or even calculated values) will fit. We believe that it is much more convenient and efficient for both the researcher and the potential reader to follow a universal approach that provides a unique and precise solution for the entire domain of the observed set of parameters.

2.2 *The special trans function* $tran_{>}(x)$ *as an exact analytical solution of DIR transcendental equation*

In this paper, STFT modelling of the exact analytical solution for the transcendental equation (10) is applied. In other words, the transcendental equation for the dimensionless injection rate (Eq. (1) or Eq. (10)) was accurately analytically solved using STFT [12–14, 18–20] and a closed-form solution was obtained for it.

Within STFT the Eq. (10), has two analytical closed-form solutions denoted as $\Psi_{<} = tran_{<}(x)$ and $\Psi_{>} = tran_{>}(x)$. These two real exact analytical solutions $tran_{<}(x)$ and $tran_{<}(x)$ are known as special trans functions [12–14, 18–20]. Therefore, according to the DIR formulas (7) and (8),

$$\alpha = e^{-tran_{>}(x)+\beta_{\varepsilon}} = \frac{C_{\varepsilon}}{tran_{>}(x)} \tag{14}$$

is obtained. In some detail, the exact analytical solution of the transcendental equation (1) (or Eq. (10)) was previously published in Ref. [12–14, 18–20]. Namely, for $\Psi(x) < 1$ the Eq. (10) has an exact analytical solution of the form:

$$\Psi_{<}(x) = tran_{<}(x) \tag{15}$$

where

$$tran_{<}(x) = \lim_{t\to\infty}\left[\ln\left(\frac{\Phi_{<}(t-1,x)}{\Phi_{<}(t,x)}\right)\right]$$

or, since $\Psi_{<} = xe^{\Psi_{<}}$, it follows that

$$\Psi_{<}(x) = tran_{<}(x) = \lim_{t\to\infty}\left[x\left(\frac{\Phi_{<}(t-1,x)}{\Phi_{<}(t,x)}\right)\right] \tag{16}$$

where

$$\Phi_{<}(t,x) = \sum_{n=0}^{[t]}\frac{(-x)^n(t-n)^n}{n!}$$

and $[t]$ denotes the greatest integer less than or equal to t. To simplify the analysis and numerical calculations, it is convenient to introduce the expression (16) in the following way:

$$\langle\Psi_{<}(x)\rangle_{P_{<}} = \left\langle\ln\left(\frac{\Phi_{<}(t-1,x)}{\Phi_{<}(t,x)}\right)\right\rangle_{P_{<}} \tag{17}$$

where $\langle\Psi_{<}\rangle_{P_{<}}$ is a value of the transcendental number $\Psi_{<}$ given with

$$P_{<} = [1 - log(G_{<})] \tag{18}$$

accurate digits, where the error function $G_{<}$ is defined as:

$$G_{<} = |\Psi_{<}(x) - xe^{\Psi_{<}(x)}| \tag{19}$$

For a fixed integer $[t]$, expression for $\langle\Psi_{<}\rangle_{P_{<}}$ takes the form:

$$\langle\Psi_{<}(x)\rangle_{P_{<}} = \left\langle x\cdot\frac{\sum_{n=0}^{[t-1]}\frac{(-x)^n(t-1-n)^n}{n!}}{\sum_{n=0}^{[t]}\frac{(-x)^n(t-n)^n}{n!}}\right\rangle_{P_{<}} \tag{20}$$

because $\Psi_{<} = xe^{\Psi_{<}}$.

Additionally, the exact analytically closed-formed solution of the transcendental equation (1) (or Eq. (10)) for $\Psi > 1$ has been published as a new and original solution within Special Trans Functions Theory in Ref. [12–14, 18–20]. Namely,

$$\Psi_{>}(x) = \Psi_{<}(x) + K_{\delta}(x,a) \tag{21}$$

where $\Psi_{<}(x)$ is defined in Eqs. (16) and (20), and $K_{\delta}(x,a)$ has the analytical closed-form solution:

$$K_{\delta}(x,a) = tran_{K_{\delta}}(x,a)$$

where $tran_{K_{\delta}}(x,a)$ is a special trans function defined as:

$$tran_{K_{\delta}}(x,a) = \lim_{x\to\infty}\left\{\ln\left(\frac{\Phi_{K_{\delta}}(u-a,b)}{\Phi_{K_{\delta}}(u,b)}\right)\right\} \tag{22}$$

and $\Phi_{K_{\delta}}(b,u)$ is given as:

$$\Phi_{K_{\delta}}(u,b) = \left(R(u,b)\cdot e^{bu}\right)' = (bR(u,b) + R'(u,b))\cdot e^{bu} \tag{23}$$

where

$$R(u) = \sum_{n=0}^{\left[\frac{u}{a}\right]} (-1)^n \frac{(u-na)^n}{n!} \left(be^{-ab}\right)^n, \quad u > 0 \tag{24}$$

and

$$b = \frac{\Psi_<(x)}{a} \tag{25}$$

where $a \in \Re^+$ and $a > 1$.

