Sādhanā (2022)47:169
https://doi.org/10.1007/s12046-022-01944-w

Piece-wise approximation for Gaussian Q-function and its applications

JYOTI GUPTA* and ASHISH GOEL

Jaypee Institute of Information Technology, Sector 62, Noida, Uttar Pradesh 201309, India
e-mail: jyotiguptas21@gmail.com; ashish.goel@jiit.ac.in

MS received 24 January 2022; revised 15 May 2022; accepted 5 July 2022

Abstract. The Gaussian Q-function and its integer powers are quite versatile in various fields of science. In this paper, we propose a novel exponential-based approximation for the Gaussian Q-function and compare the tightness of the approximation against existing methods. The proposed approximation is a piece-wise function, which is derived using a numerical method of integration. With the help of numerical results, we show that the proposed approximation yields accurate results for a wide range of Q-function arguments and at the same time outperforms existing approximations. Using the results of the proposed approximation, we evaluate the symbol error probability (SEP) of triangular quadrature amplitude modulation (TQAM) schemes for different constellations over additive white Gaussian noise (AWGN) and various fading channels including Nakagami-m, $\alpha-\mu$, $\kappa-\mu$ and $\alpha-\kappa-\mu$ fading channels. We facilitate the closed-form solution of the intractable integrals used in SEP calculation over the Nakagami-m fading channel. To further validate the results, we perform Monte Carlo simulations for SEP calculation of TQAM schemes over various fading channels and indicate a close match with the exact SEP calculation.

Keywords. Gaussian Q-function; AWGN; Nakagami-m fading; $\alpha-\kappa-\mu$ distribution; symbol error probability (SEP); triangular quadrature amplitude modulation (TQAM).

1. Introduction

The Gaussian Q-function holds paramount importance in the performance analysis of communication systems [1]. It is defined in terms of error function (erf($\cdot$)) or complementary error function (erfc($\cdot$)), which has prominent use in communication theory problems [1]. Since integral forms are tough to handle mathematically, the literature has offered different approximations of Gaussian Q-function and their bounds [2]. The closed-form expressions simplify the mathematical analysis and therefore play an important part in the performance evaluation of communication systems [3]. In wireless communication systems, Symbol Error Probability (SEP) is a widely used system metric to gauge the transmission quality over the communication channels [4]. In order to perform a simple mathematical analysis of the error probabilities, several researchers have proposed different types of approximations or bounds for Gaussian Q-function [5–14]. In [6], Borjesson and Sundberg proposed an approximation that is the ratio of an exponential function to a polynomial. In [7], Craig has provided a crucial expression for Gaussian Q-function in polar form with finite and fixed limit of integration. In [8], Chiani *et al* approximated the Gaussian Q-function in form of an exponential based bound for the (1-D and 2-D) Gaussian Q-functions as a sum of two exponential functions. It is one of the simplest forms of the expression with a lack of precision for both small and large arguments. In [9], Karagiannidis and Lioumpas have provided another analytical approximation but that is intractable and practically not useful for integrating and finding the closed-form SEP expression over fading channels. In [10], Isukapalli and Rao proposed a polynomial-type analytically tractable approximation, which is the product of an exponential function and a truncated infinite series. In [11], Loskot and Beaulieu proposed an improved approximation, which is the sum of two exponential terms, but its accuracy deteriorates at small values of arguments. In [12], Shi and Karasawa proposed a superior approximation using the N-point quadrature rule, which improves accuracy at the expense of complexity as N rises. In [13], Olabiyi and Annamalai presented an invertible exponential-type approximation, in the form of a weighted sum of powers of an exponential function. In [14], Sadhwani *et al* proposed a simple approximation that involves the sum of exponential functions using trapezoidal rule. However, its accuracy vanishes for smaller as well as larger arguments of Q-function and relative error of this approximation rises monotonically after a certain value of argument of Q-function.

All these research findings and wide applications of Q-function approximation are the prime motivating factors to carry out this work. The main objective of this work is to

*For correspondence
Published online: 20 August 2022

develop a more precise and simple expression for Q-function approximation. In this paper, we propose a novel piece-wise approximation for Q-function using trapezoidal rule utilizing unequal intervals.

1.1 *Motivation*

The main idea behind using unequal interval spacing in the trapezoidal rule for integration in a piece-wise manner is to minimize the relative error in the specified range of the Q-function argument. As a result, the piece-wise function and unequal integration intervals in trapezoidal rule for integration can provide better accuracy at lower as well as higher values of the Q-function arguments. The minimum mean square relative error is used to calculate the interval spacing of the integration function employed in the trapezoidal rule, resulting in the ideal interval spacing. The proposed approximation for Q-function offers a simple piece-wise function, which is the sum of three exponential terms. The simplicity of the proposed approximation enables it to evaluate the closed-form solution for SEP calculations of various digital modulation techniques over AWGN and fading channels.

The remainder of the paper is organized as follows. Section 2 analyses the proposed piece-wise approximation for the Gaussian Q-function and its relative error comparison with existing approximations. Mathematical analysis of SEP for various digital modulation schemes over AWGN and Nakagami-m fading channels are carried out in section 3. Section 4 presents the simulation and the numerical results for SEP calculation using the proposed piece-wise approximation over AWGN and fading channels. Along with its comparison with existing approximations and exact calculation for SEP. The conclusions are provided in section 5.

2. Q-Function and proposed piece-wise approximation for the Gaussian Q-function

Statistically, Q-function is defined as the tail distribution of standard normal distribution [1]. The one-dimensional normalized Gaussian Q-function, is defined as [1]:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} exp\left(-\frac{z^2}{2}\right) dz. \tag{1}$$

The Gaussian Q-function and complementary error function ($erfc(.)$) are related as:

$$erfc(x) = 2Q(\sqrt{2}x). \tag{2}$$

An alternative expression of the Q-function known as Craig's [7] formula, is defined as:

$$Q(x) = \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta. \tag{3}$$

and

$$Q(-x) = 1 - Q(x). \tag{4}$$

Using (3), we can write (2) in the following form

$$erfc(x) = \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} exp\left(-\frac{x^2}{\sin^2\theta}\right) d\theta. \tag{5}$$

In order to simplify the calculation, several approximations have been proposed in the literature [6–14]. Recently, Sadhwani *et al* [14] proposed an approximation for $Q(x)$ using the trapezoidal rule [14], (4) for equal interval spacing of integration. However, this approximation [14] is not able to provide the tighter bound at very small and large values of Q-function arguments (x). It can be noticed from figure 3, [14] that relative error at $x = 0$ is approximately 16.66% and for $x \geq 2.76$ it is increasing monotonically for $N = 3$. In order to overcome these drawbacks, we propose a piece-wise approximation for Gaussian Q-function using trapezoidal rule with unequal intervals of integration. The intervals spacing are found out in such a way so that the mean square relative error of piece-wise Gaussian Q-function is minimum. Therefore, the proposed approximation for Gaussian Q-function aims to provide tighter bound for $\forall x$. The trapezoidal rule is a numerical analysis approach for approximating the definite integral. For definite integration of any function, the trapezoidal rule is as follows [14]

$$\begin{aligned} I &= \int_{x_0}^{x_0+Nh} f(x)dx \\ &= \frac{h}{2}[(y_0 + y_N) + 2(y_1 + y_2 + \cdots + y_{N-1})]. \end{aligned} \tag{6}$$

where $y_0 = f(x_0)$ and $y_0 = f(x_0 + Nh)$.

The area under the integration is divided into N sub-intervals, each representing an evenly spaced trapezium, with h defining the spacing between them.

The general formula of the trapezoidal rule for unequal interval spacing can be written as [15]

$$\begin{aligned} I = \int_{x_0}^{x_N} f(x)dx = \Bigg[&(x_1 - x_0)\frac{(y_0 + y_1)}{2} + (x_2 - x_1)\frac{(y_1 + y_2)}{2} \\ &+ \cdots + (x_N - x_{N-1})\frac{(y_{n-1} + y_N)}{2}\Bigg]. \end{aligned} \tag{7}$$

where $f(x_N) = y_N$. Using (7), The $erfc(x)$ given in (5) can be evaluated as

$$I = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} exp\left(-\frac{x^2}{\sin^2\theta}\right)d\theta, \tag{8}$$

$$I = (\theta_1-\theta_0)\frac{(y_0+y_1)}{2} + (\theta_2-\theta_1)\frac{(y_1+y_2)}{2} + \cdots\cdots + (\theta_N-\theta_{N-1})\frac{(y_{N-1}+y_N)}{2}. \tag{9}$$

where $y_N = \frac{2}{\pi}e^{-\frac{x^2}{\sin^2\theta_N}}$, $\theta_0 = 0$, $\theta_N = \frac{\pi}{2}$ and $0 \le \theta_1, \theta_2, \theta_3, \ldots, \theta_{N-1} \le \frac{\pi}{2}$.

