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Graphical method to solve fuzzy linear programming

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Abstract. In this paper, the graphical method for solving linear programming is extended in fuzzy environment. Here, we dealt with the fully fuzzy linear programming (FFLP) which involves fuzzy constraints and fuzzy objective. Determining and visualizing the fuzzy feasible space in the geometrical space is one of the novel contributions of this study. Defining fuzzy constraints as fuzzy lines and finding the nature of the point of intersection between fuzzy lines are also studied. The fuzzy constraints divide the geometrical into fuzzy half planes. Intersection of such fuzzy half planes yields a fuzzy convex hull. The optimal solution of fuzzy linear programming problem is obtained at an extreme point of this fuzzy convex hull. The results obtained from the proposed method are compared with existing methodologies.

Keywords. FLPP; fuzzy optimization; fuzzy constraints; intersecting fuzzy lines; fuzzy feasible space.

1. Introduction

Linear programming is the most basic and widely useful technique in the field of operation research or mathematical programming (e.g. [1, 2]). A linear programming problem (LPP) consists of a linear objective function and a set of linear constraints. The linear objective function has to optimize under the set of linear constraints. The linear constraints are represented by linear equality or inequalities. The presence of ambiguity in objective things and the impreciseness of human thoughts, there exist many cases where the cost coefficients, resources capacity and other coefficients in linear constraints in the LPP may not be presented in a precise manner. The model of fuzzy linear programming problem (FLPP) addresses such situations and should provide interpretable solutions.

Tanaka *et al* [3] introduces the fuzzy programming based on the ground work of decision making in fuzzy environment proposed by Bellman and Zadeh [4]. Zimmermann [5] proposed the linear programming in fuzzy environment in 1978. In the recent years, there have been a significant development on FLPP. Study on FLPP involving both fuzzy equality and inequality constraints can be found. Most of the researchers convert the fuzzy constraints and objective function to obtain a crisp LPP (e.g. [6–8]) or multi-objective programming (e.g. [6, 9]). Use of goal programming to solve FLPP is also common practice among the recent studies [10]. Use of ranking operator between fuzzy numbers or expectation and mean value of fuzzy number can be found in this purpose [6–8, 11]. Jamison and Lodwick [12] introduces a penalty method for possible constraint violations to solve FLPP. An interactive method with fuzzy goal is also used to solve FLPP by [8] in 2007.

Triangular [6, 7] and trapezoidal [13, 14] fuzzy numbers are mainly used as fuzzy parameters in FLPP.

Based on the involvement of fuzzy numbers and the nature of decision variables, FLPP can be mainly categorised into the six possible groups . Recently, Ghanbari *et al* [15] studied on existing models of fuzzy linear programming which are listed below.

1. The FLPP involving fuzzy resources capacity, i.e. right side of the constraints as fuzzy numbers [14]. The decision variables are obtained as fuzzy numbers in this group.
2. The cost coefficients in the objective function are chosen as fuzzy numbers [16] in this group.
3. The FLPP in this group have the cost coefficients and the resources capacity (e.g. [17]) as fuzzy numbers. Here decision variables are considered to be non-fuzzy (crisp) number.
4. The cost coefficients and resources capacity are chosen as fuzzy numbers in this group of FLPP [18, 19]. Here decision variables are also obtained as fuzzy quantity.
5. This group are made of the FLPP which include fuzziness for the decision variables, the cost coefficents and resource quantities (e.g. [13, 20]).

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6. This group consists of fully fuzzy linear programming problem (FFLPP), i.e. all the involved entities are fuzzy sets or hesitant fuzzy sets (e.g. [6, 7, 21]).

The proposed graphical method solves the FFLPP which belong to the last group of the above listed categories.

From the study of Ghanbari *et al* [15] on existing methodologies to solve FLPP, it can be seen that the common practice is to convert the FLPP into real or crisp mathematical programming (linear/non-linear/MOP) and solve it. The main drawback of the existing methods is that they are not able to provide a mathematically closed form to solve FLPP. Due to these deficiencies for existing methods they were unable to assure a feasible and non-negative solution which is further discussed in examples 6 and 7. One of the main reason behind this deficiency may be the lack of the geometrical visualization of FLPP. The present study focus to explore the geometry of FLPP and interpretation of the solution in geometrical plane. For simplicity, FLPP with fuzzy equality and triangular fuzzy numbers (TFN) are discussed here. The proposed model can be further extended for more general FLPP.

Visualizing the feasible space of FLPP could be the first step towards the visualization of the geometry of FLPP. The geometrical interpretation of fuzzy inequalities and equality are major concern to be addressed in this endeavour. In this study, it has been shown that the fuzzy equality are indifferent from fuzzy lines in the geometrical planes. The intersection of these fuzzy lines produces the vertex of the fuzzy feasible space. The computation and visualization of the process of obtaining the point of intersection of two fuzzy lines are studied in details in the following.

The intersection of two fuzzy lines in geometrical plane is equivalent to the solution of two fuzzy linear equality. In literature, different attempts have been made by several researchers to obtain solution fuzzy linear system. The nature of linear equations involving fuzzy parameters has been discussed in the studies of Sanchez [22]. Buckley and Qu [23] analysed the characteristic of possible solution of linear and quadratic fuzzy equations. Friedman *et al* [24] were first to propose a method to solve general fuzzy linear system (FLS). In their study, the linear equations involved fuzzy numbers in the right side and the variable coefficients are considered as crisp (real) numbers. They converted the FLS into crisp linear equations by using embedding and solved it. Same approach of converting the FLS into system of crisp linear equation has been also adopted by several authors, e.g. Dehghan [25]. The method proposed by Dehgan [25] obtain positive solutions of the FLS. Apart from that, converting FLS into system of non-linear equation has been attempted by Malkawi *et al* [26]. An α-level based approach is used to solve FLS has been proposed by Moloudzadeh *et al* [27]. But, this method can not ensure solution for general form of FLS [28]. Behera and Chakraverty [29, 30] applied finite element method to find solution of FLS. A fully FLS with an arbitrary fuzzy coefficient is studied by Kumar *et al* [6]. They converted the FLS into an optimization problem. In another study by Otadi and Mosleh [31], FLS is solved by converting it into a linear programming problem and found the non-negative solution of a fully fuzzy matrix equation involving triangular fuzzy numbers. A method to fuzzy linear equation involving symmetric trapezoidal fuzzy number can be found in [32, 33]. Liu [17] proposed a numerical method to obtain an approximate solution of FLS. Muzzioli [34] proposed the solution method of the type of equations $\widetilde{A}x_1 + \widetilde{B} = \widetilde{A}x_2 + \widetilde{B}$. Cao *et al* [35] proposed a geometry based method to solve FLS which is used to solve vegetarian diet problem. In this method, only the right side constant of the linear equations were considered as fuzzy numbers and co-coefficients of the variables as crisp numbers. In the present study, all the variable coefficients and right side of the linear equations are treated as triangular fuzzy numbers.