$$\langle \Psi_>(x) \rangle_{P_>} = \left\langle x \left(\frac{\Phi_<(t-1,x)}{\Phi_<(t,x)} \right) \right\rangle_{P_>} + \left\langle \ln \left(\frac{\Phi_{K_\delta}(u-a,b)}{\Phi_{K_{\delta_<}}(u,b)} \right) \right\rangle_{P_>} \tag{27}$$

or form

$$\langle \Psi_>(x) \rangle_{P_>} = \left\langle x \frac{\sum_{n=0}^{[t-1]} \frac{(-x)^n (t-1-n)^n}{n!}}{\sum_{n=0}^{[t]} \frac{(-x)^n (t-n)^n}{n!}} + \ln \left(\frac{be^{-ab} \sum_{n=0}^{\left[\frac{u-a}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-a-na)^n}{n!} + e^{-ab} \sum_{n=1}^{\left[\frac{u-a}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-a-na)^{n-1}}{(n-1)!}}{b \sum_{n=0}^{\left[\frac{u}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-na)^n}{n!} + \sum_{n=1}^{\left[\frac{u}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-na)^{n-1}}{(n-1)!}} \right) \right\rangle_{P_>} \tag{28}$$

In this paper, the value for a is taken from the monograph [18], but any other real value $a > 1$ would be also appropriate. For example, in the article [14], the value $a = 2$ was used. It is worth noting that the precision of the solution does not depend on the value of a, the only important thing is that a is chosen from the R^+ and that a has a value greater than 1. Practically applicable formula (21) has the form:

$$\langle \Psi_>(x) \rangle_{P_>} = \langle \Psi_<(x) \rangle_{P_<} + \langle K_\delta(x,a) \rangle_{P_>} \tag{26}$$

where $\langle \Psi_>(x) \rangle_{P_>}$ is the value of the transcendental number $\Psi_>(x)$ given with $[P_>]$ accurate digits. Analogously to the previous case $P_> = [1 - log(G_>)]$, where the error function is defined as $G_> = |\Psi_> - xe^{\Psi_>}|$, and $P_{K_\delta} = [1 - log(G_{K_\delta})]$ where G_{K_δ} takes the form

$$G_{K_\delta} = |K_\delta - ab\left(e^{K_\delta} - 1\right)|$$

More precisely, formula (21) takes the form:

where $a = 2\ln 3$ and $b = \frac{\Psi_<}{2\ln 3}$ [18]. Note that Eq. (28) is an exact solution of Eq. (10). After a simple structural modification of equation (28), the following form is obtained:

$$\langle \Psi_>(x) \rangle_{P_>} = \left\langle \Psi_<(x) \frac{b \sum_{n=0}^{\left[\frac{u-a}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-a-na)^n}{n!} + \sum_{n=1}^{\left[\frac{u-a}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-a-na)^{n-1}}{(n-1)!}}{b \sum_{n=0}^{\left[\frac{u}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-na)^n}{n!} + \sum_{n=1}^{\left[\frac{u}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-na)^{n-1}}{(n-1)!}} \right\rangle_{P_>} \tag{29}$$

2.3 *Determining the exact analytical formula for dimensionless injection rate*

In this subsection, the goal is to get an explicit expression for DIR. From Eqs. (28) and (8) directly follows:

$$\langle\alpha\rangle_{P_>} = \left(\left\langle C_\varepsilon e^{-\beta_\varepsilon} \frac{\sum_{n=0}^{[t-1]} \frac{\left(-C_\varepsilon e^{-\beta_\varepsilon}\right)^n (t-1-n)^n}{n!}}{\sum_{n=0}^{[t]} \frac{\left(-C_\varepsilon e^{-\beta_\varepsilon}\right)^n (t-n)^n}{n!}} \right. \right. \\ \left. \left. + \ln\left(\frac{be^{-ab} \sum_{n=0}^{\left[\frac{u-a}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-a-na)^n}{n!} + e^{-ab} \sum_{n=1}^{\left[\frac{u-a}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-a-na)^{n-1}}{(n-1)!}}{b \sum_{n=0}^{\left[\frac{u}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-na)^n}{n!} + \sum_{n=1}^{\left[\frac{u}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-na)^{n-1}}{(n-1)!}} \right) \right\rangle_{P_>} \right)^{-1} \tag{30}$$

and $b = \frac{\Psi_<}{a}$. Equation (2.3) takes the following structure:

$$\langle\alpha\rangle_{P_>} = \left\langle \frac{C_\varepsilon}{\Psi_<} \frac{b \sum_{n=0}^{\left[\frac{u}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-na)^n}{n!} + \sum_{n=1}^{\left[\frac{u}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-a-na)^{n-1}}{(n-1)!}}{b \sum_{n=0}^{\left[\frac{u-a}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-a-na)^n}{n!} + \sum_{n=1}^{\left[\frac{u-a}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-a-na)^{n-1}}{(n-1)!}} \right\rangle_{P_>} \tag{31}$$

Based on equations (29) and (7), it can be concluded that

$$\langle\alpha(x,a)\rangle_{P_\alpha} = \exp\left(\beta_\varepsilon - \left\langle \Psi_<(x) \frac{b \sum_{n=0}^{\left[\frac{u-a}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-a-na)^n}{n!} + \sum_{n=1}^{\left[\frac{u-a}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-a-na)^{n-1}}{(n-1)!}}{b \sum_{n=0}^{\left[\frac{u}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-na)^n}{n!} + \sum_{n=1}^{\left[\frac{u}{a}\right]} \frac{\left(-be^{-ab}\right)^n (u-na)^{n-1}}{(n-1)!}} \right\rangle_{P_\alpha} \right) \tag{32}$$

Equations (31) and (32) are exact analytical solutions for DIR (α). These formulas are simple, practical, and do not require a significant computation time, although formulas (28)–(32) seem robust and complex.

