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Inverse analysis for parameter estimation of sandy soil with axially loaded pile using nonlinear programming

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Abstract. A computerized mathematical procedure based on nonlinear programming is presented for the purpose of a sandy soil parameter prediction problem by inverse analysis. The task of inverse analysis is to evaluate the values of input parameters for a specified output or response of the system. This inverse problem to determine the values of system input parameters is mathematically formulated as a constrained optimization problem in this study. The input parameters are considered as design variables. The constraint set is developed through the implementation of lower and upper bound on design variables. The objective function is determined by evaluating the error or mismatch between the specified output (reference value) and the model predicted value for a given design vector. The resulting nonlinear programming problem is solved using the Interior Penalty Function method coupled with the Davidon-Fletcher-Powell method and quadratic interpolation scheme. To demonstrate the proposed methodology, two illustrative examples related to axially loaded piles installed in sandy soil medium are considered. The numerical results indicate that the back analyzed values are in good agreement with their respective reference values.

Keywords. Parameter estimation; inverse analysis; nonlinear programming; reference value; model predicted value; pile system.

1. Introduction

In recent years with dramatic developments in computational mechanics and computational tools, a new direction has been found in the modeling of geotechnical engineering systems. For doing this it needs to have a very good knowledge of soil behavior and soil mechanics. Practically due to several reasons like nature and variability of soils, unknown boundary conditions, the mathematical model may not truly represent the actual problem. After all, the experimental data obtained from real and imperfect instruments may not match with the assumed model developed for identified values of system parameters. In order to mitigate the problem of imperfections in the assumed model and the measured data, the problem of system identification may be represented as the evaluation of parameters (or parameter estimation). This would make the mathematical model the best representation of the system.

In parameter estimation, physical measurements are used in conjunction with theoretical equations of the mathematical model representing the physical system to evaluate empirical constants. In a forward analysis, the response of an engineering system is determined from the known design parameters. But in parameter estimation, in contrast to the above, from the known response of the system, the objective is to predict the design parameters. Thus, parameter estimation can be visualized as inverse analysis or back analysis.

Generally, the inverse problems are ill-posed i.e., they may not have any solution, the solutions may or may not be unique and the solutions, even when they exist, may not continuously depend on the data. As a result, research need is felt in developing a procedure to determine meaningful solutions by using some optimization principles like minimum error, maximum likelihood, etc. However, the parameters are sequentially adjusted following definite optimization schemes based on mathematical theories till the required match between the predicted and observed response should be as best as possible.

In the geotechnical engineering field, inverse analysis procedures have been employed extensively for consolidation settlement, failed slopes to estimate the relevant soil parameters. Attempts were also made in tunneling works and other areas. The potential of the back analysis technique is increasingly been realized and appreciated by the engineering community. Some of the studies reported in the field of geotechnical engineering are reviewed here e.g. a

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master equation of settlement-time relationship is used under one-dimensional consolidation of both constant and increasing external consolidation pressure [1]. The coefficients of an autoregressive model are statistically identified from settlement observations at the early stage of construction and these identified parameters are used later for backward calculation of prominent eigenvalues of consolidation. Fenelli and Galateri [2] have used the results of several horizontal load tests on piles to back analyze the soil modulus distribution with depth and stress level. A numerical technique is adopted for estimating Young's modulus and Poisson's ratio of soils from in situ measurements of displacements using the Fletcher and Reeves method of optimization [3]. Gioda and Sakurai [4] have surveyed recent developments in numerical techniques for back analysis in the geomechanics field. A constitutive model based on the disturbed state concept is used to evaluate the response of interfaces in dynamic soil-structure interaction problems [5]. The model predictions are compared with the test data used for the determination of parameters and the independent test not used for finding the parameters. Yamagami *et al* [6] have presented a nonlinear programming technique for slope stability analysis using the limiting equilibrium method. Another method is developed to estimate the slope failure due to rainfall based on the back analysis of 24 case records of failure [7]. Honjo *et al* [8] have used the maximum likelihood method in developing an inverse analysis procedure for a single pile settlement model based on linear elastic theory. A global solution is obtained using unconstrained minimization by analyzing a total of thirteen pile loading test results from two projects in the Bangkok area. Yi *et al* [9] have adopted an inverse model to identify the dynamic response from the substructure under seismic excitation considering pile-soil interaction. Validation is done by an idealized frame structure separately in this model by equivalent linearization method and considering rigid foundation. Adachi and Kojima [10] have presented a numerical procedure for inverse analysis to obtain more reliable design parameters for the design of earth tunnels. The responses of transparent granular soil due to penetration of different types of piles are studied and compared [11]. Gazetas [12] has back analyzed the shear modulus of rockfill by referring to the data from full-scale lateral load tests on piles.