The geometrical interpretation of the fuzzy linear equation is the area which is being addressed in this study to obtain the intersection (solution) of two fuzzy linear equations. It has been observed that linear equation of two variables with fuzzy coefficients represent fuzzy line in geometrical plane. Geometrically, the solution of a two fuzzy linear equations or fuzzy lines can be viewed as a fuzzy point of intersection. Bukley and Eslami [23, 36] have introduced the concept of fuzzy geometry. Chakraborty and Ghosh [37, 38] redefined the understanding and visualization of fuzzy geometry. Latest advancement on fuzzy geometry can be found in the studies [37, 39–42]. Visualization and understanding the topological properties of fuzzy point lying on fuzzy line is first step towards obtaining fuzzy points of intersection of fuzzy lines. A detail study on the interrelation between fuzzy number, fuzzy point and fuzzy line has been done in our previous study [42]. In this previous study [42], nature of fuzzy point at every position on the fuzzy line is obtained through the concept of fuzzy function and geometrical transformation of fuzzy point. Here, the point of intersection between two fuzzy lines is obtained with the understanding of the belongings of a fuzzy point on fuzzy line. This concept of point of intersection of fuzzy lines is used to obtain the vertex of the fuzzy feasible space (FFS) of the FLP.

Recently Kumar *et al* [6] proposed a new method to solve fully fuzzy linear programming problem. This method convert the FLPP into a crisp LPP by introducing a new set of a constraints in the form of linear equality. There are two major issues with this method which should be addressed. Introduction of new set of constraints create the chance of nonexistence of solution which is addressed in this present study with numerical example (see example 6). Other issue with the method [6] is that there is possibility of obtaining negative solution which has been addressed by Najafi *et al* [7]. Najafi *et al* [7] added a new set of non-negativity constraints on the model of Kumar *et al* [6] to guarantee the non-negative solution. The proposed method has no scope of obtaining negative solution which is discussed in example 7.

A new geometry based method to find fuzzy point of intersection between two fuzzy lines is presented in next section 2. Section 3 introduces the graphical method to solve fuzzy linear programming problem along with numerical examples. Finally, section 4 concludes the works with future directions.

2. Intersection of fuzzy lines

In this section, we will focus to an interpretable intersection of fuzzy lines. In this search, we could think of three possible geometrical ways to compute the intersection of fuzzy lines. (i) The fuzzy lines $\widetilde{L}_1$ and $\widetilde{L}_2$ are expressed as a group of lines with bounded by a well defined mathematical membership function. Then, the point of intersections obtained as a collection the of intersection of the same lines using the definition of same points. (ii) Another way of visualizing the point of intersection of $\widetilde{L}_1$ and $\widetilde{L}_2$ could be the intersection fuzzy set using fuzzy set theory. (iii) Finally, a new geometry based technique for visualization of fuzzy point of intersection between $\widetilde{L}_1$ and $\widetilde{L}_2$ is proposed.

2.1 *Collection of intersection of lines*

Existing study on fuzzy geometry [37, 38, 43] portray the fuzzy line $\widetilde{L}:\ \tilde{a}x+\tilde{b}y=\tilde{c}$ as group of lines with bounded by the following membership function

$$\widetilde{L}=\bigcup_{\alpha\in[0,1]}\{l^{\alpha}:\ a^{\alpha}x+b^{\alpha}y=c^{\alpha}\} \tag{1}$$

where

$$\begin{cases} a^{\alpha},\ b^{\alpha} \text{ and } c^{\alpha} \text{ are same points of } \tilde{a},\tilde{b} \text{ and } -\tilde{c} & \text{if } xy\geq 0\\ a^{\alpha},\ b^{\alpha} \text{ and } c^{\alpha} \text{ are same points of } \tilde{a},-\tilde{b} \text{ and } -\tilde{c} & \text{if } x\geq 0, y\leq 0\\ a^{\alpha},\ b^{\alpha} \text{ and } c^{\alpha} \text{ are same points of } -\tilde{a},\tilde{b} \text{ and } -\tilde{c} & \text{if } x\leq 0, y\geq 0 \end{cases}$$

Based on the above equation (1) , the lines l_1^{α} and l_2^{α} are called same lines of $\widetilde{L}_1$ and $\widetilde{L}_2$. One way to think about the intersection $\widetilde{S}$ of two fuzzy lines $\widetilde{L}_1$ and $\widetilde{L}_2$ can be as a collection of the points S^{α}, i.e. $\widetilde{S}=\underset{\alpha\in[0,1]}{\cup}\{S^{\alpha}\}$, where S^{α} is the point of intersection of the same lines l_1^{α} and l_2^{α}. It should be noted that the membership value of S^{α} is also equal to α, i.e. $\mu(S^{\alpha}\mid\widetilde{S})=\alpha$. The computational process of this method are very easy to understand. Also, this method has a very beautiful property which is discussed below.

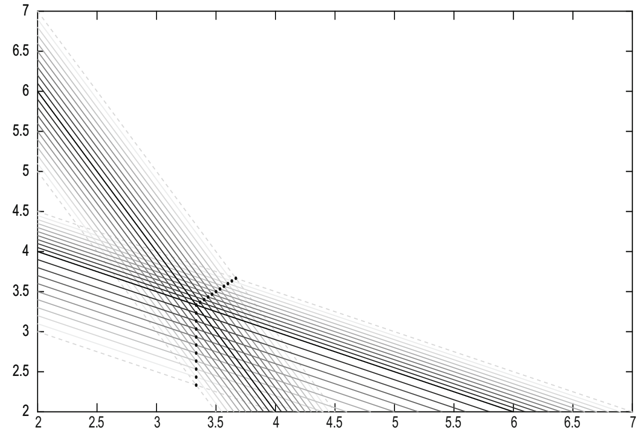

Figure 1. Intersection of fuzzy lines.

Applying the extension principle, the membership function of the set $\widetilde{S}$ should be defined as follows:

$$\mu\Big((x,y)\mid\widetilde{S}\Big)=Sup\Big\{\alpha:\ \ (x,y)\text{ is point of intersection of the same lines } l_1^{\alpha}\text{ and } l_2^{\alpha}\Big\} \tag{2}$$

Since, the point (x,y) has no possibility to lie on more than one line l^{α}, the operator $'Sup'$ becomes redundant which makes this method very adaptable. But, this particular technique does not provide an interpretable fuzzy point in geometrical space which is illustrated in the following example 1 and figures 1 and 2.

Example 1 Let us consider two fuzzy lines $(1,1,1)x+(2,2,2)y=(8,10,11)$ and $(2,2,2)x+(1,1,1)y=(9,10,11)$. For α, the same lines are obtained as $l_1^{\alpha}:\ x+2y=8+2*\alpha$ and $l_2^{\alpha}:\ 2x+y=9+\alpha$ on the lower side of the cores $L_1:\ x+2y=10$ and $L_1:\ 2x+y=10$. Whereas the same lines on the upper side of the cores L_1 and L_2 are obtained as $l_1^{\alpha}:\ x+2y=10+\alpha$ and $l_2^{\alpha}:\ 2x+y=10+\alpha$. Subsequently, the points of intersections S^{α} are obtained as $(\frac{10}{3},\frac{7+3\alpha}{3})$ and $(\frac{10+\alpha}{3},\frac{10+\alpha}{3})$. From the figures 1 and 2, it can be observed that the collection $\widetilde{S}=\underset{\alpha\in[0,1]}{\cup}\{S^{\alpha}\}$ does not form an interpretable fuzzy set in geometrical space.