The integration function given in (8) has been divided into N sub-intervals, representing N unequally spaced trapeziums. We have done analysis for $N = 2, 3$and4. For $N = 2$, there will be two intervals for integration i.e., 0 to θ_1 and θ_1 to $\frac{\pi}{2}$. Therefore, for this case, we can write (9) in the following form

$$I(x,\theta_1) = \frac{1}{2}\left[\frac{\pi}{2}y_1 + \left(\frac{\pi}{2}-\theta_1\right)y_2\right]. \tag{10}$$

Similarly, for $N = 3$, we have taken three unequal intervals as 0 to θ_1, θ_1 to θ_2 and θ_2 to $\frac{\pi}{2}$ in the range $0 \le \theta_1$, $\theta_2 \le \frac{\pi}{2}$. Here θ_1 and θ_2 are variables and $0 \le \theta_1 < \theta_2 \le \frac{\pi}{2}$. Therefore, integration of (9) for $N = 3$, can be written as:

$$I(x,[\theta_1,\theta_2]) = \frac{1}{2}\left[\theta_2 y_1 + \left(\frac{\pi}{2}-\theta_1\right)y_2 + \left(\frac{\pi}{2}-\theta_2\right)y_3\right] \tag{11}$$

For $N = 4$, there will be four unequal intervals for integration i.e., 0 to θ_1, θ_1 to θ_2, θ_2 to θ_3 and θ_3 to $\frac{\pi}{2}$.

$$I(x,[\theta_1,\theta_2,\theta_3]) = \frac{1}{2}\left[\theta_2 y_1 + (\theta_3-\theta_1)y_2 + \left(\frac{\pi}{2}-\theta_2\right)y_3 + \left(\frac{\pi}{2}-\theta_3\right)y_4\right]. \tag{12}$$

Hence, for the general case of N unequal intervals of integration for Trapezoidal rule, approximation of *erfc*(.) function is calculated using (9).

The percentage relative error (RE) can be calculated as:

$$RE = \frac{|erfc(x) - I(x,[\theta_1,\theta_2,\ldots,\theta_{N-1}])|}{erfc(x)} \times 100. \tag{13}$$

where $I(x,[\theta_1,\theta_2,\ldots,\theta_{N-1}])$ is the approximated $erfc(x)$ function for N intervals.

To achieve high accuracy of the approximation function, the optimal values of the parameters $\theta_1, \theta_2, \ldots, \theta_{N-1}$ are calculated by the mean square error (MSE) optimization method to minimize the relative error defined in (13) as:

$$[\theta_{1opt}, \theta_{2opt}, \ldots, \theta_{N-1opt}] = argmin_{\theta_1,\theta_2,\ldots,\theta_{N-1}} E\left\{\left[\frac{|erfc(x) - I(x,[\theta_1,\theta_2,\ldots\theta_{N-1}])|}{erfc(x)}\right]^2\right\} \tag{14}$$

The proposed approximation is a piece-wise function, therefore the optimal values of θ_1, $\theta_2, \ldots,$ and θ_{N-1} are calculated by applying (14) for desired ranges of x.

It has been found that relative error of [14] for $N = 2, 3$, and 4 is monotonically increasing after $x_1 = 1.95$, $x_1 = 2.76$ and $x_1 = 3.70$, respectively.

Therefore, in this paper, we have considered two pieces of $erfc(x)$ approximation function i.e., for $x < x_1$ and $x \ge x_1$. For $N = 2$, the obtained optimal value of the parameter θ_1 is calculated using (14), and the obtained value of θ_1 for $x < x_1 = 1.95$ is $\frac{423\pi}{1800}$ and for $x \ge x_1$ its calculated value is $\frac{623\pi}{1800}$.

Similarly, for $N = 3$, the $erfc(x)$ is also defined in two pieces i.e., for $x < x_1 = 2.76$ and $x \ge x_1 = 2.76$. Using (14), the obtained optimal values of the parameters θ_1 and θ_2 for $x < x_1$ are $\frac{27\pi}{200}$ and $\frac{191\pi}{600}$, respectively and for $x \ge x_1$ their calculated values are $\frac{553\pi}{1800}$ and $\frac{723\pi}{1800}$, respectively.

For $N = 4$, we have also calculated the values of θ_1, θ_2, and θ_3, the calculated optimal values for $x < x_1 = 3.7$ are $\frac{151\pi}{1800}, \frac{411\pi}{1800}$, and $\frac{82\pi}{225}$ respectively and for $x \ge x_1$ their calculated values are $\frac{553\pi}{1800}, \frac{661\pi}{1800}$ and $\frac{139\pi}{450}$, respectively.

On substituting the values of θ_1, θ_2 and θ_{n-1} in (11), the piece-wise approximation for $erfc(x)$ for N intervals can be derived in the following form:

$$erfc(x) = \begin{cases} \sum_{i=1}^{N} a_i e^{-c_i x^2}, & 0 \le x \le x_1 \\ \sum_{i=1}^{N} a'_i e^{-c'_i x^2}, & x_1 \le x \end{cases} \tag{15}$$

where a_i, c_i, a'_i and c'_i , $1 \le i \le N$ are the parameters of the proposed piece-wise approximation for $erfc(x)$, the values of these parameters are given in the table 1.

Using (15), the approximated Gaussian Q-function can be written as:

$$Q(x) = \begin{cases} \frac{1}{2}\sum_{i=1}^{N} a_i e^{-\frac{c_i x^2}{2}}, & 0 \le x < x_1 \\ \frac{1}{2}\sum_{i=1}^{N} a'_i e^{-\frac{c'_i x^2}{2}}, & x_1 \le x. \end{cases} \tag{16a}$$

For simplicity, we denote

$$Q_1(x) = \frac{1}{2}\sum_{i=1}^{N} a_i e^{-\frac{c_i x^2}{2}}, \quad 0 \le x < x_1, \tag{16b}$$

$$Q_2(x) = \frac{1}{2}\sum_{i=1}^{N} a'_i e^{-\frac{c'_i x^2}{2}}, \quad x_1 \le x \tag{16c}$$

Figure 1 depicts the comparison of the proposed approximation for $erfc(x)$ with exact and other existing approximations [8, 9, 11, 12] and [14]. It can be easily observed from figure 1 that the plot of the proposed approximation for $erfc(x)$ is closest to the exact plot for $erfc(x)$ function, whereas the plot of other approximations [8, 9, 11, 12] and [14] are showing deviation from the exact plot for $erfc(x)$ function. We can also say that the proposed approximation and the exact $erfc(x)$ function plots are virtually indistinguishable. A comparison of proposed and

Table 1. Parameters of proposed piecewise approximation for $erfc(x)$.

x	Coefficients for approximated $erfc(.)$ function								
$0 \leq x \leq x_1$	N	a_1	c_1	a_2	c_2	a_3	c_3	a_4	c_4
	2	0.5	2.207769	0.265	1	–	–	–	–
	3	0.318333	5.905137	0.365000	1.412149	0.181667	1	–	–
	4	0.228333	14.735657	0.280556	2.314053	0.271667	1.205771	*0.135556*	*1*
$x_1 \leq x \leq 5$	N	a'_1	c'_1	a'_2	c'_2	a'_3	c'_3	a'_4	c'_4
	2	*0.5*	1.275637	0.153889	1	–	–	–	–
	3	0.401667	1.479463	0.192778	1.101851	0.098333	1	–	–
	4	0.308889	1.468834	0.12500	1.196372	0.132778	1.044408	0.066111	1

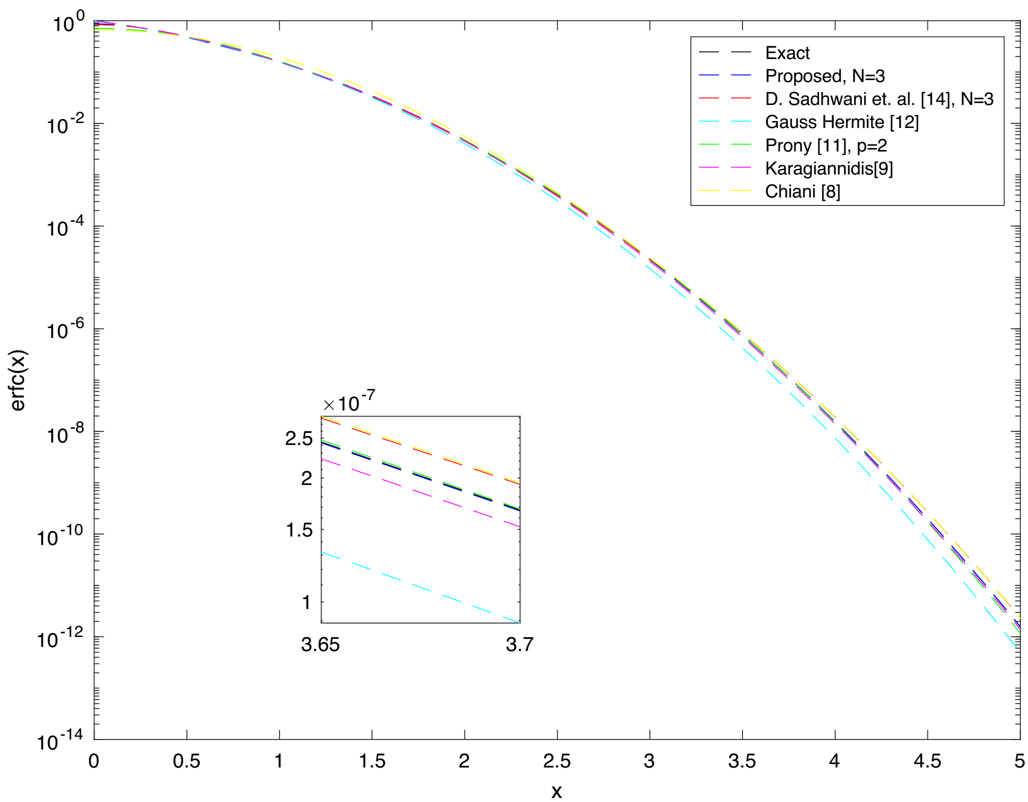

Figure 1. Comparison of proposed approximated $erfc(.)$ function with existing approximations.

existing approximations for $erfc(x)$ function at few values of x is also presented in table 2. Table 2 confirms the accuracy of the proposed approximation. It can be seen from table 2 that value of the $erfc(x)$ function using the proposed approximation at $x = 5$, is 1.57551×10^{-12}, which is found to be closest to the corresponding value of the exact $erfc(x)$ function which is 1.5374×10^{-12}. Whereas other approximations are found to have more errors in approximation than that of the proposed approximation function.