Formulas (31) and (32) imply the introduction of new gradient coefficients as follows $\frac{\partial\alpha}{\partial x}\frac{\partial x}{\partial\beta}, \frac{\partial\alpha}{\partial x}\frac{\partial x}{\partial\varepsilon}, \frac{\partial\alpha}{\partial x}\frac{\partial x}{\partial P}$ and $\frac{\partial\alpha}{\partial q_i}$, where notations and explanations of the parameter q_i are given in table 1. The modelling of gradient coefficients requires structurally very complex further analysis.

Formulas that are similar to equations (28)–(32) appear when considering the problem of precision of the numerical structure of special trans functions corresponding to exact analytical solutions in different scientific fields ([12–14, 18–20] and many other references).

There is a significant difference between the concept of STFT exact analytical solutions and the concept of conventional numerical techniques and analytical methods, primarily because by varying the upper bounds of the sum ($[t]$ or $[u/a]$) in STFT formulas, corresponding solutions can be obtained with arbitrarily high precision. The accuracy of the special trans function depends on the upper bound of the sum [12–14, 18–20]. This claim was experimentally confirmed by the numerical results shown in tables 3-6.

Finally, in table 2, we give a comparative overview of some characteristics of the conventional analytical method used in Ref. [1] (SAF based on the W function) and STF Theory used in this article.

3. Numerical and graphical simulations of formulas derived in section 2

New original methods, theories and the essential importance of their application cannot be neglected in serious research and scientific approaches. The subject of theoretical analysis that follows is the comparison of the new STFT approach for modelling exact analytical DIR solutions and the approaches described in Ref. [1]. More precisely, within this section, a numerical and graphical comparison of approximate formula (12) with formulas (28)–(32) is given for different values of the functional parameter x.

The main goal of this paper is to compare DIR solutions obtained using STFT and approximate formula (12) for SAF. For the selected set of parameter values from table 1, the approximate method (12) applied in Ref. [1] is quite

good. Unfortunately, the spectrum of application of such parameter values is very narrow. In addition, it is unjustified to apply the approximation when the exact method (STFT) as a theoretical standard has already been developed and successfully applied in many areas.

3.1 *Numerical results for $tran_<(x)$ function solutions Eq. (20)*

Bearing in mind the formula (29), it is clear that the value of the $tran_>(x)$ function depends on the value and precision of $tran_<(x)$. Therefore, at the beginning of this section, the results of the numerical test for $tran_<(x)$ are given for some of the selected values of the parameter x as well as for some selected values of the upper limits of the sum in the formula (20).

It is not difficult to see that the $tran_<(x)$ functions have excellent accuracy for all selected x values (see table 3). Additionally, the numerical precision of $tran_<(x)$ depends on the selected values for the upper limit $[t]$ in the sum (21). Figure 3 shows the graphical representations of the numerical results for $tran_<(x)$ (table 3) and their dependence on the values of the parameters x and $[t]$.

Note that these numerical results (table 3) are not essential to the analyzed example from Ref. [1], but in an educational sense, they may be useful for articles where the special function $tran_<(x)$ appears as an analytical solution. However, it has to be emphasized that to determine the special trans function $tran_>(x)$, within STFT it is necessary to define the special trans function $tran_<(x)$ as well.

In theoretical terms, STFT is a comprehensive, consistent theory for determining the exact closed-form of the analytical solution for a wide family of nonlinear functional equations [12–15, 17–20], as well as in some other papers.

3.2 *Comparison of $W_{-1}(x)$ function (Eq. (12)) and $tran_>(x)$ function (Eqs. (28) and (29))*

Within this subsection, some numerical results based on analytical exact formulas (28) and (29) and on the asymptotic analytical formula (12) are presented. The obtained results are shown in table 4. Differences in the results obtained from equations (28) and (29) concerning the solutions of equation (12) occur due to the approximate structure of formula (12) and the exact analytical nature of formulas (28) and (29).

Note that the STFT numerical calculations of formulas (28) and (29) are simple, practical and do not take much time. Accordingly, it can be argued that formulas (28) and (29) are dominant concerning the solution of the W function (12).

Table 2. Comparative presentation of the SAF and STFT characteristics.

Applied Method	Nature of the Solution	Precision	The Solution Definition Domain	Gradient Coefficients	Standard Method
SAF	Asymptotic Analytical	Low	$0 < x < 0.0007$ i.e., 0.2%	No	No
STFT	Exact Analytical–in Closed Form (Special Trans Functions)	Highest (Unlimited)	$0 \leq x \leq 1/e$ i.e., 100%	$\frac{\partial z}{\partial x}\frac{\partial x}{\partial \beta}$, $\frac{\partial z}{\partial x}\frac{\partial x}{\partial c}$, $\frac{\partial z}{\partial x}\frac{\partial x}{\partial P}$, $\frac{\partial z}{\partial q_i}$	Yes

Table 3. The numerical results of the $tran_{<}(x)$ function (Eq. (21)) for some of the selected x values.