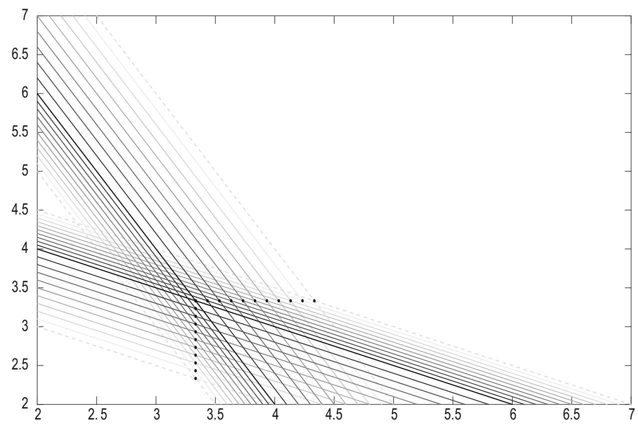

Figure 2. Intersection of fuzzy lines.

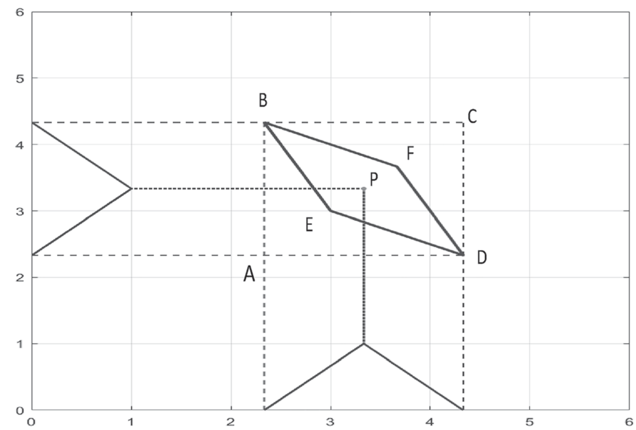

Figure 4. Projection of intersection.

2.2 *Intersection of fuzzy lines.*

The most basic and easiest way to define the point of intersection of $\widetilde{L}_1$ and $\widetilde{L}_2$ could be the set $\widetilde{L}_1 \cap \widetilde{L}_2$. This idea was adopted by Cao *et al* [35] to solve fuzzy linear programming. Here, the fuzzy lines $\widetilde{L}_1$ and $\widetilde{L}_2$ are viewed as fuzzy sets in space. The membership function of the fuzzy set $\widetilde{L}_1 \cap \widetilde{L}_2$ can be defined as the following way:

$$\mu\Big((h,k) \mid \widetilde{L}_1 \cap \widetilde{L}_2\Big) = min\Big\{\mu((h,k) \mid \widetilde{L}_1), \mu((h,k) \mid \widetilde{L}_2)\Big\} \tag{3}$$

The main concern of this process should be the geometrical interpretation of the fuzzy set $\widetilde{L}_1 \cap \widetilde{L}_2$. The set $\widetilde{L}_1 \cap \widetilde{L}_2$ lie on both of the lines $\widetilde{L}_1$ and $\widetilde{L}_2$. In the figures 3 and 4, it can be observed that the area bounded by *EBFD* represents the support of the fuzzy set $\widetilde{L}_1 \cap \widetilde{L}_2$. The main concern about $\widetilde{L}_1 \cap \widetilde{L}_2$ lies within its physical interpretation or practical significance. It can be observed the projection of the fuzzy set $\widetilde{I} = \widetilde{L}_1 \cap \widetilde{L}_2$ on the x and y axes produces two fuzzy sets $\widetilde{I}_x$ and $\widetilde{I}_y$ respectively. From figure 4, it can be observed that the fuzzy sets $\widetilde{I}_x$ and $\widetilde{I}_y$ do not reconstruct the intersection set $\widetilde{I} = \widetilde{L}_1 \cap \widetilde{L}_2$ under the composition operator. So, this incompatibility of the intersection set $\widetilde{I}$ drive us to find other way to visualize the point of intersection of fuzzy lines.

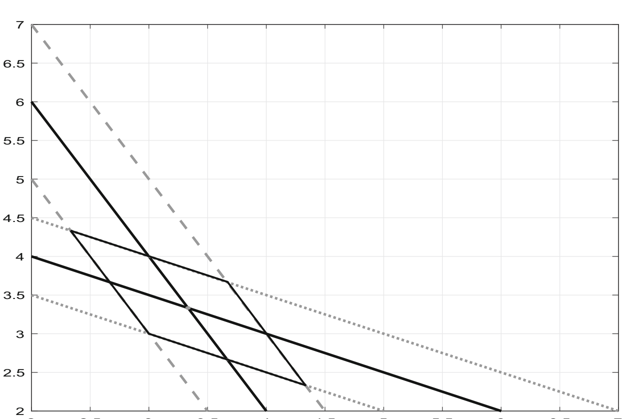

Figure 3. Intersection of fuzzy lines.

2.3 *Proposed point of intersection*

The proposed method is centered on the desire of finding an interpretable fuzzy point of intersection which would be contained within the fuzzy lines $\widetilde{L}_1$ and $\widetilde{L}_2$. This proposition is completely based on our previous study [42] where the interrelation between the fuzzy point and fuzzy line is established. The study [42] suggests that we can obtain a fuzzy point $\widetilde{P}$ at every location (x, y) on the core $ax + by = c$ of a fuzzy line $\tilde{a}x + \tilde{b}y = \tilde{c}$ as an ordered pair $(\tilde{x}, \tilde{y})$ of two fuzzy numbers $\tilde{x}$ and $\tilde{y}$. For a details understanding of the process of obtaining a fuzzy point on a fuzzy line, the readers may refer to Das and Chakraborty [42].

This proposed process follows two simple steps. In the first step, two fuzzy points $\widetilde{P}_1(h,k)$ and $\widetilde{P}_2(h,k)$ are obtained on $\widetilde{L}_1$ and $\widetilde{L}_2$ respectively. Here, the point (h, k) is the point of intersection of the core lines L_1 and L_2. In the final step, the fuzzy point of intersection $\widetilde{P}(h,k)$ is obtained as the intersection of the fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$, i.e. $\widetilde{P} = \widetilde{P}_1 \cap \widetilde{P}_2$. So, the the fuzzy point $\widetilde{P}$ should be defined by its membership function induced by the conjunction operator (also known as $t-norm$) 'min'', i.e.

$$\mu\big((x,y) \mid \widetilde{P}\big) = min\Big\{\mu_1\big((x,y) \mid \widetilde{P}_1\big), \mu_2\big((x,y) \mid \widetilde{P}_2\big)\Big\} \tag{4}$$

Algorithm 1 Algorithm for obtaining point of intersection of fuzzy lines

Require: Fuzzy lines $\widetilde{L}_1$: $(\underline{a}_1, a_1, \bar{a}_1)x + (\underline{b}_1, b_1, \bar{b}_1)y = (\underline{c}_1, c_1, \bar{c}_1)$ and $\widetilde{L}_2$: $(\underline{a}_2, a_2, \bar{a}_2)x + (\underline{b}_2, b_2, \bar{b}_2)y = (\underline{c}_2, c_2, \bar{c}_2)$.
1: Obtain the point of intersection $P(x, y)$ of the cores $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$.
2: Obtain the fuzzy point $\widetilde{P}_1(\tilde{x}', \tilde{y}')$ on $\widetilde{L}_1$ using algorithm 1 of [42] where $\tilde{x}' = (\underline{x}', x, \bar{x}')$ and $\tilde{y}_1 = (\underline{y}', y, \bar{y}')$.
3: Obtain the fuzzy point $\widetilde{P}_2(\tilde{x}'', \tilde{y}'')$ on $\widetilde{L}_2$ using algorithm 1 of [42] where $\tilde{x}'' = (\underline{x}'', x, \bar{x}'')$ and $\tilde{y}_1 = (\underline{y}'', y, \bar{y}'')$.
4: Construct fuzzy numbers $\tilde{x} = (\underline{x}, x, \bar{x})$ and $\tilde{y} = (\underline{y}, y, \bar{y})$ where $\underline{x} = min\{\underline{x}', \underline{x}''\}$, $\bar{x} = min\{\bar{x}', \bar{x}''\}$, $\underline{y} = min\{\underline{y}', \underline{y}''\}$ and $\bar{y} = min\{\bar{y}', \bar{y}''\}$.
5: Return the fuzzy point of intersection $\widetilde{P}(\tilde{x}, \tilde{y})$.