Figure 2 depicts the comparison of percentage relative error for the proposed and Sadhwani *et al* [14] for $N = 2, 3,$ and4 intervals, respectively.

To show the outperformance of the proposed approximation, we have also calculated the percentage relative error using (13) for all of the schemes taken into consideration [8, 9, 11, 12] and [14]. Figure 3 shows the comparison of relative errors of the proposed approximation for $erfc(x)$ function with other existing approximations [8, 9, 11, 12] and [14]. It can also be observed from figure 3 that the proposed approximation (16) for $N = 3$ is having the least approximation error than that of the all other schemes [8, 9, 11, 12] and [14] especially in the higher arguments region.

To retain a reasonable trade-off, we set an optimum value of $N = 3$, to validate the accuracy of the proposed

Table 2. Comparison of proposed and existing approximations for $erfc(x)$.

x	$erfc(x)$	Proposed N = 3	Sadhwani *et al* [14] N = 3	Gauss Hermite N = 2 [12]	Prony p = 2 [11]	Karagiannidis [9]	Chiani *et al* [8]
0.3	0.671373	0.674567	0.680521	0.671179	0.616780	0.678248	0.595782
1.5	0.033895	0.034367	0.034203	0.031636	0.032956	0.033136	0.042460
2.5	0.000407	0.000404	0.000402	0.000312	0.000417	0.000381	0.000442
3.5	7.43098e−7	7.40848e−07	8.24398e−07	4.23146e−07	7.62510e−7	6.78936e−07	8.37837e−07
4.5	1.96616e−10	1.97229e−10	2.68165e−10	7.78411e−11	1.71459e−10	1.77294e−10	2.68478e−10
5	1.5374e−12	1.57551e−12	2.3157e−12	4.9890e−13	1.1698e−12	1.3806e−12	2.3163e−12

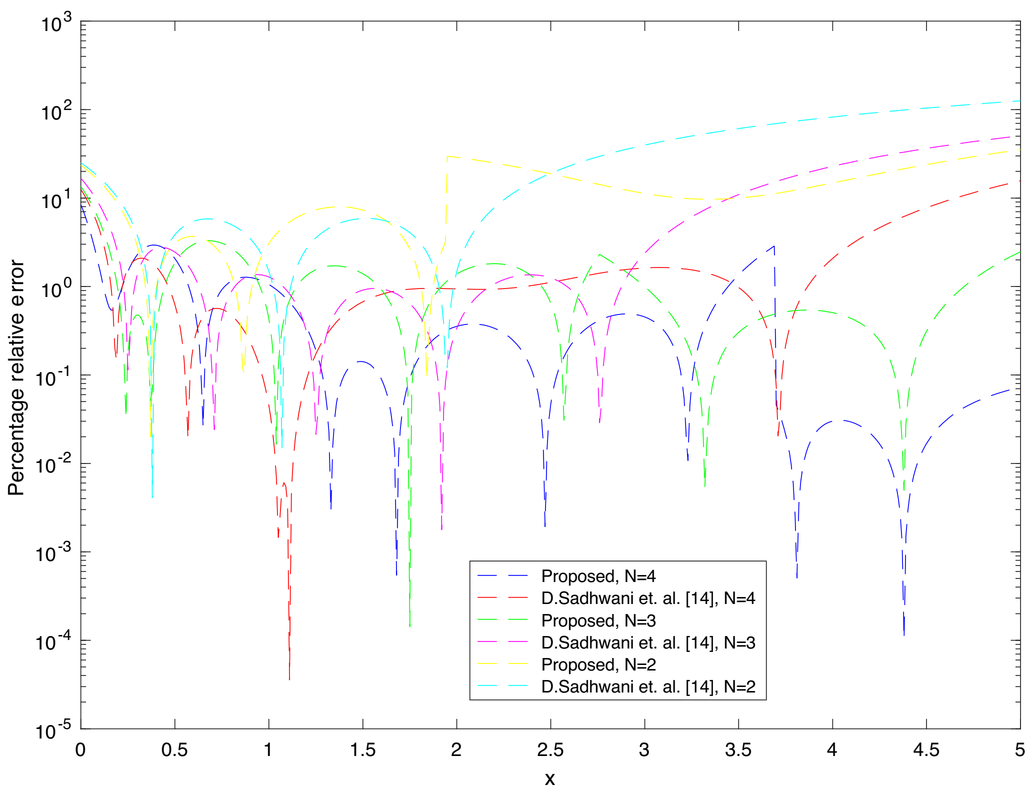

Figure 2. Comparison of Relative Error of proposed and Sadhwani *et al* [14] approximation.

approximation. The proposed approximation for Q-function is defined for $N = 3$ intervals and resulting only has only three exponential terms like [14], but at the same time it achieves higher accuracy than existing approximations [8, 9, 11, 12] and [14].

3. SEP of digital modulation schemes over AWGN and Nakagami-m fading channel

This section deals with the general and closed-form SEP calculations for various TQAM constellations over AWGN and fading channels. The SEP calculation for TQAM over AWGN channel is very important in wireless communication [16]. We first calculate the SEP of a general class of TQAM, over AWGN channel, using the proposed approximations for Gaussian Q-function. The general SEP equation for AWGN channel is given by [16]

$$\begin{aligned} SEP = P_{AWGN} \\ &= KQ\left(\sqrt{\beta\gamma}\right) + \frac{2}{3}K_C Q^2\left(\sqrt{\frac{2\beta\gamma}{3}}\right) \\ &- 2K_C Q\left(\sqrt{\beta\gamma}\right) Q\left(\sqrt{\frac{\beta\gamma}{3}}\right). \end{aligned} \tag{17}$$

where γ represents the signal-to noise ratio (SNR); β, K, and K_C are defined as system parameter, average number of nearest-neighbors and average number of couples of adjacent nearest-neighbors, respectively [16].

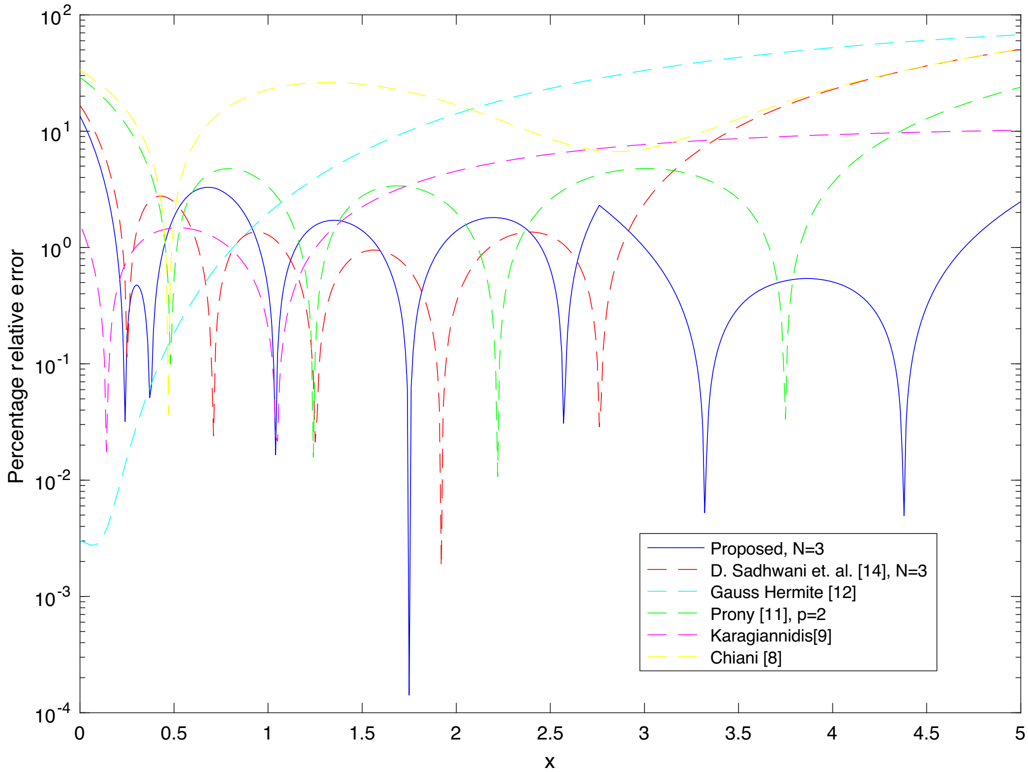

Figure 3. Comparison of Relative Error of the proposed and existing approximation.