A general formulation is developed based on Maximum Likelihood optimization criteria to back analyze a test tank and an excavation problem [13]. Young's modulus of soil for the former case and Young's modulus and K_0 (the ratio between the horizontal and vertical in situ stresses) for the latter case are considered in the back analysis. Shoji *et al* [14] have proposed a procedure to predict the ultimate bearing capacity during the construction of an embankment. Bhattacharya [15] has analyzed the problem of slope stability using the nonlinear programming method. Bergado *et al* [16] study the deformations, strength, and flow parameters by inverse analysis where geotechnical data are obtained from two full-scale test embankments constructed on improved soft Bangkok clay. Attempts were also made to report the stability of slopes formed by the excavation of the soft Bangkok clay [17]. Venclik [18] has developed an inverse back analysis code based on the algorithm suggested by Gioda [19, 20]. Warrington [21] has back-analyzed the soil properties through an inverse analysis by minimizing the difference between the computed and actual displacement-time data for the pile-soil interaction problem. Honjo *et al* [22] have inversely analyzed the seismic records taken from a seismometer array located in downtown Tokyo Japan for about ten years for estimation of dynamic soil parameters. Raju [23] has evaluated soil parameters, namely Young's modulus and Poisson's ratio of soil through an inverse analysis from pile load test results using Powell's method of optimization. Dey *et al* [24] have used an inverse formulation to estimate the Burger model parameters and results reveal that the developed technique is quite efficient in predicting the model parameters.

A review of literature pertaining to the back analysis of axially loaded piles reveals that the work done in this field is limited. In this study, the sandy soil parameters are estimated from the response of an axially loaded pile structure using inverse analysis based on an optimization algorithm. For this purpose, the sandy soil parameter estimation problem is first mathematically formulated as a constrained optimization problem and then a methodology based on an interior penalty function approach is proposed to evaluate input parameters.

2. Description of the problem

The ultimate load-carrying capacity of an axially loaded pile equals the sum of the loads taken by pile shaft and pile base respectively. The ultimate load (equal to the product of working load & factor of safety) depends on the working load as well as the factor of safety. In the present study, the drilled shaft is considered. A simple schematic diagram of an axially loaded pile system is shown in figure 1.

The ultimate load (Q_{ult}) is balanced by the loads carried by pile shaft ($Q_{s,ult}$) and pile base ($Q_{b,ult}$) respectively. It may be expressed as

$$Q_{ult} = Q_{b,ult} + Q_{s,ult} \tag{1}$$

In general ultimate load causes 10% relative settlement of pile head (i.e. ratio of pile head settlement and pile diameter = 0.1) [25] and shaft resistance gets its limit capacity during this settlement [26]. Limit shaft resistance $Q_{s,ult}$ is written as

$$Q_{s,ult} = \sum_{i=1}^{n} q_{sLi} A_{si} \tag{2}$$

where, q_{sLi} represents limit unit shaft resistance due to pile penetration within any soil layer i, A_{si} is pile shaft area