Example 2 Buckley and Que [23] have shown in their study that the system of fuzzy equations $\widetilde{L}_1 : (-4, -2, 0)\tilde{x} + (1, 2, 3)\tilde{y} = (-1, 0, 1)$ and $\widetilde{L}_2 : (-3, -2, -1)\tilde{x} + (0, 0, 0)\tilde{y} = (-1, 0, 1)$ have no solution. But, following the proposed method a solution of this system can be found. Applying the algorithm 1 of [42] fuzzy points $\widetilde{P}_1(0, 0)$ and $\widetilde{P}_2(0, 0)$ are calculated on $\widetilde{L}_1$ and $\widetilde{L}_2$ respectively. $\widetilde{P}_1$ is obtained as ordered pair of the fuzzy numbers $(-1, 0, 1)$ and $(-5, 0, 5)$. Also, the fuzzy point $\widetilde{P}_2$ is obtained as ordered pair of the fuzzy numbers $(-1, 0, 1)$ and $(-1, 0, 1)$.

Thus, the fuzzy point $\widetilde{P}$ is obtained as ordered of the fuzzy numbers $(-1, 0, 1)$ and $(-1, 0, 1)$ from equation 4.

The study of Muzzioli and Reynaerts [34] reported the the solution of the above system of linear equations as $\tilde{x}_1 = (-1, 0, 1)$ and $\tilde{x}_2 = (-5, 0, 5)$. Form the following figure 5 it can be concluded that solution obtained from proposed method is geometrically more fitted in the fuzzy lines $\widetilde{L}_1$ and $\widetilde{L}_2$.

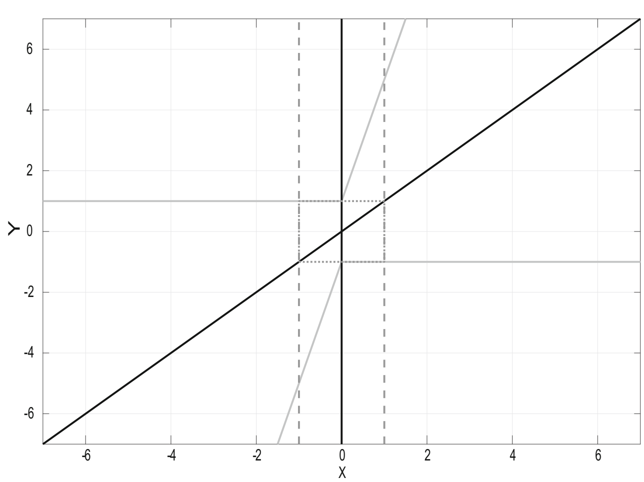

Figure 5. Intersection of fuzzy lines.

Example 3 Let us consider the fuzzy lines given the equations $\widetilde{L}_1 : (4, 5, 6)\tilde{x} + (5, 6, 8)\tilde{y} = (40, 50, 67)$ and $\widetilde{L}_1 : (6, 7, 7)\tilde{x} + (4, 4, 5)\tilde{y} = (43, 48, 55)$. Following the proposed method the point of intersection or solution of the above system of fuzzy equations can be obtained as $\tilde{x}_1 = (3.28, 4, 4.9)$ and $\tilde{x}_2 = (4, 5, 6.3)$.

In the study of Dehghan *et al* [25], the solution of this above system of fuzzy equation is reported as $x = (\frac{43}{11}, 4, 4)$ and $y = (\frac{54}{11}, 5, \frac{11}{2})$.

From figures 6 and 7 depict that the fuzzy region of intersection of $\widetilde{L}_1$ and $\widetilde{L}_2$ (known as fuzzy point of intersection) obtained from proposed method fits geometrically on $\widetilde{L}_1$ and $\widetilde{L}_2$. On the other hand, the solution obtained by Dehghan et al [25] geometrical not well fitted on $\widetilde{L}_1$ and $\widetilde{L}_2$.

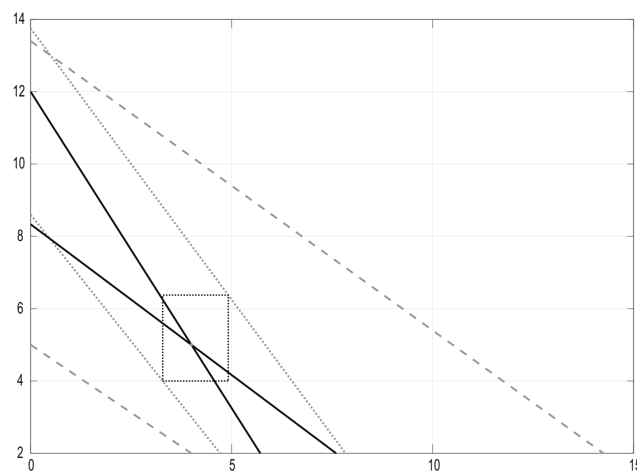

Figure 6. Proposed point of intersection.

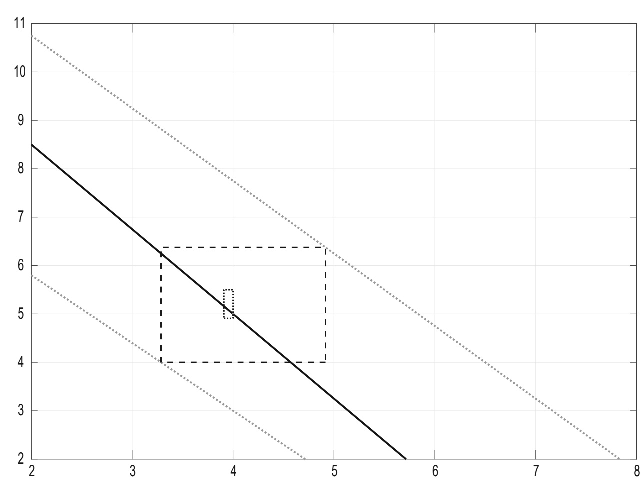

Figure 7. Point of intersection: proposed and Dehghan *et al.* [25].