These parameters for several TQAM constellations are summarized in table 3.

The general form of the SEP expression for TQAM modulation over AWGN channel using the proposed approximation is a piece-wise function defined in four different intervals and these intervals are derived from the intervals of the proposed piece-wise approximation for Q-function:

where $\gamma_1 = \frac{x_1^2}{\beta}$, $\gamma_2 = \frac{3x_1^2}{2\beta}$ and $\gamma_3 = \frac{3x_1^2}{\beta}$. The values of the functions $Q_1(.)$ and $Q_2(.)$ are already defined in (16b) and (16c), respectively. In (16), Q-function with three different arguments is involved. Based on three arguments and piece-wise Q-function defined in (18), the values of γ_1, γ_2 and γ_3 are calculated.

$$P_{PS\_AWGN} = \begin{cases} KQ_1(\sqrt{\beta\gamma}) + \frac{2}{3}K_C Q_1^2\left(\sqrt{\frac{2\beta\gamma}{3}}\right) - 2K_C Q_1\left(\sqrt{\beta\gamma}\right)Q_1\left(\sqrt{\frac{\beta\gamma}{3}}\right); & 0 \le \gamma \le \gamma_1 \\ KQ_2(\sqrt{\beta\gamma}) + \frac{2}{3}K_C Q_1^2\left(\sqrt{\frac{2\beta\gamma}{3}}\right) - 2K_C Q_2\left(\sqrt{\beta\gamma}\right)Q_1\left(\sqrt{\frac{\beta\gamma}{3}}\right); & \gamma_1 \le \gamma \le \gamma_2 \\ KQ_2(\sqrt{\beta\gamma}) + \frac{2}{3}K_C Q_2^2\left(\sqrt{\frac{2\beta\gamma}{3}}\right) - 2K_C Q_2\left(\sqrt{\beta\gamma}\right)Q_1\left(\sqrt{\frac{\beta\gamma}{3}}\right); & \gamma_2 \le \gamma \le \gamma_3 \\ KQ_2(\sqrt{\beta\gamma}) + \frac{2}{3}K_C Q_2^2\left(\sqrt{\frac{2\beta\gamma}{3}}\right) - 2K_C Q_2\left(\sqrt{\beta\gamma}\right)Q_2\left(\sqrt{\frac{\beta\gamma}{3}}\right); & \gamma_3 \le \gamma \end{cases} \quad (18)$$

Table 3. SEP parameters for TQAM constellations [16].

Constellation, M	TQAM, M = 16	TQAM, M = 64	TQAM, M = 256
β	$\frac{2}{9}$	$\frac{2}{37}$	$\frac{2}{149}$
K	$\frac{33}{8}$	$\frac{161}{32}$	$\frac{705}{128}$
K_C	$\frac{27}{8}$	$\frac{147}{32}$	$\frac{675}{128}$

where M is the constellation size

Typically, linear combinations of the following two integrals or their special variants are employed for SEP representation of digital modulation schemes [1].

$$I_1 = \int_0^\infty Q(a\sqrt{\gamma})Q(b\sqrt{\gamma})p_\gamma(\gamma)d\gamma, \tag{19a}$$

$$I_2 = \int_0^\infty Q^v(a\sqrt{\gamma})p_\gamma(\gamma)d\gamma. \tag{19b}$$

where $p_\gamma(\gamma)$ is probability density function of fading channel, a and b are the real positive constants dependent on the digital modulation scheme and v is the order of $Q(.)$. To evaluate the SEP of various modulation schemes listed in table 3 over Nakagami-m fading channel, the Nakagami-m fading channel with following Probability Density Function (PDF) is considered [1]

$$p_\gamma(\gamma) = \frac{m^m}{\overline{\gamma}^m\Gamma(m)}\gamma^{m-1}exp\left(-\frac{m\gamma}{\overline{\gamma}}\right) \tag{20}$$

where $\overline{\gamma}$ denotes the average SNR, m is the fading-parameter $(0.5 \le m \le \infty)$ and $\Gamma(.)$ represents the Gamma function.

By definition, SEP of any modulation scheme over fading channel can be expressed as:

$$\begin{aligned} SEP_{fading} &= P_{fading} \\ &= \int_0^\infty P_{AWGN} \cdot p_\gamma(\gamma)d\gamma. \end{aligned} \tag{21}$$

The general expression of SEP over fading channel (20) can be written using (17), (19) and (20) as follows:

$$\begin{aligned} SEP_{fading} &= P_{fading} \\ &= K\Im_1 + \frac{2}{3}K_C\Im_2 - 2K_C\Im_3. \end{aligned} \tag{22}$$

where

$$\Im_1 = \int_0^\infty Q(a\sqrt{\gamma})p_\gamma(\gamma)d\gamma, \tag{23}$$

$$\Im_2 = \int_0^\infty Q^2(a\sqrt{\gamma})p_\gamma(\gamma)d\gamma, \tag{24}$$

and

$$\Im_3 = \int_0^\infty Q(a\sqrt{\gamma})Q(b\sqrt{\gamma})p_\gamma(\gamma)d\gamma. \tag{25}$$

The integrals $\Im_1, \Im_2$ and $\Im_3$ given in (22) are the general form of the integrals defined in (17). The values of the integrals $\Im_1, \Im_2$ and $\Im_3$ used to calculate the SEP over fading channel depends on 1st, 2nd and the 3rd term of (17), respectively. Therefore, there is a separate relationship of the coefficients a and b that used to define (23), (24) and (25) with the coefficient β used in 1st, 2nd and 3rd term of (17) respectively, and this relationship is presented in table 4.

Further, the integrals $\Im_1, \Im_2$ and $\Im_3$ should be calculated as piece-wise function according to the given range of Q-function arguments. Three cases have been made to solve the three integrals $\Im_1, \Im_2$ and $\Im_3$ these are described as below:

Case I: The integral $\Im_1$ is defined as:

$$\begin{aligned} \Im_1 &= \int_0^\infty Q(a\sqrt{\gamma})p_\gamma(\gamma)d\gamma, \\ \Im_1 &= \int_0^{l_1} \{Q_1(a\sqrt{\gamma})\}p_\gamma(\gamma)d\gamma + \int_{l_1}^\infty \{Q_2(a\sqrt{\gamma})\}p_\gamma(\gamma)d\gamma, \end{aligned} \tag{26}$$

where limit $l_1 = \frac{2x_1^2}{a^2}$; the value of a will change accordingly to the constellation diagram of the modulation scheme. Based on the proposed approximation for Q-function (.) and using classical Meijer's G-function [17–20], we can write (26) into following form:

Table 4. Value of limits for different TQAM Constellations [16].

Constellation (M)	TQAM, M = 16	TQAM, M = 64	TQAM, M = 256
Case I	$a = \sqrt{\beta} = \sqrt{2/9}$ $l_1 = 68.56$	$a = \sqrt{\beta} = \sqrt{2/37}$ $l_1 = 281.85$	$a = \sqrt{\beta} = \sqrt{2/149}$ $l_1 = 1135.02$
Case II	$a = \sqrt{\frac{2\beta}{3}} = \sqrt{4/27}$ $l_1 = 102.8376$	$a = \sqrt{\frac{2\beta}{3}} = \sqrt{4/111}$ $l_1 = 422.78$	$a = \sqrt{\frac{2\beta}{3}} = \sqrt{4/447}$ $l_1 = 1702.5336$
Case IIII	$a = \sqrt{\beta} = \sqrt{2/9}$ $b = \sqrt{\frac{\beta}{3}} = \sqrt{2/27}$ $l_1 = 68.56$ $l_2 = 205.68$	$a = \sqrt{\beta} = \sqrt{2/37}$ $b = \sqrt{\frac{\beta}{3}} = \sqrt{2/111}$ $l_1 = 281.85$ $l_2 = 845.55$	$a = \sqrt{\beta} = \sqrt{2/149}$ $b = \sqrt{\frac{\beta}{3}} = \sqrt{2/447}$ $l_1 = 1135.02$ $l_2 = 3405.07$

$$\Im_1 = \frac{m^m}{2\overline{\gamma}^m\Gamma(m)}\sum_{i=1}^{3}\left(\left\{\frac{a_i}{(f_i)^m}\left(G_{12}^{11}\left(f_i l_1\middle|\begin{matrix}1\\ m,0\end{matrix}\right)\right.\right.\right.$$
$$\left.\left.\left.-G_{12}^{11}\left(0\middle|\begin{matrix}1\\ m,0\end{matrix}\right)\right)\right\}+\left\{\frac{a_i'}{(f_i')^m}\left(G_{12}^{20}\left(f_i' l_1\middle|\begin{matrix}1\\ 0,m\end{matrix}\right)\right)\right\}\right) \tag{27}$$

where $f_i = \frac{c_i a^2}{2} + \frac{m}{\overline{\gamma}}$; $f_i' = \frac{c_i' a^2}{2} + \frac{m}{\overline{\gamma}}$.