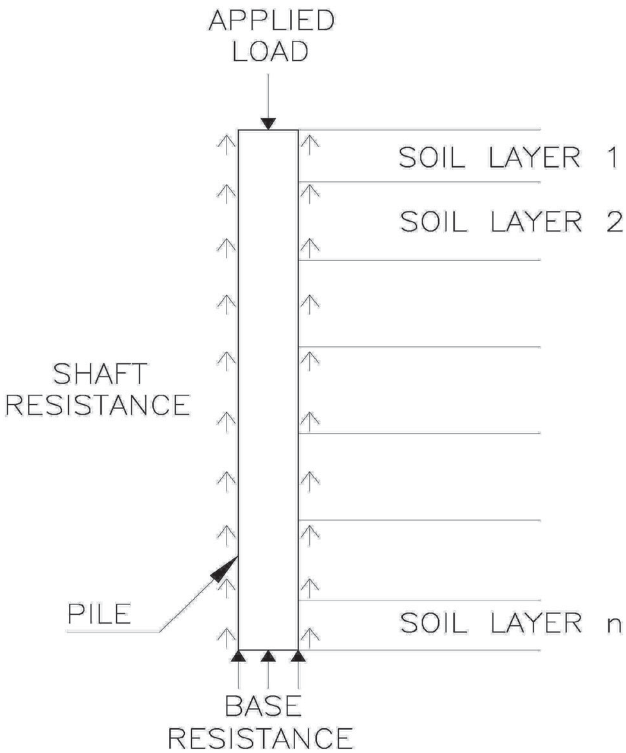

Figure 1. Axially loaded pile structure.

adjacent with i^{th} soil layer, and n is the total number of soil layers adjacent with the pile shaft. The ultimate base resistance $Q_{b,ult}$ is represented by

$$Q_{b,ult} = q_{b,10\%}A_b \tag{3}$$

where, A_b is the cross-sectional area of pile base and $q_{b,10\%}$ indicates ultimate unit base resistance corresponding to 10% relative settlement of the pile head. Finally, the ultimate load-carrying capacity of the pile is given by

$$Q_{ult} = q_{b,10\%}A_b + \sum_{i=1}^{n} q_{sLi}A_{si} \tag{4}$$

The equations used for designing drilled shafts in sandy soil are given below.

2.1 *Equations of limit unit shaft resistance and ultimate unit base resistance for drilled shaft installed in sandy soil*

Limit unit shaft resistance for the drilled shaft in sandy soil is written as [25–27]

$$q_{sL} = K\sigma_v' \tan\phi_c \tag{5}$$

where,

$$K = 0.7K_0 \exp\left[\left\{0.0114 - 0.0022\ln\left(\frac{\sigma_v'}{p_A}\right)\right\}D_R\right] \tag{6}$$

in which σ_v' is effective vertical stress where limit capacity is required, ϕ_c is critical state friction angle of sand, D_R is the relative density of sand denoted as a percentage, K_0 is earth pressure coefficient at rest and p_A is reference stress. Ultimate unit base resistance for the drilled shaft in sandy soil is expressed as [26]

$$q_{b,10\%} = [0.23\exp(-0.0066D_R)]q_{bL} \tag{7}$$

where,

$$\frac{q_{bL}}{p_A} = 1.64\exp[0.1041\phi_c + (0.0264 - 0.0002\phi_c)D_R] \times \left(\frac{\sigma_h'}{p_A}\right)^{(0.841-0.0047D_R)} \tag{8}$$

in which σ_h' is effective horizontal stress at pile base and q_{bL} is limit unit base resistance calculated at the depth of pile base.

Using the above equations one can obtain the foundation response (e.g. average skin resistance, pile diameter, etc.) through forward analysis for given values of sandy soil parameters. Now the task is to estimate the sandy soil parameters using inverse analysis when foundation response is known. In the present study, the inverse analysis is done through an optimization method which is presented in the following section.