3. Solving fuzzy LPP

Let us consider the standard fuzzy linear programming problem of the following form:

$$\begin{cases} Maximize: \ f(x,\tilde{c}) = \tilde{c}_1 x_1 + \tilde{c}_2 x_2 + \cdots + \tilde{c}_n x_n \\ subject\ to: \\ \quad g_1(x,\tilde{a}_1) = \tilde{a}_{11}x + \tilde{a}_{12}x_2 + \cdots \tilde{a}_{1n}x_n \le \tilde{b}_1 \\ \quad g_2(x,\tilde{a}_2) = \tilde{a}_{21}x + \tilde{a}_{22}x_2 + \cdots \tilde{a}_{2n}x_n \le \tilde{b}_2 \\ \quad \vdots \\ \quad g_m(x,\tilde{a}_m) = \tilde{a}_{m1}x_1 + \tilde{a}_{m2}x_2 + \cdots \tilde{a}_{mn}x_n \le \tilde{b}_m \\ (x_1, x_2 \ldots x_n) \in \mathbb{R}^n_{\geqq} \end{cases} \tag{5}$$

where $x = (x_1, x_2, \ldots x_n)$; $\tilde{a}_j = (\tilde{a}_{1j}, \tilde{a}_{2j}, \ldots \tilde{a}_{nm})$; $\tilde{a}_{ij}$ and $\tilde{b}_j$ are fuzzy numbers.

So, the fuzzy feasible space of the FLPP is given as $\widetilde{X} = \{x \in \mathbb{R}^n_{\geqq} : g_1(x,\tilde{a}_1) \le \tilde{b}_1, g_2(x,\tilde{a}_2) \le \tilde{b}_2, \ldots g_m(x,\tilde{a}_m) \le \tilde{b}_m\}$.

3.1 *Fuzzy Feasible Space (FFS) of FLP*

In general, the fuzzy feasible space (FFS) $\widetilde{X}$ (say) of the FLPP induced by fuzzy linear inequalities and non-negativity conditions. First step towards the geometrical visualization of feasible space should be the geometrical visualization of fuzzy inequalities. A detail study on computation and geometrical visualization of fuzzy inequalities are given in [37, 44].

In order to visualize the feasible space $\widetilde{X}$, the calculation of the membership degree of belongingness of each point $x \in \widetilde{X}$ should be explained. Zadeh's extension principle [4] can be applied in this regard.

The FFS can be expressed as $\widetilde{X} = \bigcap_{j=1}^{m} \{x \in \mathbb{R}^n_{\geqq} : g_j(x,\tilde{a}_j) \le \tilde{b}_j\} = \bigcap_{j=1}^{m} \widetilde{X}_j$.

Applying fuzzy set theory, $\widetilde{X}$ can be redefined by its membership function as follows:

$$\mu(x \mid \widetilde{X}) = \min_j \left\{ \mu(x \mid \widetilde{X}_j) \right\} \tag{6}$$

Again, the individual membership functions $\mu(x \mid \widetilde{X}_j)$ can be defined as the following equation (7).

$$\mu(x \mid \widetilde{X}_j) = \sup_{a'_{j1}x_1 + a'_{j2}x_2 + \ldots a'_{jn}x_n \le b'_j} \min\Big\{ \mu(a'_{j1} \mid \tilde{a}_{j1}), \mu(a'_{j2} \mid \tilde{a}_{j2}), \ldots \mu(a'_{jm} \mid \tilde{a}_{jm}), \mu(b'_j \mid \tilde{b}_j) \Big\} \tag{7}$$

It has to be clear that the '*supremum*' used in the above equation applied over all the possible combinations of $(a'_{j1}, a'_{j2}, \ldots a'_{jm}, b'_j)$ for which the inequality $a'_{j1}x_1 + a'_{j2}x_2 + \cdots a'_{jn}x_n \le b'_j$ satisfy. Here, $a'_{j1} \in \tilde{a}_{j1}(0), a'_{j2} \in \tilde{a}_{j2}(0), \ldots a'_{jm} \in \tilde{a}_{jm}(0)$ and $b'_j \in \tilde{b}_j(0)$.

So, we have to search for all possible combination of $(a'_{j1}, a'_{j2}, \ldots a'_{jm}, b'_j)$ for which the inequality $a'_{j1}x_1 + a'_{j2}x_2 + \ldots a'_{jn}x_n \le b'_j$ satisfy and provides the maximum of $min\Big\{\mu(a'_{j1} \mid \tilde{a}_{j1}), \mu(a'_{j2} \mid \tilde{a}_{j2}), \ldots \mu(a'_{jm} \mid \tilde{a}_{jm}), \mu(b'_j \mid \tilde{b}_j)\Big\}$.

To implement this procedure computationally, the supports of $\tilde{a}_{j1}(0) = [\underline{a}_{j1}, \bar{a}_{j1}], \tilde{a}_{j2}(0) = [\underline{a}_{j2}, \bar{a}_{j2}], \ldots \tilde{a}_{jn}(0) = [\underline{a}_{jn}, \bar{a}_{jn}]$, and $\tilde{b}_j(0) = [\underline{b}_j, \bar{b}_j]$ should be discretized into a certain (preferably high) number of grid points. These grid points are tested to obtain the optimum degree of $\mu(x \mid \widetilde{X}_j)$. A sequential algorithmic procedure have been provided in the following.

Algorithm 2 Algorithm for obtaining $\mu(x \mid \widetilde{X})$ using EP

Require: The number k in which the supports of $a_{11}, a_{12}, \ldots a_{mn}, b_1, b_2, \ldots b_m$ will be discretize.

Initialize $\alpha_1 = 0, \alpha_2 = 0 \ldots \alpha_m = 0$.
for $i = 1$ to m **do**
  for $a'_{i1} = \underline{a}_{i1}$ to $\bar{a}_{i1}$ with step length $\frac{\bar{a}_{i1} - \underline{a}_{i1}}{k}$ **do**
    for $a'_{i2} = \underline{a}_{i2}$ to $\bar{a}_{i2}$ with step length $\frac{\bar{a}_{i2} - \underline{a}_{i2}}{k}$ **do**
      ...
      for $a'_{in} = \underline{a}_{in}$ to $\bar{a}_{in}$ with step length $\frac{\bar{a}_{in} - \underline{a}_{in}}{k}$ **do**
        for $b'_i = \underline{b}_i$ to $\bar{b}_i$ with step length $\frac{\bar{b}_i - \underline{b}_i}{k}$ **do**
          if $a'_{j1}x_1 + a'_{j2}x_2 + \ldots a'_{jn}x_n \le b'_j$ **then**
            $temp = min\big\{\mu(a'_{j1} \mid \tilde{a}_{j1}), \mu(a'_{j2} \mid \tilde{a}_{j2}), \ldots \mu(a'_{jm} \mid \tilde{a}_{jm}), \mu(b'_j \mid \tilde{b}_j)\big\}$
          end if
          if $temp > \alpha_i$ **then**
            $temp = \alpha_i$
          end if
        end for
      end for
      ...
    end for
  end for
end for
Return membership value $\mu(x \mid \widetilde{X}) = \alpha = min\{\alpha_1, \alpha_2 \ldots \alpha_m\}$.

From the algorithm 2, it can be understood that a tiresome computation has to be done in the search of $\mu(x \mid \widetilde{X})$. Numerically, the computational complexity of the procedure could be presented as $m \times \mathcal{O}(k) \times \mathcal{O}(k^n) = \mathcal{O}(k^{n+1})$, usually $k \gg m$.