Case II: For $a = b$, the integral $\Im_2$ for $k \geq 2$ can be defined as:

$$\Im_2 = \int_0^\infty Q^k(a\sqrt{\gamma})p_\gamma(\gamma)d\gamma,$$
$$\Im_2 = \int_0^{l_1}\{Q_1^k(a\sqrt{\gamma})\}p_\gamma(\gamma)d\gamma + \int_{l_1}^\infty\{Q_2^k(a\sqrt{\gamma})\}p_\gamma(\gamma)d\gamma, \tag{28}$$

where limits $l_1 = \frac{2x_1^2}{a^2}$; here also the value of a will depend on the constellation diagram of the modulation scheme. Based on the proposed approximation for Q-function (.) and using multinomial theorem and classical Meijer's G-function, we can write (28) into following form:

$$\Im_2 = \frac{m^m}{2^k\overline{\gamma}^m\Gamma(m)}\sum_{i=0}^{k}\sum_{j=0}^{k-i}\frac{k!}{i!j!k-i-j!}$$
$$\left[\left\{\frac{a_1^i a_2^j a_3^{k-i-j}}{f^m}\left(G_{12}^{11}\left(fl_1\middle|\begin{matrix}1\\ m,0\end{matrix}\right)-G_{12}^{11}\left(0\middle|\begin{matrix}1\\ m,0\end{matrix}\right)\right)\right\}\right.$$
$$\left.+\left\{\frac{(a_1')^i(a_2')^j(a_3')^{k-i-j}}{(f')^m}\left(G_{12}^{20}\left(f'l_1\middle|\begin{matrix}1\\ 0,m\end{matrix}\right)\right)\right\}\right]. \tag{29}$$

where

$$f = \frac{c_1a^2i}{2}+\frac{c_2a^2j}{2}+\frac{c_3a^2(k-i-j)}{2}+\frac{m}{\overline{\gamma}};$$
$$f' = \frac{c_1'a^2i}{2}+\frac{c_2'a^2j}{2}+\frac{c_3'a^2(k-i-j)}{2}+\frac{m}{\overline{\gamma}}.$$

Case III: For $a \neq b$, the integral $\Im_3$ can be written as:

$$\Im_3 = \int_0^\infty Q(a\sqrt{\gamma})Q(b\sqrt{\gamma})p_\gamma(\gamma)d\gamma,$$
$$\Im_3 = \int_0^{l_1}\{Q_1(a\sqrt{\gamma})\}\{Q_1(b\sqrt{\gamma})\}p_\gamma(\gamma)d\gamma$$
$$+\int_{l_1}^{l_2}\{Q_2(a\sqrt{\gamma})\}\{Q_1(b\sqrt{\gamma})\}p_\gamma(\gamma)d\gamma$$
$$+\int_{l_2}^{\infty}\{Q_2(a\sqrt{\gamma})\}\{Q_2(b\sqrt{\gamma})\}p_\gamma(\gamma)d\gamma, \tag{30}$$

where limits are $l_1 = \frac{2x_1^2}{a^2}$ and $l_2 = \frac{2x_1^2}{b^2}$; like previous two cases in this case also the value of a and b will change accordingly to the constellation diagram of the modulation scheme.

Based on the proposed approximation for Q-function (.) and using classical Meijer's G-function, we can write (30) in the following form:

$$\Im_3 = \frac{m^m}{4\overline{\gamma}^m\Gamma(m)}\left[\sum_{i=1}^{9}\left(\left\{\frac{A_i}{f_i^m}\left(G_{12}^{11}\left(f_il_1\middle|\begin{matrix}1\\ m,0\end{matrix}\right)\right.\right.\right.\right.$$
$$\left.\left.-G_{12}^{11}\left(0\middle|\begin{matrix}1\\ m,0\end{matrix}\right)\right)\right\}$$
$$+\left\{\frac{B_i}{(f_i')^m}\left(G_{12}^{11}\left(f_i'l_1\middle|\begin{matrix}1\\ m,0\end{matrix}\right)-G_{12}^{11}\left(f_i'l_2\middle|\begin{matrix}1\\ m,0\end{matrix}\right)\right)\right\}$$
$$\left.\left.+\left\{\frac{C_i}{(f_i'')^m}\left(G_{12}^{20}\left(f_i''l_2\middle|\begin{matrix}1\\ 0,m\end{matrix}\right)\right)\right\}\right)\right]. \tag{31}$$

where

$$[A_i]_{i=1}^9 = \left[a_1^2, a_2^2, a_3^2, a_1a_2, a_1a_3, a_1a_2, a_2a_3, a_1a_3, a_2a_3\right].$$
$$[B_i]_{i=1}^9 = \left[a_1a_1', a_2a_2', a_3a_3', a_1a_2', a_1a_3', a_2a_1', a_2a_3', a_3a_1', a_3a_2'\right].$$
$$[C_i]_{i=1}^9 = \left[(a_1')^2, (a_2')^2, (a_3')^2, a_1'a_2', a_1'a_3', a_1'a_2', a_2'a_3', a_1'a_3', a_2'a_3'\right].$$

$$[f_i]_{i=1}^9 = \left[\left(\frac{c_1a^2+c_1b^2}{2}+\frac{m}{\overline{\gamma}}\right), \left(\frac{c_2a^2+c_2b^2}{2}+\frac{m}{\overline{\gamma}}\right),\right.$$
$$\left(\frac{c_3a^2+c_3b^2}{2}+\frac{m}{\overline{\gamma}}\right), \left(\frac{c_2a^2+c_1b^2}{2}+\frac{m}{\overline{\gamma}}\right),$$
$$\left(\frac{c_3a^2+c_1b^2}{2}+\frac{m}{\overline{\gamma}}\right), \left(\frac{c_1a^2+c_2b^2}{2}+\frac{m}{\overline{\gamma}}\right),$$
$$\left(\frac{c_3a^2+c_2b^2}{2}+\frac{m}{\overline{\gamma}}\right), \left(\frac{c_1a^2+c_3b^2}{2}+\frac{m}{\overline{\gamma}}\right),$$
$$\left.\left(\frac{c_2a^2+c_3b^2}{2}+\frac{m}{\overline{\gamma}}\right)\right].$$

$$[f_i']_{i=1}^9 = \left[\left(\frac{c_1'a^2+c_1b^2}{2}+\frac{m}{\overline{\gamma}}\right), \left(\frac{c_2'a^2+c_2b^2}{2}+\frac{m}{\overline{\gamma}}\right),\right.$$
$$\left(\frac{c_3'a^2+c_3b^2}{2}+\frac{m}{\overline{\gamma}}\right), \left(\frac{c_2'a^2+c_1b^2}{2}+\frac{m}{\overline{\gamma}}\right),$$
$$\left(\frac{c_3'a^2+c_1b^2}{2}+\frac{m}{\overline{\gamma}}\right), \left(\frac{c_1'a^2+c_2b^2}{2}+\frac{m}{\overline{\gamma}}\right),$$
$$\left(\frac{c_3'a^2+c_2b^2}{2}+\frac{m}{\overline{\gamma}}\right), \left(\frac{c_1'a^2+c_3b^2}{2}+\frac{m}{\overline{\gamma}}\right),$$
$$\left.\left(\frac{c_2'a^2+c_3b^2}{2}+\frac{m}{\overline{\gamma}}\right)\right].$$

Table 5. Comparison of accuracy of SEP for various TQAM Constellations over AWGN channel. (a): SEP performance comparison for TQAM-16, (b): SEP performance comparison for TQAM-64, (c): SEP performance comparison for TQAM-256.