x=0.000001			
[t]	$tran_{<}(x)$	Precision	
50	0.000001000001000001500002666671875010800023343 1	$P_{tran<}(x)$	361
100	0.000001000001000001500002666671875010800023343 1	$P_{tran<}(x)$	723
200	0.000001000001000001500002666671875010800023343 1	$P_{tran<}(x)$	1445
300	0.000001000001000001500002666671875010800023343 1	$P_{tran<}(x)$	2167
400	0.000001000001000001500002666671875010800023343 1	$P_{tran<}(x)$	2889
500	0.000001000001000001500002666671875010800023343 1	$P_{tran<}(x)$	3611
x=0.001			
[t]	$tran_{<}(x)$	Precision	
50	0.001001001502671885823395187155416002577266698 1	$P_{tran<}(x)$	198
100	0.001001001502671885823395187155416002577266698 1	$P_{tran<}(x)$	396
200	0.001001001502671885823395187155416002577266698 1	$P_{tran<}(x)$	792
300	0.001001001502671885823395187155416002577266698 1	$P_{tran<}(x)$	1188
400	0.001001001502671885823395187155416002577266698 1	$P_{tran<}(x)$	1584
500	0.001001001502671885823395187155416002577266698 1	$P_{tran<}(x)$	1980
x=0.1			
[t]	$tran_{<}(x)$	Precision	
50	0.111832559158962964833569456820265842272645362	$P_{tran<}(x)$	76
100	0.111832559158962964833569456820265842272645362	$P_{tran<}(x)$	151
200	0.111832559158962964833569456820265842272645362	$P_{tran<}(x)$	301
300	0.111832559158962964833569456820265842272645362	$P_{tran<}(x)$	452
400	0.111832559158962964833569456820265842272645362	$P_{tran<}(x)$	602
500	0.111832559158962964833569456820265842272645362	$P_{tran<}(x)$	753
x=0.2			
[t]	$tran_{<}(x)$	Precision	
50	0.259171101819073745056651950215406705713588340	$P_{tran<}(x)$	50
100	0.259171101819073745056651950215406705713588340	$P_{tran<}(x)$	100
200	0.259171101819073745056651950215406705713588340	$P_{tran<}(x)$	199
300	0.259171101819073745056651950215406705713588340	$P_{tran<}(x)$	298
400	0.259171101819073745056651950215406705713588340	$P_{tran<}(x)$	397
500	0.259171101819073745056651950215406705713588340	$P_{tran<}(x)$	496
x=0.3			
[t]	$tran_{<}(x)$	Precision	
50	0.489402227180214969036231251921738938180576111	$P_{tran<}(x)$	29
100	0.489402227180214969036231251996293368923410006	$P_{tran<}(x)$	57
200	0.489402227180214969036231251996293368923410006	$P_{tran<}(x)$	113
300	0.489402227180214969036231251996293368923410006	$P_{tran<}(x)$	169
400	0.489402227180214969036231251996293368923410006	$P_{tran<}(x)$	225
500	0.489402227180214969036231251996293368923410006	$P_{tran<}(x)$	281
x=0.36			
[t]	$tran_{<}(x)$	Precision	
50	0.806084315646218822932124985602502609945258489	$P_{tran<}(x)$	10
100	0.806084315970817777995032459581475159830775360	$P_{tran<}(x)$	18
200	0.806084315970817778285521361620992001764814606	$P_{tran<}(x)$	37
300	0.806084315970817778285521361620992001997459968	$P_{tran<}(x)$	56
400	0.806084315970817778285521361620992001997459968	$P_{tran<}(x)$	74
500	0.806084315970817778285521361620992001997459968	$P_{tran<}(x)$	92

It is not difficult to observe that the function $tran_{>}(x)$ has superior accuracy (see tables 4 and 5) as well as superior time calculation efficiency for all values of x. Based on the proposed analytical structure of the formulas and from the numerical values shown in tables 4 and 5, it becomes clear that STFT can produce a superior numerical calculation. The SAF approach to equation (12) becomes unusable if the sequential conditions for the functional parameter x are not met, while the function $tran_{>}(x)$ becomes a unique analytical expression for solving the transcendental equation (10).

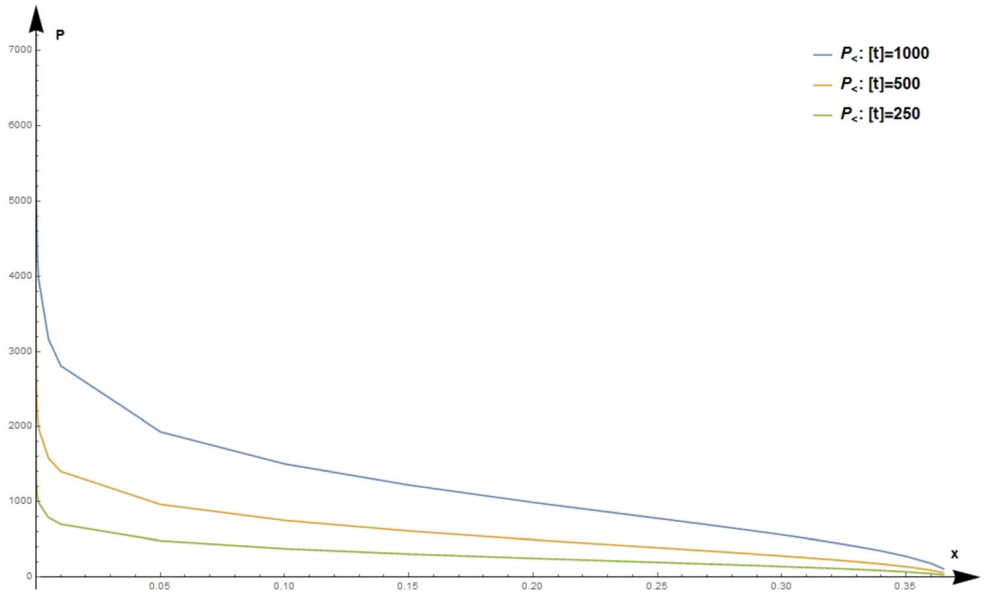

Figure 3. Graphical representation of the $tran_<(x)$ function solution precision for some selected values of the parameter [t] from table 3.