Another way of defining $\mu(x \mid \widetilde{X})$ could be done by using the concept of same and inverse points. A detailed study by Ghosh and Chakraborty [37, 44] show that the feasible region $\widetilde{X}_j$ can be written as $\widetilde{X}_j = \bigcup_{\alpha \in [0,1]} \{\widetilde{X}_j(\alpha)\}$ where $\widetilde{X}_j(\alpha)$ can be expressed as follows.

$$\widetilde{X}_j(\alpha) = \left\{ x \in \mathbb{R}^n_{\geqq} \ : \ g_j(x, a_j^\alpha) \leq b_j^\alpha \right\} \tag{8}$$

where a_j^α and b_j^α are inverse points of $\tilde{a}_j$ and $\tilde{b}_j$.

Based on the above equation (8), the membership function of $\widetilde{X}_j$ can be described as follows:

$$\mu\big(x \mid \widetilde{X}_j\big) = \underset{\alpha \in [0,1]}{Sup} \left\{ \alpha : \ a_{j1}^\alpha x_1 + a_{j2}^\alpha x_2 + \ldots a_{jn}^\alpha x_n \leq b_j^\alpha \right\} \tag{9}$$

where $(a_{j1}^\alpha, a_{j2}^\alpha, \ldots a_{jn}^\alpha)$ and b_j^α are inverse points.

In order to find the membership value $\mu\big(x \mid \widetilde{X}_j\big)$, the following two steps will be sufficient.

- We have check for a fix α whether the inequality $a_{j1}^\alpha x_1 + a_{j2}^\alpha x_2 + \cdots a_{jn}^\alpha x_n \leq b_j^\alpha$ satisfies or not.
- Take the supremum of those α for which the inequality $a_{j1}^\alpha x_1 + a_{j2}^\alpha x_2 + \cdots a_{jn}^\alpha x_n \leq b_j^\alpha$ hold.

A sequential algorithm of this method is listed in algorithm 3.

Algorithm 3 Algorithm for obtaining $\mu(x \mid \widetilde{X})$ using Same and Inverse points

Require: The number k in which the unit interval [0, 1] will be discretize.

1: **for** $\alpha = 1$ to 0 with step length $\frac{1}{k}$ **do**
2:     **for** $i = 1$ to m **do**
3:         temp=0
4:         Find the inverse points $(a_{i1}^l, a_{i2}^l, \ldots a_{im}^l)$ and b_i^r or $(a_{i1}^r, a_{i2}^r, \ldots a_{im}^r)$ and b_i^l.
5:         **if** $a_{j1}^l x_1 + a_{j2}^l x_2 + \ldots a_{jn}^l x_n \leq b_j^r$ or $a_{j1}^r x_1 + a_{j2}^r x_2 + \ldots a_{jn}^r x_n \leq b_j^l$ **then**
6:             temp=temp+0
7:         **else**
8:             temp=temp+1
9:         **end if**
10:        **if** $temp = 0$ **then**
11:            Go to step 15
12:        **end if**
13:    **end for**
14: **end for**
15: Return membership value $\mu(x \mid \widetilde{X}) = \alpha$.

Numerically the computational complexity of the above procedure can be given as $m \times \mathcal{O}(k) = \mathcal{O}(k)$, since $k \gg m$.

If we compare the computational complexities of the above described procedures, it would be clear that exercising the procedure of inverse point is more preferable. The detailed study of Ghosh and Chakraborty [37, 44] show the equivalency of the above described methods.

Example 4 Let us consider the fuzzy inequalities $C_1 : (2,2,4)x + (3,4,6)y \leq (10,12,14)$ and $C_2 : (3,4,6)x + (2,2,4)y \leq (10,12,14)$. So, the feasible space $\widetilde{X}$ presented by the inequalities C_1 and C_2 along with the non-negativity conditions can be written as $\widetilde{X} = \big\{(x,y) \in \mathbb{R}^2_{\geqq} : (2,2,4)x + (3,4,6)y \leq (10,12,14)\big\} \cap \big\{(x,y) \in \mathbb{R}^2_{\geqq} : (3,4,6)x + (2,2,4)y \leq (10,12,14)\big\} = \big\{(x,y) \in \mathbb{R}^2_{\geqq} : (2x+3y, 2x+4y, 4x+6y) \leq (10,12,14)\big\} \cap \big\{(x,y) \in \mathbb{R}^2_{\geqq} : (3x+2y, 4x+2y, 6x+4y) \leq (10,12,14)\big\} = \widetilde{X}_1 \cap \widetilde{X}_2$.

For an α, the inverse points of $(2x+3y, 2x+4y, 4x+6y)$ and (10, 12, 14) are given as $2x + 3y + \alpha y$ and $14 - 2\alpha$ or $4x + 6y - \alpha(2x+2y)$ and $10 + 2\alpha$. Applying concept of inverse points the α-cut of the feasible space $\widetilde{X}(\alpha)$ can written as equation (10).

$$\begin{aligned} \widetilde{X}_1(\alpha) &= \bigcup_{\gamma \in [\alpha,1]} \Big[\big\{(x,y) \in \mathbb{R}^2_{\geqq} : 2x + 3y + \gamma y \leq 14 - 2\gamma\big\} \cup \\ &\qquad \big\{(x,y) \in \mathbb{R}^2_{\geqq} : 4x + 6y - \gamma(2x+2y) \leq 10 + 2\gamma\big\}\Big] \\ &= \bigcup_{\gamma \in [\alpha,1]} \Big[\big\{(x,y) \in \mathbb{R}^2_{\geqq} : 2x + (3+\gamma)y \leq 14 - 2\gamma\big\} \cup \\ &\qquad \big\{(x,y) \in \mathbb{R}^2_{\geqq} : (4-2\gamma)x + (6-2\gamma)y \leq 10 + 2\gamma\big\}\Big] \end{aligned} \tag{10}$$

So, the core of the region $\widetilde{X}_1$ and $\widetilde{X}_2$ would be determined by the inequalities $2x + 4y \leq 12$ and $4x + 2y \leq 12$ respectively.

Suppose, we want to determine $\mu\big((2,3) \mid \widetilde{X}_1\big)$. The first inequality of equation (10) suggests for $\gamma \leq 0.2$, the point (2, 3) satisfies $2x + (3+\gamma)y \leq 14 - 2\gamma$. The second inequality gives $\gamma \geq \frac{4}{3}$ for the belonging ness of (2, 3) within $(4-2\gamma)x + (6-2\gamma)y \leq 10 + 2\gamma$ which is not possible. Thus, the membership value of (2, 3) is obtained as $\mu\big((2,3) \mid \widetilde{X}_1\big) = 0.2$.

Similarly, it can be obtained that $\mu\big((2,3) \mid \widetilde{X}_2\big) = 0.5$

Finally, from (6) we can conclude that $\mu\big((2,3) \mid \widetilde{X}\big) = min\{\mu\big((2,3) \mid \widetilde{X}_1\big), \mu\big((2,3) \mid \widetilde{X}_2\big)\} = 0.2$.

3.2 *Obtaining vertex of the Fuzzy feasible space*

The optimal solution of classical linear programming occurs at a vertex of the feasible space. The analogical sense suggests that, the optimality of the FLP should occur at vertex of fuzzy feasible space. The fuzzy vertices should be obtained as point of intersections of the linear constraints or fuzzy lines. The procedure of obtaining fuzzy points of intersection described in section 2 should be followed in this requirement.

The vertices of the feasible spaces should be obtained from the set of the point of intersections based on geometrical visualization.