$\overline{\gamma}$(dB)	Exact	Proposed N = 3	Sadhwani *et al* N = 3 [14]	Gauss Hermite N = 2 [12]	Prony p = 2 [11]	Karagiannidis [9]	Chiani *et al* [8]
TQAM-16							
0	0.745584	0.758342	0.780670	0.745331	0.781171	0.752521	0.781310
10	0.219288	0.221177	0.215415	0.214478	0.224034	0.218375	0.267771
20	4.99001e−06	4.98943e−06	5.36977e−06	3.00342e−06	5.18162e−06	4.57396e−06	5.49143e−6
TQAM-64							
5	0.857330	0.883207	0.907099	0.857163	0.916753	0.864743	0.917011
15	0.345535	0.342492	0.336900	0.340685	0.360105	0.345483	0.415445
30	4.90664e−13	5.09098e−13	7.67195e−13	1.46692e−13	3.50279e−13	4.40014e−13	7.67289e−13
TQAM-256							
10	0.913234	0.950897	0.974718	0.913126	0.989987	0.921011	0.989667
20	0.455843	0.444776	0.444746	0.451653	0.482351	0.457016	0.540519
30	0.000669	0.000669	0.000662	0.000502	0.000691	0.000625	0.000716

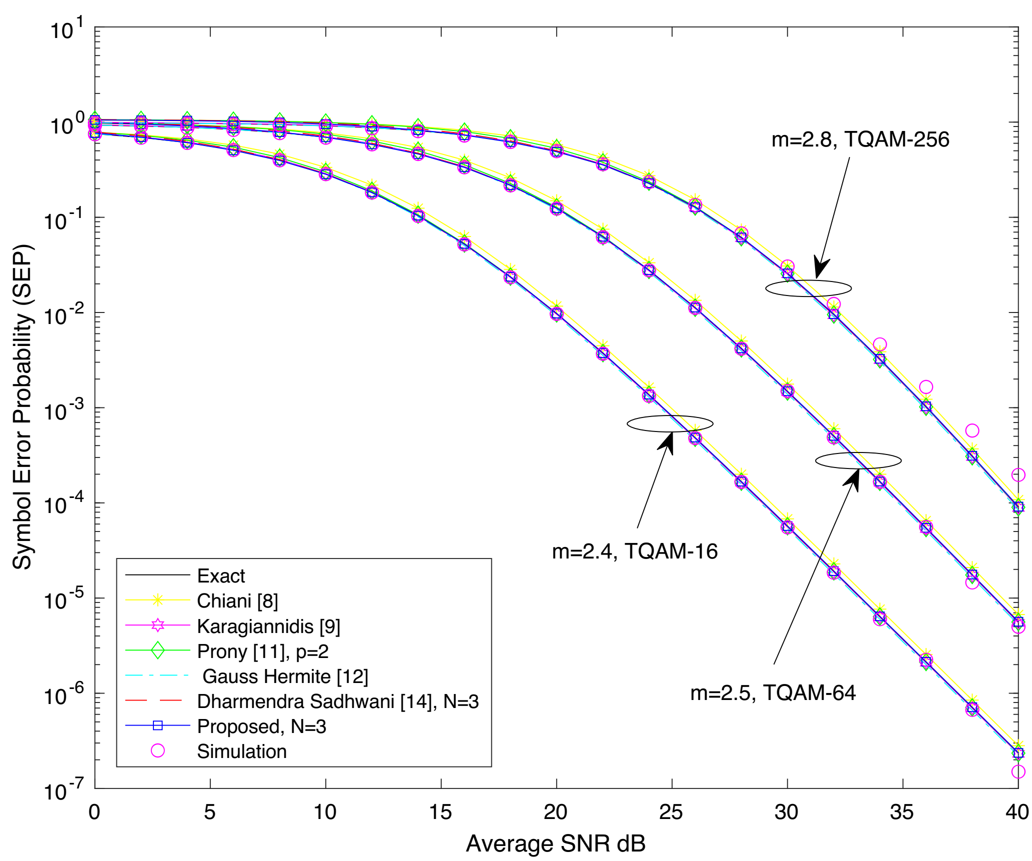

Figure 4. SEP of different TQAM constellations over Nakagami-m fading channel.

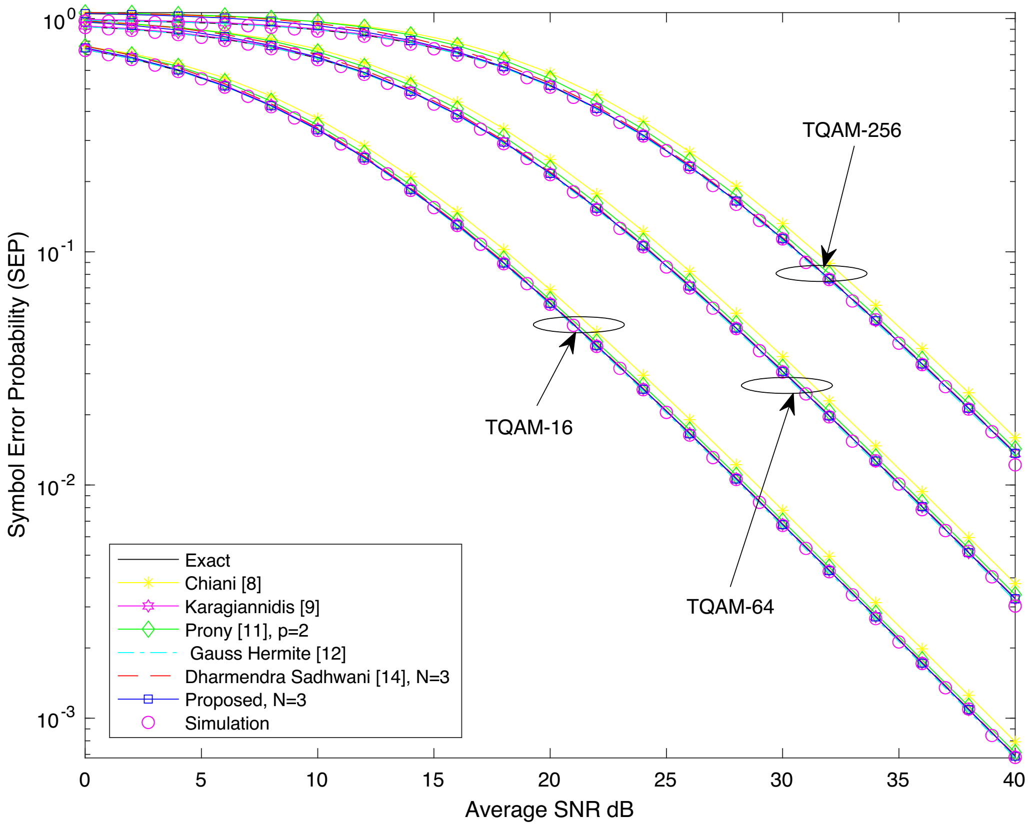

Figure 5. SEP of different TQAM constellations over $\alpha-\kappa-\mu$ fading channel for $\alpha=1$, $\kappa=1$ and $\mu=2$.

$$[f_i'']_{i=1}^{9} = \left[\left(\frac{c_1'a^2+c_1'b^2}{2}+\frac{m}{\overline{\gamma}}\right), \left(\frac{c_2'a^2+c_2'b^2}{2}+\frac{m}{\overline{\gamma}}\right), \left(\frac{c_3'a^2+c_3'b^2}{2}+\frac{m}{\overline{\gamma}}\right), \left(\frac{c_2'a^2+c_1'b^2}{2}+\frac{m}{\overline{\gamma}}\right), \left(\frac{c_3'a^2+c_1'b^2}{2}+\frac{m}{\overline{\gamma}}\right), \left(\frac{c_1'a^2+c_2'b^2}{2}+\frac{m}{\overline{\gamma}}\right), \left(\frac{c_3'a^2+c_2'b^2}{2}+\frac{m}{\overline{\gamma}}\right), \left(\frac{c_1'a^2+c_3'b^2}{2}+\frac{m}{\overline{\gamma}}\right), \left(\frac{c_2'a^2+c_3'b^2}{2}+\frac{m}{\overline{\gamma}}\right)\right].$$

The symbol error probability (SEP) expression of TQAM modulation scheme over fading channel can be calculated using (22). The proposed closed form solutions of integrals $\Im_1$, $\Im_2$ and $\Im_3$ given by (27), (29) and (31) are the simple representations in terms of Meijer G-functions, which can be easily implemented in MATLAB R2019b using in-built Meijer G-function.

Because of their adaptability, generalised fading models such as $\alpha-\mu$, $\kappa-\mu$, $\eta-\mu$,$\alpha-\eta-\kappa-\mu$, $\alpha-\kappa-\mu$, $\alpha-\eta-\mu$ and have been the subject to several investigations in recent years [21–23]. In this work also, we examine the SEP performance of various TQAM modulation schemes taken into consideration over $\alpha-\kappa-\mu$ fading channel. $\alpha-\kappa-\mu$ fading channel is also utilised to generate $\alpha-\mu$, $\kappa-\mu$, Rician, Nakagami-m, Rayleigh, one-sided Gaussian, and Weibull fading channels as special cases [24, 25]. The PDF of $\alpha-\kappa-\mu$ fading channel is represented as [26]

$$p_\gamma(\gamma) = \sum_{j=0}^{\infty} \frac{0.5\alpha\mu^{\mu+2j}\kappa^j(1+\kappa)^{\mu+j}}{\Gamma(\mu+j)j!e^{\kappa\mu}\overline{\gamma}^{0.5\alpha(\mu+j)}}\gamma^{0.5\alpha(\mu+j)-1}e^{-\mu\frac{(1+\kappa)}{\overline{\gamma}^{0.5\alpha}}\gamma^{0.5\alpha}} \tag{32}$$

where $\overline{\gamma}$ denotes the average SNR, γ defines the instantaneous signal-to-noise ratio (SNR), α, κ and μ represent for the nonlinearity of the medium, the ratio between the total power of the dominant components and the total power of the scattered waves, and the total number of the multipath clusters, respectively [26].