Analysis of the results presented in tables 4 and 5 shows the advantage of the $tran_>(x)$ function when $x > 0.15$. It is more than clear that the asymptotic formula (12) has its limitations of use, and is therefore only applicable to some specific x values. These limitations lead to the selection of specific sets of synthetic functional parameters and strong limitations without which it is not possible to apply the asymptotic formula (12). This approach makes formula (12) unreliable for entire intervals of values of functional parameters.

All this is a sufficient reason to use accurate analytical solutions of $tran_>(x)$ in the form of a trans function ($tran_<(x)$ and $tran_>(x)$), i.e., to make trans functions a standard approach for solving this type of nonlinear functional equation. Numerical results and their comparison with W functions show the credibility of STF theory. A graphical comparison of these results are given in figures 4 and 5.

It is important to emphasize that the declining trend of the accuracy of the $tran_>(x)$ solution for x values close to 0.35 (see figure 4) did not occur due to poor performance of formulas (28) and (29). Namely, the value of solution $tran_>(x)$ depends on the value (i.e., precision) of solution $tran_<(x)$. As a result, the decrease in accuracy of solution $tran_>(x)$ can be very easily corrected by simply increasing the accuracy of solution $tran_<(x)$, which would achieve a constantly growing trend in accuracy $tran_>(x)$ shown in figure 4.

Examples of Perovich's Special Trans Function Theory (STFT) application are found in articles concerning the solutions genesis of closed-form in many scientific areas, such as the theory of neutron slowing down [25], nonlinear circuits theory [26], theory of dispersive equations [27], physics of nuclear reactor [17, 28], theory of materials engineering [14], theory of STFT conductive fluid typology [29], numerical simulations and computations of special trans functions [20], theory of some families of transcendental equations [30, 31], modelling plasma processes [32], Plutonium temperature estimation [12] and theory of the nuclear fuel burn-up [33], the inverse problem of temperature estimation [34], theory of certain Lambert transcendental equations [15], equilibrium critical thickness [13], theory of solar cell [35], in the theory of Kepler's transcendental equation [19], etc.

3.3 *On a $W_{-1}(x)$ function solution and $tran_>(x)$ function solution for DIR estimation*

This subsection will present some numerical results based on the proposed analytical exact solution (31) and (32) and the asymptotic formula for injection rate α from paper [1]. This comparison is shown in table 6, for very limited values of functional parameters characterizing this physical phenomenon. Note that the corresponding parameters within equations (31) and (32) are taken from the table 1.

Despite this fairly good match of the results in table 6, the error still exists and is not negligible. The dominance of STFT is indisputable and can therefore be treated as a theoretical standard in the analysis of CO_2 storage phenomena. In other words, in article [1] the parameters (table 1) are strictly limited. Their values were chosen to satisfy the application of an approximate SAF approach based on Lambert's W functions (12). The last column of table 6 shows the differences in the quantities of injected CO_2 measured as [kg/h], obtained if the STFT approach

Table 4. The numerical comparison of the $W_{-1}(x)$ function (Eq. (12)) and $tran_{>}(x)$ function (Eq. (28) and (29)) for some of the selected x values $x \geq 0.15$.

x=0.15		$tran_{<}(x) = 0.17949126834798473361699561937$	$P_{<}(x) = 1223$
Function	$[t]$	Result	Precision $P_{>}$
$tran_{>}(x)$	50	2.99359498679771815603012977805120293632927062	11
	100	2.99359498677437786687743997717816103333168477	23
	200	2.99359498677437786687749239304535975648703360	43
	500	2.99359498677437786687749239304535975648703359	112
	1000	2.99359498677437786687749239304535975648703359	223
$W_{-1}(x)$	–	N/A	N/A
x=0.2		$tran_{<}(x) = 0.25917110181907374505665195021$	$P_{<}(x) = 992$
Function	$[t]$	Result	Precision $P_{>}$
$tran_{>}(x)$	50	2.54264135777379324961573736564341310777768348	13
	100	2.54264135777352642429380606773495086702380713	25
	200	2.54264135777352642429380615666184829016147491	51
	500	2.54264135777352642429380615666184829016147491	128
	1000	2.54264135777352642429380615666184829016147491	256
$W_{-1}(x)$	–	N/A	N/A
x=0.25		$tran_{<}(x) = 0.35740295618138890306881110405$	$P_{<}(x) = 781$
Function	$[t]$	Result	Precision $P_{>}$
$tran_{>}(x)$	50	2.15329236411034609466176867355862917416983603	15
	100	2.15329236411034964916909915010384925773742032	28
	200	2.15329236411034964916909915009298137553620649	58
	500	2.15329236411034964916909915009298137553620649	146
	1000	2.15329236411034964916909915009298137553620649	291
$W_{-1}(x)$	–	N/A	N/A
x=0.3		$tran_{<}(x) = 0.48940222718021496903623125199$	$P_{<}(x) = 562$
Function	$[t]$	Result	Precision $P_{>}$
$tran_{>}(x)$	50	1.78133702342162748823614651367549713344431646	16
	100	1.78133702342162761197417028151275063008704670	32
	200	1.78133702342162761197417028151274526082155836	66
	500	1.78133702342162761197417028151274526082155836	165
	1000	1.78133702342162761197417028151274526082155836	330
$W_{-1}(x)$	–	N/A	N/A
x=0.35		$tran_{<}(x) = 0.71663881645607385058816980000$	$P_{<}(x) = 276$
Function	$[t]$	Result	Precision $P_{>}$
$tran_{>}(x)$	50	1.34971725219224883316244601646497154362334353	19
	100	1.34971725219224883338314445944635709819604858	39
	200	1.34971725219224883338314445944635709819003431	78
	500	1.34971725219224883338314445944635709819003431	195
	1000	1.34971725219224883338314445944635709819003431	276
$W_{-1}(x)$	–	N/A	N/A
x=0.365		$tran_{<}(x) = 0.87982009141595381117248400076$	$P_{<}(x) = 111$
Function	$[t]$	Result	Precision $P_{>}$
$tran_{>}(x)$	50	1.13065531336673958516811039292825122368414146	20
	100	1.13065531336673958517127545393645012731974164	42
	200	1.13065531336673958517127545393645012731974089	86
	500	1.13065531336673958517127545393645012731974089	111
	1000	1.13065531336673958517127545393645012731974089	111
$W_{-1}(x)$	–	N/A	N/A