Let us consider the FLPP involving two decision variables x and y of the following form:

$$\begin{cases} Maximize:\ f(x,\tilde{c}) = \tilde{c}_1 x + \tilde{c}_2 y \\ subject\ to: \\ g_1(x,y,\tilde{a}_1) = \tilde{a}_{11}x + \tilde{a}_{12}y \leq \tilde{b}_1 \\ g_2(x,y,\tilde{a}_2) = \tilde{a}_{21}x + \tilde{a}_{22}y \leq \tilde{b}_2 \\ \vdots \\ g_m(x,y,\tilde{a}_m) = \tilde{a}_{m1}x + \tilde{a}_{m2}y \leq \tilde{b}_m \\ (x,y) \in \mathbb{R}^2_{\geqq} \end{cases} \tag{11}$$

Similar to Corner method, out of $\binom{m+2}{2} = \frac{(m+2)(m+1)}{2}$ points of intersections, the contributing vertices of the feasible space has to be sorted from graphical analysis. Let us consider that $\mathbf{S} = \{\widetilde{P}_1, \widetilde{P}_2 \ldots\}$ is the set of the vertices of FFS. From the figure 9, it can be observed that the constraints are represented by the lines $\widetilde{L}_1, \widetilde{L}_2$ and the axes x and y. So, the point of intersections occur at O(0, 0), A(0, 3), B(0, 6), P(2, 2), C(0, 3) and D(0, 6). But, out of these six points, the vertices which form the feasible are O(0, 0), A(0, 3), P(2, 2) and C(0, 3).

After obtaining the set $\mathbf{S}$ of vertices, the optimal solution solution is found at certain vertices. To find the optimal vertices, the functional value of the objective function $f(x, y)$ are calculated at each vertex $\{\widetilde{P}_1, \widetilde{P}_2 \ldots\}$. Denote $f(\widetilde{P}_i) = \tilde{f}_i$. The optimal solution occurs at some corner point $\widetilde{P}_k(say)$ for which the inequalities hold $\Re(\tilde{f}_k) \geq \Re(\tilde{f}_i);\ i = 1, 2 \ldots n, i \neq k$ (see definition 1).

Definition 1 (*Ranking of Triangular Fuzzy Numbers* [11]):

The ranking function $\Re$ of a FTN $\widetilde{A}(\underline{a}, a, \bar{a})$ is defined from fuzzy universe $F(\mathbb{R})$ into the set of real numbers $\mathbb{R}$. The function $\Re : F(\mathbb{R}) \to \mathbb{R}$ is defined as $\Re(\widetilde{A}) = \frac{\underline{a}+2a+\bar{a}}{4}$.

So, the ordering between two triangular fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ will be defined as $\widetilde{A} \gtrsim \widetilde{B}$ if and only if $\Re(\widetilde{A}) \geq \Re(\widetilde{B})$ i.e., $\frac{\underline{a}+2a+\bar{a}}{4} \geq \frac{\underline{b}+2b+\bar{b}}{4}$.

In the following the proposed method of solving FLPP using graphical method is illustrated with several numerical examples. An algorithm summarization of the proposed method is given in algorithm 4.

Example 5 Let us focus on the following FLPP:

$$\begin{cases} Maximize:\ f(x,y) = \tilde{2}x + \tilde{3}y \\ subject\ to: \\ g_1(x,y) = \tilde{2}x + \tilde{4}y \leq \tilde{12} \\ g_2(x,y) = \tilde{4}x + \tilde{2}y \leq \tilde{12} \\ (x,y) \geq (0,0) \end{cases} \tag{12}$$

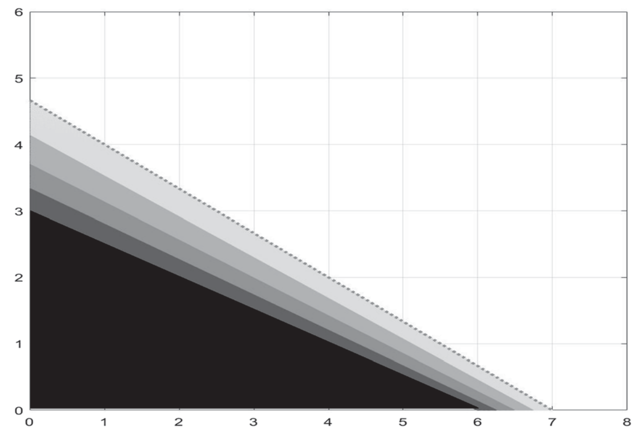

Figure 8. Fuzzy inequality.

The first step towards solving the above FLPP should be determining fuzzy feasible space formed by the constraints g_1, g_2 and non-negativity conditions $x \geq 0$ and $y \geq 0$. Plotting the fuzzy lines $g_1 = 0$ and $g_2 = 0$ in the geometrical space should give the geometrical visualization of the FFS. The point to point membership value calculation can be executed by using the algorithm 3. The fuzzy region implicated by the constraint $g_1 \leq 0$ is shown in the figure 8 and the fuzzy feasible space implicated by the constraints $g_1 \leq 0, g_2 \leq 0$ and $x \geq 0, y \geq 0$ is shown in the figure 9.

The second step would be determining the vertices of FFS. Out of total $\binom{4}{2} = 6$ corner points, four vertices situated at O(0, 0); A(3, 0); B(2, 2) and C(0, 3) contributed to form the FFS. The nature of the fuzzy points $\widetilde{O}(0,0); \widetilde{A}(3,0); \widetilde{B}(2,2)$ and $\widetilde{C}(0,3)$ are determined from the algorithm 1.

According to algorithm 1, the points $\widetilde{O}(0,0); \widetilde{A}(3,0)$ and $\widetilde{C}(0,3)$ have zero spread lengths along the directions of both axes, i.e. the points $\widetilde{O}(0,0); \widetilde{A}(3,0)$ and $\widetilde{C}(0,3)$ are crisp. Whereas, the point $\widetilde{B}(2,2)$ have symmetric

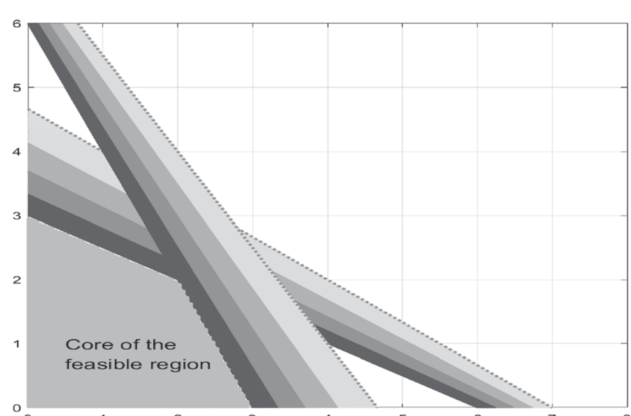

Figure 9. Feasible space of FLPP.

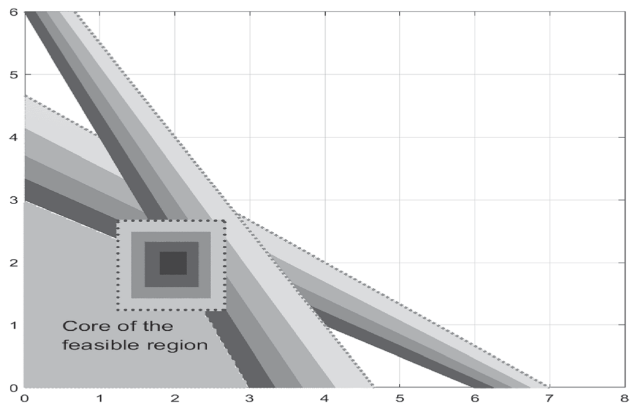

Figure 10. The vertices of FFS.