In this paper, three special cases of $\alpha-\kappa-\mu$ distribution are used to validate the proposed work. In order to validate the tightness of the proposed approximation, some comparative analysis of SEP expressions over $\alpha-\kappa-\mu$ fading channel and their special cases are demonstrated for three different TQAM constellations. The $\kappa-\mu$ distribution can be achieved from the $\alpha-\kappa-\mu$ distribution by fixing $\alpha=2$. The Nakagami-m distribution can be achieved from the $\kappa-\mu$ distribution by fixing $\kappa=0$ and $\mu=m$. The

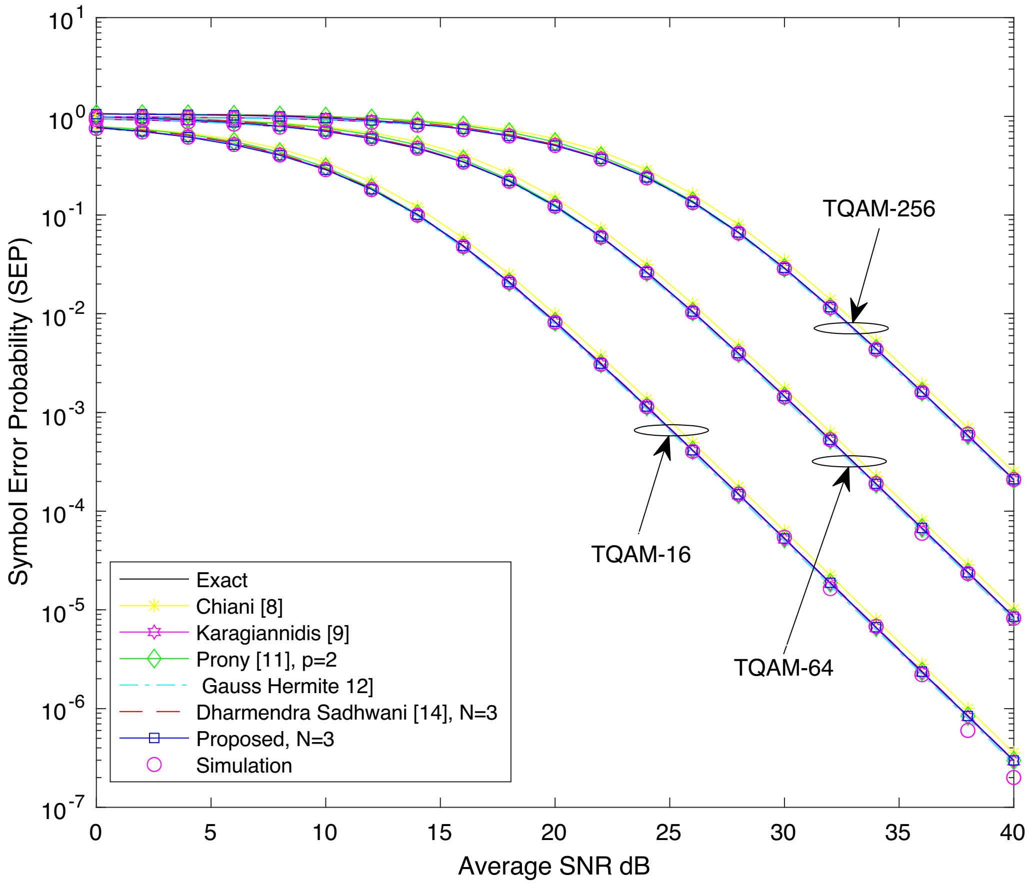

Figure 6. SEP of different TQAM constellations over $\alpha-\mu$ fading channel for $\alpha=3$, $\kappa=0$ and $\mu=1.5$.

$\alpha-\mu$ distribution can be achieved from the $\alpha-\kappa-\mu$ distribution by fixing $\kappa=0$.

4. Simulation results and discussion

The complexity of any Q-function approximation depends upon the number of exponential terms used to represent it. As it can be found that the Q-function approximation proposed in [8, 9, 11, 12] and [14] requires three terms. Our proposed approximation for $\mathrm{N}=3$, also requires three exponential terms. Hence, its complexity is found to be the same but the proposed scheme provides better accuracy than existing schemes [8, 9, 11, 12] and [14].

In [9] Karagiannidis and Lioumpas [9] derived the closed-form solution for SEP performance, it involves 12 different hypergeometric functions, which make it difficult to evaluate and therefore results in increased complexity. Further, the SEP derived using [9] is only valid for $m \geq 1$ due to the presence of $Q^2(.)$. On the other hand, the SEP expression employing [12], necessitates 18 distinct hyper-geometric functions for evaluation of SEP performance. As a result, when it comes to the actual evaluation of SEP, both of these variants of [9] and [12] have limited significance.

To compare the accuracy of the SEP results for TQAM-16, TQAM-64 and TQAM-256 modulation schemes over AWGN channel, the numerical results are presented in table 5.

It can be seen from table 5(a) that the exact values of SEP for TQAM-16 at 10 dB and 20 dB SNR are 0.219288 and 4.99001×10^{-6}, respectively and their corresponding values using proposed approximation are 0.221177 and 4.98943×10^{-6}, however their corresponding values of SEP using existing approximations [8, 9, 11, 12] and [14] are found to be inaccurate in comparison to the SEP obtained using the proposed approximation. As a result, we can say that the proposed approximation is more accurate than existing methods over the entire range of SNR.

Similar observations can also be drawn from tables 5(b)-(c), which present the SEP performance comparison of TQAM-64 and TQAM-256 over AWGN channel using proposed and existing schemes.

We have also plotted the analytical results and performed the Monte Carlo simulations to evaluate the SEP performance of our proposed Gaussian Q-function approximation

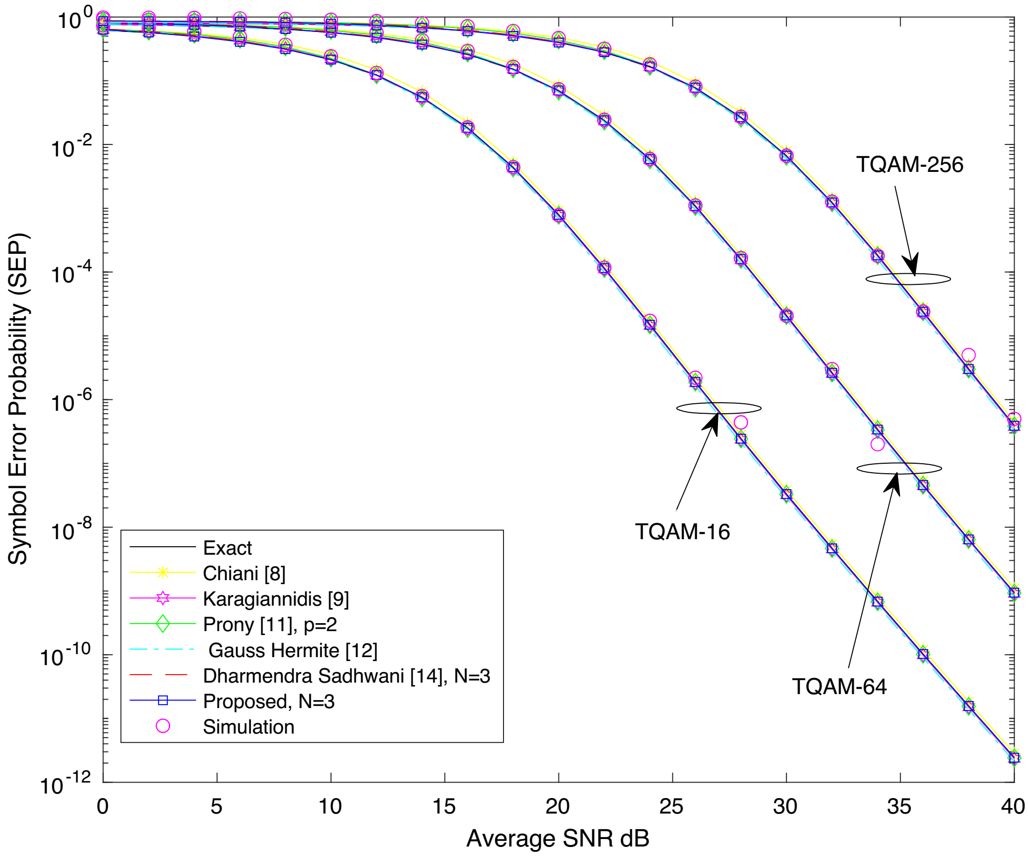

Figure 7. SEP of different TQAM constellations over $\kappa - \mu$ fading channel for $\alpha = 2$, $\kappa = 2$ and $\mu = 4$.

for different modulation schemes over Nakagami-m, $\alpha - \kappa - \mu$, $\alpha - \mu$ and $\kappa - \mu$ fading channels. To check the accuracy of the proposed approximation, we have evaluated the SEP of various TQAM constellations over AWGN, Nakagami-m, $\alpha - \kappa - \mu$, $\alpha - \mu$ and $\kappa - \mu$ fading channels using (19) and (30) and compared their results with the standard SEP calculations using exact expression for Q-function and existing approximations [8, 9, 11, 12] and [14].

Figure 4 demonstrates the SEP performance of TQAM-16, TQAM-64 and TQAM-256 constellations over Nakagami-m channel for $m = 2.4$, 2.5 and 2.8 respectively using proposed and existing approximations of Q-function.