proposed in this paper is used instead of the SAF approach proposed in [1]. More precisely, $\Delta = |M_0 - M_0{}^{tran_{>}}| \cdot 3600$ [kg/h]. In table 6, M_0 and $M_0^{tran_{>}}$ values are shown in kg/s, while pressure P is expressed in MPa.

The STFT approach and the results given in tables 3, 4, and 5 clearly show the conceptual and practical advantages of the trans function. The impractical application of a very complex SAF approximation model based on the Lambert W function is unnecessary, especially when there is a clear,

Table 5. The numerical comparison of the $W_{-1}(x)$ function (Eq.(12)) and $tran_{>}(x)$ function (Eq.(28) and (29)) for some of the selected x values $x<0.15$.

x=0.000001		$tran_{<}(x) = 0.000001000001000001500002666 67$	$P_{<}(x) = 7221$
Function	$[t]$	Result	Precision $P_{>}$
$tran_{>}(x)$	600	16.626508901405603058079824577321821060426 3707	11
	700	16.626508901373202966464923126996175645 3715402	13
	800	16.626508901372489051024015720557518117 5091537	15
	900	16.626508901372473716859048663969328565 7188062	17
	1000	16.626508901372473394494311650732854837 5475609	18
$W_{-1}(x)$	–	16.62650	7
x=0.00001		$tran_{<}(x) = 0.000010000100001500026667 18751$	$P_{<}(x) = 6152$
Function	$[t]$	Result	Precision $P_{>}$
$tran_{>}(x)$	600	14.163600815810184513653461255013134124661 0537	16
	700	14.163600815810183001399809631311219416039 2163	19
	800	14.163600815810183009148567569074607824896 3331	21
	900	14.163600815810183009109360919509209987343 8859	23
	1000	14.163600815810183009109557273900040421526 6117	25
$W_{-1}(x)$	–	14.16360	7
x=0.0001		$tran_{<}(x) = 0.000100010001500266718760 80233$	$P_{<}(x) = 5067$
Function	$[t]$	Result	Precision $P_{>}$
$tran_{>}(x)$	600	11.667114532566354418377253941630482362835 5508	22
	700	11.667114532566354418378828604631504759443 0216	25
	800	11.667114532566354418378827447741762830495 3409	28
	900	11.667114532566354418378827446970631527617 6218	32
	1000	11.667114532566354418378827446970473651866 9761	35
$W_{-1}(x)$	–	11.6671	6
x=0.001		$tran_{<}(x) = 0.001001001502671885823395 18715$	$P_{<}(x) = 3960$
Function	$[t]$	Result	Precision $P_{>}$
$tran_{>}(x)$	600	9.118006470402740121258337182046826193323 48155	32
	700	9.118006470402740121258337182046814274290 43219	38
	800	9.118006470402740121258337182046814274270 43486	43
	900	9.118006470402740121258337182046814274270 43497	47
	1000	9.118006470402740121258337182046814274270 43497	53
$W_{-1}(x)$	–	9.1180	5
x=0.01		$tran_{<}(x) = 0.010101527198538753273 92018767$	$P_{<}(x) = 2807$
Function	$[t]$	Result	Precision $P_{>}$
$tran_{>}(x)$	600	6.472775124394004694741057892724488037104 34559	55
	700	6.472775124394004694741057892724488037104 34559	65
	800	6.472775124394004694741057892724488037104 34559	73
	900	6.472775124394004694741057892724488037104 34559	82
	1000	6.472775124394004694741057892724488037104 34559	92
$W_{-1}(x)$	–	6.47277	6
x=0.1		$tran_{<}(x) = 0.111832559158962964833569 45682$	$P_{<}(x) = 1505$
Function	$[t]$	Result	Precision $P_{>}$
$tran_{>}(x)$	600	3.577152063957297218409391963511994880401 79626	114
	700	3.577152063957297218409391963511994880401 79626	133
	800	3.577152063957297218409391963511994880401 79626	151
	900	3.577152063957297218409391963511994880401 79626	170
	1000	3.577152063957297218409391963511994880401 79626	190
$W_{-1}(x)$	–	3.57	3

widely applicable and appropriate STFT formula as a separate trans function. SAF can be easily avoided because there is STFT which does not depend on the value of functional parameters (table 1), nor on the process being analyzed.