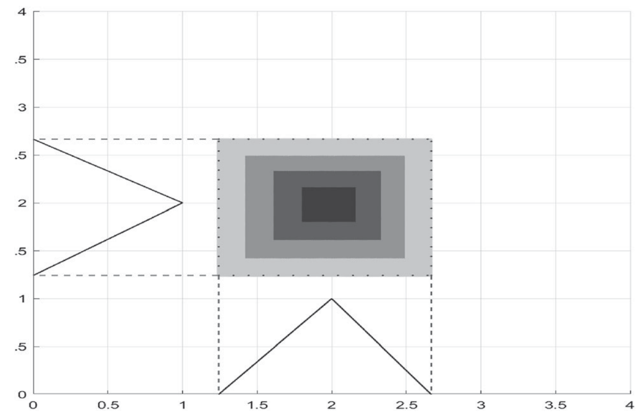

Figure 11. Optimal solution of FLPP.

spread lengths of 1 unit and 1 unit along x and y axes respectively.

The next step is to calculate the objective value $f(\widetilde{O})$, $f(\widetilde{A}), f(\widetilde{B})$, and $f(\widetilde{C})$ at the points $\widetilde{O}(0,0); \widetilde{A}(3,0); \widetilde{B}(2,2)$ and $\widetilde{C}(0,3)$. Needless to say, the objective values $f(\widetilde{O})$, $f(\widetilde{A}), f(\widetilde{B})$, and $f(\widetilde{C})$ at the points $\widetilde{O}(0,0); \widetilde{A}(3,0); \widetilde{B}(2,2)$ and $\widetilde{C}(0,3)$ will be fuzzy numbers. The optimal (maximum) value among $f(\widetilde{O}) = 0, f(\widetilde{A}) = 6, f(\widetilde{B}) = \widetilde{10}, f(\widetilde{C}) = 9$ at the points $\widetilde{O}(0,0); \widetilde{A}(3,0); \widetilde{B}(2,2)$ and $\widetilde{C}(0,3)$ provide the optimal solution of the above FLPP.

So, it is clear that at fuzzy point $\tilde{B}(2,2)$, we obtain the optimal value of the objective function $\tilde{f}(x,y)$ and the optimal solution obtained as $x = \tilde{2}$ and $y = \tilde{2}$. The optimal point $\tilde{B}(2,2)$ is depicted in the figures 10 and 11.

Algorithm 4 Algorithm for Graphical Method to Solve Fuzzy Linear Programming

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Require: The cost coefficients $\tilde{c}_1, \tilde{c}_2$ and boundary coefficients $\tilde{a}_{11}, \tilde{a}_{12}, \tilde{a}_{21}, \ldots \tilde{a}_{mn}$.
1: **for** ı = 1 to m **do**
2:     **for** $k = i + 1$ to m **do**
3:         Find the fuzzy point of intersection $\widetilde{P}_{ik}$ between the constraint L_i and L_k (Use algorithm 1).
4:     **end for**
5: **end for**
6: Find the vertices of the feasible space.
7: Compute $F_{ij} = \tilde{f}(c, P_{ij})$.
8: Find the optimal solution as=$max\{F_{ij}\}$.

Example 6 The following fuzzy linear programming problem is taken from the study of Kumar *et al* [6].

$$\begin{cases} Maximize: \ f(x,y) = (1,6,9)x + (2,3,8)y \\ \quad subject\ to: \\ g_1(x,y) = (2,3,4)x + (1,2,3)y \le (6,16,30) \\ g_2(x,y) = (-1,1,2)x + (1,3,4)y \le (1,17,30) \\ \quad (x,y) \ge (0,0) \end{cases} \tag{13}$$

The optimal solution of the above problem is reported in [6] as $\tilde{x}_A = (1,2,3), \tilde{y}_A = (4,5,6)$ and $\tilde{f}_K = (9,27,75)$.

Solving the same problem with the proposed method provides an alternative solution $\tilde{x}_P = (0,2,7.25), \tilde{y}_P = (2.86,5,15.5)$ and $\tilde{f}_P = (5.72,27,193.48)$.

It is interesting to observe that the optimal value obtained in the proposed method is better than the optimal solution obtained in the study of Kumar *et al* [6], because $\Re(\tilde{f}_P) = 63.3 \ge \Re(\tilde{f}_K) = 34.5$.

Apart from the improved results, the method of Kumar *et al* [6] has risk of in existence. If we slightly change the coefficients of the constraints g_1 and g_2, the method of Kumar et al [6] fail to exist.

For example if we change the constraint g_2 as $(-1,1,3)x + (1,3,4)y \le (1,17,30)$ then the method of Kumar et al. [6] fails to provide any solution whereas proposed method provide a solution $\tilde{x}_P = (0,2,7.25), \tilde{y}_P = (2.93,5,15.5)$ and $\tilde{f}_P = (2.71,27,189.25)$.

Example 7 The following fuzzy linear programming problem is taken from the study of Najafi *et al* [7].

$$\begin{cases} Maximize: \ f(x,y) = (1,3,9)x + (1,2,8)y \\ \quad subject\ to: \\ g_1(x,y) = (1,3,5)x + (2,3,4)y \le (1,9,22) \\ g_2(x,y) = (1,2,3)x + (2,3,4)y \le (1,8,18) \\ \quad (x,y) \ge (0,0) \end{cases} \tag{14}$$

If the method proposed by Kumar *et al* [6] is applied to this problem then non-negative solution $\tilde{x}_K = (-1,1,2), \tilde{y}_K = (1,2,3)$ is obtained.

The same problem has positive solution $\tilde{x}_N = (0, 1, 2), \tilde{y}_N = (.05, 2, 3)$ when solve by the method proposed by Najafi et al. [7]. The optimal value of is $\tilde{f_N} = (0.5, 7, 42)$ and $\Re(\tilde{f_N}) = 14.125$.

Solving the same problem with the proposed method provides an alternative solution $\tilde{x}_P = (0, 1, 7.5), \tilde{y}_P = (1.167, 2, 5.25)$ and $\tilde{f_P} = (1.167, 7, 109.59)$; $\Re(\tilde{f_P}) = 31.19$.

It is interesting to observe that the optimal value obtained in the proposed method is better than the optimal solution obtained in the study of Najafi et al. [7], i.e. $\Re(\tilde{f_P}) = 81.94 \geq \Re(\tilde{f_N}) = 23.33$

4. Conclusion

A graphical method to solve fuzzy linear programming problem is introduced. Geometrical visualization of the fuzzy feasible space induced by fuzzy linear inequalities has been achieved using same and inverse points. The formation of fuzzy points from intersection of two fuzzy lines has been analyzed. The proposed geometry based FLPP obtain the optimal solution as an interpretable fuzzy point.

This study is just an initiation towards analytical solution of fuzzy linear programming problem. A firm geometrical ground has been laid here for the extension of simplex method for multi-dimensional problem.

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