The SEP plots using the proposed approximation over Nakagami-m fading channel for $m = 2.4$, 2.5 and 2.8 are almost coinciding with their corresponding plots of exact SEP calculation as well as with the simulation results. It can be easily noticed that the corresponding SEP results using the approximation proposed by Sadhwani *et al* [14] show deviation from the exact plots. It can be also seen from figure 4 that SEP results using the approximation proposed by Loskot and Beaulieu [11] shows slight improvement for higher arguments but this gain vanishes at low SNR. Though, Q-function approximation expression proposed by Chiani *et al* [8] is simple, its accuracy vanishes at lower as well as higher arguments of the Q-function. Hence, it confirms that the proposed piece-wise approximation provides better tightness in comparison to [8, 9, 11, 12] and [14].

Figure 5 illustrates the accuracy of the proposed approximation (16) which evaluate the SEP performance (21) of various TQAM constellations over $\alpha - \kappa - \mu$ fading channel (32). The accuracy of the proposed work is also validated through Monte Carlo Simulation. SEP performance of different TQAM constellations over $\alpha - \kappa - \mu$ fading channel for $\alpha = 1$, $\kappa = 1$ and $\mu = 2$ using the proposed approximation is also found to be nearly identical with their respective SEP performance using exact *Q*-function. However, the approximation proposed by Karagiannidis and Lioumpas [9] slightly improves the SEP performance only at lower arguments but at higher values it has no gain as compared to the proposed approximation.

Similar work has also been done [27] which is indicated in figure 2.

Figure 6 depicts the exact SEP plots for different TQAM modulation schemes over $\alpha - \mu$ fading channel for $\alpha = 3$, $\mu = 1.5$ and its comparison with the corresponding results using existing approximations. Figure 6 clearly shows that the SEP performance using the proposed approximation is closer to the exact performance than that achieved by the existing approximations [8, 9] and [12]. Despite the fact that approximation proposed by Shi and Karasawa [12] shows better result at very low SNR, while its results at high SNR show significant deviation from the exact results because relative error of this approximation increases monotonically for $x \geq 0.91$. On the other hand, the approximation function proposed by Karagiannidis and Lioumpas [9] is only valid for larger values of m. The validation of all of these results is performed using Monte Carlo simulations.

Figure 7 depicts the SEP plots for TQAM-16, TQAM-64 and TQAM-256 constellations over $k - \mu$ fading channel for $k = 2$, $\mu = 4$ using the proposed approximation and its SER performance comparison with existing methods. It is observed that the SEP results using proposed approximation exactly matches with the simulated results and thus confirming the validity and accuracy of the proposed approximation.

When comparing figures 5, 6 and 7, it can be noticed that the SEP performance of TQAM modulation schemes initially improves with increase in the fading parameter α. However, SEP performance is dominated by fading parameter μ, since We can notice from figures 5, 6 and 7 that the SEP performance is improved as we are increasing the fading parameter μ. The exact SEP performance and the corresponding results using the proposed approximation and the Monte-Carlo simulations are also virtually indistinguishable, as shown in figures 5, 6 and 7.

5. Conclusion

This paper presents a tractable and more accurate approximation for Gaussian Q-function, which is a sum of three exponential functions defined in a piece-wise manner. The proposed approximation employs a novel concept of unequal intervals of integration function used in trapezoidal rule, which aid in minimizing relative error, the optimal value of these intervals is obtained by minimizing the mean square relative error.

The accuracy and validity of the proposed approximation are demonstrated through the plot of $erfc(x)$ function and its relative error performance. The tightness of the proposed approximation has been proved by showing the SEP calculation of various modulation schemes such as TQAM-16, TQAM-64 and TQAM-256 over AWGN, Nakagami-m, $\alpha - \kappa - \mu$, $\alpha - \mu$ and $\kappa - \mu$ fading channels. The closed-form solution of integrals $\Im_1$, $\Im_2$ and $\Im_3$ are valid for the entire range of m, unlike [9]. Finally, we can conclude that the proposed piecewise approximation is not only a simple function but also more accurate than other existing Q-function approximations, and it is applicable to evaluate the closed form solution of SEP calculation of all types of modulation techniques over fading channels.

6. Future Work

The closed form expression for SEP over $\alpha - \kappa - \mu$ fading channel can also be derived.

References

[1] Simon M K and Alouini M S 2005 Digital Communication Over Fading Channels, 2nd edn, Chapter 4. Wiley, Hoboken
[2] Beaulieu N C 1989 A simple series for personal computer computation of the error function Q(.). *IEEE Trans. Commun.* 37: 989–991
[3] Bao V N Q, Tuyen L P and Tue H H 2015 A survey on approximations of one-dimensional Gaussian Q-function. *REV J. Electron. Commun.* 5: 1–14
[4] Salahat E and Abualhaol I 2013 General BER analysis over Nakagami-m fading channels. In: *Proceedings of 6th Joint IFIP Wireless and Mobile Networking Conference (WMNC),* pp. 1–4
[5] Aggarwal S 2019 A survey-cum-tutorial on approximations to Gaussian Q function for symbol error probability analysis over Nakagami-m fading channels. *IEEE Commun. Surv. Tutor.* 21: 2195–2223
[6] Borjesson P and Sundberg C-E 1979 Simple approximations of the error function Q(x) for communications applications. *IEEE Trans. Commun.* 27: 639–643
[7] Craig J W 1991 A new, simple and exact result for calculating the probability of error for two-dimensional signal constellations. In: *Proceedings of MILCOM 91 - Conference record* 2, pp. 571–575
[8] Chiani M, Dardari D and Simon M K 2003 New exponential boundsand approximations for the computation of error probability in fading channels. *IEEE Trans. Wirel. Commun.* 2: 840–845
[9] Karagiannidis G K and Lioumpas A S 2007 An improved approximation for the Gaussian Q-function. *IEEE Commun. Lett.* 11: 644–646
[10] Isukapalli Y and Rao B D 2008 An analytically tractable approximation for the Gaussian Q-function. *IEEE Commun. Lett.* 12: 669–671
[11] Loskot P and Beaulieu N C 2009 Prony and polynomial approximations for evaluation of the average probability of error over slow-fading channels. *IEEE Trans. Veh. Technol.* 58: 1269–1280
[12] Shi Q and Karasawa Y 2011 An accurate and efficient approximation to the Gaussian Q-function and its applications in performance analysis in Nakagami-*m* fading. *IEEE Commun. Lett.* 15: 479–481
[13] Olabiyi O and Annamalai A 2012 Invertible exponential-type approximations for the Gaussian probability integral Q(x) with applications. *IEEE Wirel. Commun. Lett.* 1: 544–547

[14] Sadhwani D, Yadav R N and Aggarwal S 2017 Tighter bounds on the Gaussian Q function and its application in Nakagami-m fading channel. *IEEE Wirel. Commun. Lett.* 6: 574–577

[15] Mamun-Ur-Rashid Khan Md, Hossain M R and Parvin Selina 2017 Numerical integration schemes for unequal data spacing. *Am. J. Appl. Math.* 5: 48–56

[16] Rugini L 2016 Symbol error probability of hexagonal QAM. *IEEE Commun. Lett.* 20: 1523–1526

[17] WOLFARM. Wolfram function site. Available at: https://functions.wolfram.com/introductions/PDF/Gamma.pdf

[18] Gradshteyn I S and Ryzhik I M 1980 Table of Integrals, Series, and Products. 7th edn. Academic Press, New York

[19] Prudnikov A P, Brychkov Y A and Marichev O I 1986 Integrals and Series: More special functions, 3. Oxford University Press, New York

[20] WOLFARM. Wolfram function site. Available at: https://functions.wolfram.com/PDF/MeijerG.pdf

[21] Sadhwani D, Yadav R N, Aggarwal S and Raghuvanshi D K 2018 Simple and accurate SEP approximation of hexagonal-QAM in AWGN channel and its application in parametric $\alpha - \mu$, $\eta - \mu$, $\kappa - \mu$ fading, and log-normal shadowing. *IET Commun.* 12: 1454–1459

[22] Proakis J G and Saleh M 2008 *Digital Communications*. 5th edn. McGraw-Hill, New York

[23] Hosur S, Mansour M F and Roh J C 2013 Hexagonal constellations for small cell communication. In: *Proceedings of IEEE Global Communications Conference (GLOBECOM),* pp. 3270–3275

[24] Park S 2012 Performance analysis of triangular quadrature amplitude modulation in AWGN channel. *IEEE Commun. Lett.* 16: 765–768

[25] Mallik R K 2008 Average of product of two Gaussian Q-functions and its application to performance analysis in Nakagami fading. *IEEE Trans. Commun.* 56: 1289–1299

[26] Moualeu J M, da Costa D B, Hamouda W, Dias U S and de Souza R A A 2019 Performance analysis of digital communication systems over fading channels. *IEEE Commun. Lett.* 23: 192–195

[27] Salahat E and Hakam A 2014 Performance analysis of α-η-μ and α-κ-μ generalized mobile fading channels. In: *Proceedings of 20th European Wireless Conference*, pp. 1–6