Table 6 provides an overview of the strictly designed parameters of the CO_2 storage system, to be able to apply a complicated sequential asymptotic formula based on the Lambert W function. The corresponding values for the variable x are, respectively: $x = 0.0006865602; x =$

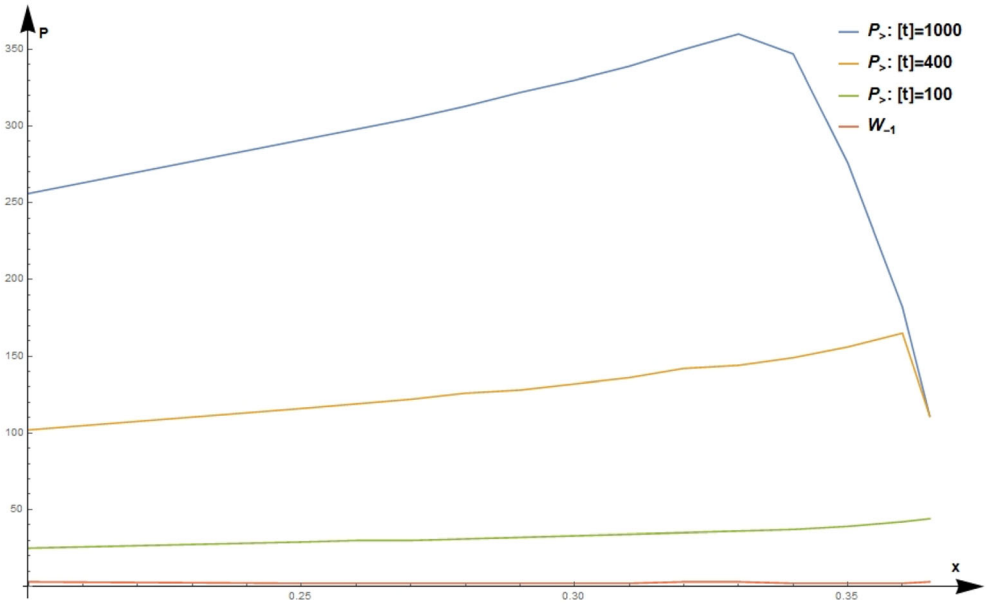

Figure 4. Graphical representation of the precision of the functions $W_{-1}(x)$ and $tran_{>}(x)$ for $x > 0.15$.

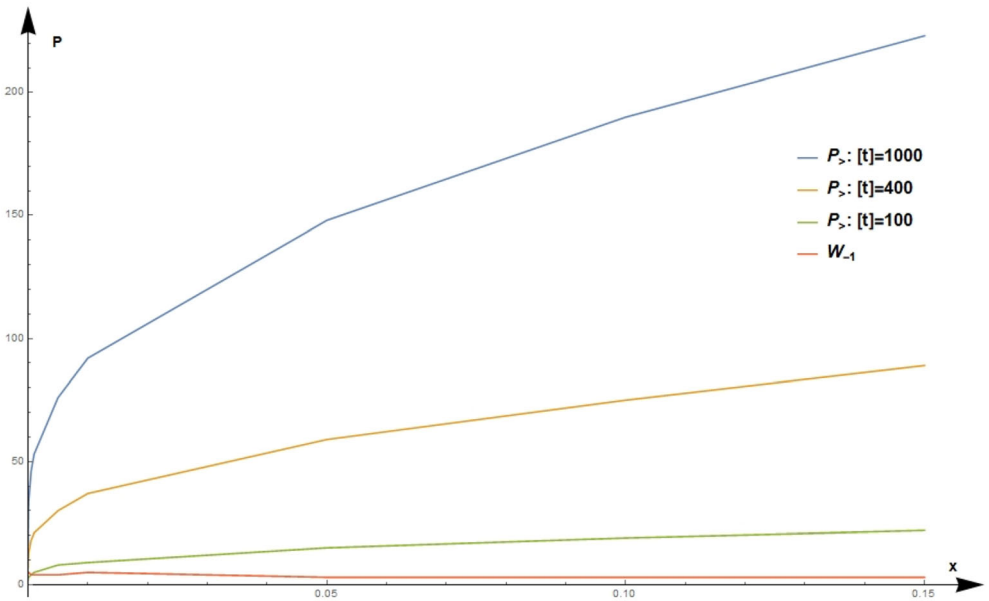

Figure 5. Graphical representation of the precision of the functions $W_{-1}(x)$ and $tran_{>}(x)$ for $x < 0.15$.

0.0003930209; $x = 0.0001076489$; $x = 0.0000034345$; $x = 0.0000019061$; $x = 0.0000004896$. Some of the projected parameters given in table 1 are of the order of 10^{-10}. In connection with this fact, the possibility of applying a sequential asymptotic formula can be seriously questioned, having in mind its accuracy. If we consider a real dynamic system of CO_2 storage that does not imply a strict design of parameters or their absolute adaptation to the mathematical model related to the sequential asymptotic formula based on the W function, and taking into account the real characteristics of physical processes, significant correction factors are obtained. Observing the real ranges in which the values of P, μ_b, μ_c, and the ratio of β and ε are, the correction factors for x values can be obtained. By applying such correction factors, real values of x can be calculated. The results of the comparative analysis of the sequential

Table 6. The numerical comparison analysis of a $W_{-1}(x)$ function solution and $tran_{>}(x)$ special tran function applied to the DIR (α) for selected functional parameters values.

			W_{-1} function				$tran_{>}$ function			
P	β	ε	$\ln(-x)$	$W_{-1}(x)$	α	M_0	$tran_{>}$ (prec)	$\alpha^{tran_{>}}$	$M_0^{tran_{>}}$	Δ
14.0	20	12.0	−7.3	−9.5	0.000384	14.9	9.557305(48)	0.000380336	14.7596	505.44
14.4	28.1	12.2	−7.8	−10.2	0.000389	15.1	10.11391(44)	0.000388864	15.0905	34.20
15.1	45.7	12.2	−9.1	−11.6	0.000392	15.2	11.54637(34)	0.000394811	15.3213	436.68
16.5	89.4	12.0	−12.6	−15.3	0.000385	14.9	15.32979(20)	0.000385318	14.9529	190.44
16.9	98.6	12.2	−13.2	−16.0	0.000386	15.0	15.97076(18)	0.000369854	14.3528	2329.92
17.5	116.2	12.2	−14.6	−17.4	0.000387	15.0	17.45990(15)	0.000383958	14.9001	359.64

Table 7. The numerical comparison analysis of a $W_{-1}(x)$ function solution and $tran_{>}(x)$ special tran function applied to some realistic x values.

	W_{-1} function			$tran_{>}$ function			difference	
x	$W_{-1}(x)$	P_{-1}	α_{-1}	$tran_{>}$	$P_{>}$	$\alpha_{>}$	$P_{>} - P_{-1}$	$\lvert\alpha_{>} - \alpha_{-1}\rvert$
0.32294670000000003	N.A.	–	N.A.	1.600724	353	0.0028479	N.A.	N.A.
0.2152978	N.A.	–	N.A.	2.419148	266	0.0018844	N.A.	N.A.
0.16147335000000002	N.A.	–	N.A.	2.881846	231	0.0015818	N.A.	N.A.
0.14823264604819494	−3.01	3	0.001514	3.011366	222	0.0015138	222	$6.8707 \cdot 10^{-7}$
0.01437333904038395	−6.040	4	0.000977	6.040939	101	0.0009778	97	$1.5215 \cdot 10^{-7}$
0.00472929220038439	−7.348	4	0.000803	7.348472	75	0.00080382	71	$5.165 \cdot 10^{-8}$
0.00262478311672856	−8.025	4	0.000768	8.025364	66	0.00076851	62	$3.4862 \cdot 10^{-8}$
0.00067419015994286	−9.559	4	0.000701	9.559537	48	0.00070128	44	$3.9449 \cdot 10^{-8}$
0.00797728202142994	−6.739	5	0.000915	6.739081	86	0.00091519	81	$1.1015 \cdot 10^{-8}$
0.00204900930963027	−8.307	4	0.000807	8.307565	62	0.00080696	58	$5.4931 \cdot 10^{-8}$

asymptotic formula and STFT over the values of the variable x calculated in this way are shown in table 7.

Observing the presented results of the comparative analysis, it becomes clear why SAF is not comparable to STFT. It can be said with certainty that STFT is the standard for solving problems of analogous forms of functional nonlinearity.

4. Conclusions

Conceptually, the conclusion contains the stated contributions of this article concerning the modelling of the solution of the transcendental functional equation for the dimensionless injection rate (DIR) within the analysis of the CO_2 storage phenomenon. Namely, based on the previous sections and subsections as well as Ref. [12–14, 18–20], the contributions of this article are defined, as follows:

1. in the article, an exact analytical formula for Dimensionless Injection Rate (DIR) (Eqs. (31) and (32)) was obtained, valid in the defined solution domain of the DIR transcendental equation (10)) i.e., for $0 \leq x \leq e^{-1}$ by using the Perovich's Special Trans Functions Theory. Theoretically, this leads to the fact that for these families of transcendental equations ((1) and (10)) we consider the problem closed in the real domain. So, this is the most important contribution of this article, from which the other contributions derive;
2. extremely high precision (practically unlimited) of STFT DIR values for the entire domain of definition. This implies the invariance of the DIR formula concerning the CO_2 storage system parameters given in table 1. For example, Tables 3-6 show that the formulas derived in this article based on STFT and applied to DIR estimations ((31) and (32)) are completely numerically valid. Also, the STFT approach is novel and precise, while the computations are simple and practical and do not require significant computation time. Note that STFT ensures reaching extreme precisions in numerical results so the highest precision in defining the Dimensionless Injection Rate has been achieved in this article until now;
3. defining gradient coefficients based on Eqs. (31) and (32), which enable predictions of the dynamics and stability of the CO_2 storage system. However, the detailed modelling of gradient coefficients for

Dimensionless Injection Rate, within CO_2 storage, requires structurally very complex further analysis;

4. based on the previously defined contributions of the article, the STFT exact analytical approach is without a doubt the theoretical standard for the analysis of the DIR transcendental functional equation. Let us note that based on the STFT theoretical, mathematical and computational detailed analysis presented in articles [12–14, 18–20] and the application in this paper, as well as the deductive derivation of proofs as an STFT paradigm, it is evident that STFT supported by Wolfram Mathematica program should be used as a theoretical standard for solving transcendental functional equations for DIR, (such as Eqs. (1) and (10) the STFT analysis presented in this article can be applied to essentially different physical phenomena represented by an identical transcendental functional equation (Eq.(10), or even some more complex (not simple) transcendental equation that can be reduced to Eq. (10) by some structural modifications). Having this in mind, the significance of this article and the possibility of its generalization becomes clear.

Finally, it should be emphasized that new effective methods and theories, as well as the importance of their application, cannot be ignored in serious research and scientific approaches. Regardless of the importance of existing methods, the fact that new approaches or exact formulas appear in the meantime must be accepted. The relationship between Lambert's W function (as well as sequential asymptotic formulas based on the W function) and Perovich's Special Trans Functions (obtained within STF Theory) is analogous.

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