3. Optimization method

Optimization is the act of obtaining the best result under given circumstances and satisfying all limitations and restrictions imposed on it. In any optimization problem there exist three major components e.g. the design variables, the constraint set, and the objective functions. In the analysis or design of any engineering system, the ultimate goal is to find out the design variables satisfying the constraint set and minimize the objective functions.

In the present investigation, the interior penalty function (IPF) method is used for the constrained minimization problem. In optimization literature, there are several methods to solve the constrained optimization problem. However, it is found from the literature that the interior penalty function (IPF) method has some special merits to solve the constrained optimization problem. To solve the constrained optimization problem based on the IPF method, the first task is to convert the constrained optimization problem into an equivalent unconstrained optimization problem and the second task is sequential minimization of an unconstrained optimization problem with the Davidon-Fletcher-Powell (DFP) method for the direction of movement and quadratic interpolation method (QIM) for step length along the search direction.

In the IPF method, the objective function is augmented with a penalty term which is small at points away from the constraints in the feasible region but 'blow up' as the boundary is approached. An optimization problem can be stated as:

$$\begin{aligned} &\text{minimize}\, f(\vec{X}) \text{ subject to} \\ &g_j(\vec{X}) \le 0 \; where, \; j = 1, 2, \ldots, m \end{aligned} \tag{9}$$

where, $\vec{X}$ is the vector of design variables. In the IPF method, the most commonly used function as presented below is

$$\phi(X, r_k) = f(X) - r_k \sum_{j=1}^{m} \frac{1}{g_j(X)} \tag{10}$$

The task is now to minimize $\phi(X, r_k)$. Here r_k is a penalty parameter. It should be noted here that if r_k is positive its effect is to add a positive penalty to $f(X)$ and this is because, at an interior point, all the terms in the sum are negative. Using a reduction factor C, the penalty parameter r_k is made successively smaller to evaluate the unconstrained minimum of $\phi(X, r_k)$. Thus

$$r_{k+1} = C.r_k \tag{11}$$

During the optimization process, starting from an initial point first task is to find out the search direction and the second is to determine optimum step length along the direction of search. To obtain search direction, Davidon-Fletcher-Powell (DFP) method is used in the present study. In this method, starting from an initial guess point in the function space under consideration there is a preferred direction in which the values are changed systematically by a well-defined iterative procedure. To find out the one-dimensional step length along the search direction, the quadratic interpolation method (QIM) is used. The algorithm IPF + DFP + QIM is described in Appendix-A.

4. Formulation of the problem in optimization format

The problem described in section 2 can be mathematically represented in an optimization format. During forward analysis foundation responses are evaluated for specified values (reference value) of sandy soil parameters. In backward analysis, the response is known, the objective is to estimate the sandy soil parameters (reference value).

The sum of absolute difference (error) of responses at the reference point and its predicted value at the design point can be treated as an objective function ($f(X)$) in the problem i.e. $f(X)$ equal to the sum of the absolute value of error in responses (reference values-model predicted values). The model predicted values of response are generated by the optimization procedure for any given design vector. The model predicted values of response are nonlinear with design variables in nature. Therefore, the nature of the objective function is also nonlinear. The design variables (X) of the problem are considered as the sandy soil parameters and represented as $X = \{x_1, x_2, x_3, \ldots, x_n\}^T$ where, x_j = j^{th} design variable. The constraint set of the system is formed by implementing a lower and an upper bound on each of the design variables (x_j). These constraints can be mathematically represented as $x_{jl} \le x_j \le x_{ju}$ for all j = 1,2,3,...,n, where, 'l' and 'u' denote the lower and upper bound values respectively. Finally, the optimization problem is stated as follows:

Find X which minimizes $f(X)$ subject to the constraints

$$g_j(X) \le 0 \; where, j = 1, 2, \ldots m \tag{12}$$

The computational steps to evaluate the optimum solution are described in the next section.

5. Computational procedure

Computational steps involved in the optimization procedure are described as follows:

Step 1: Select an initial design point ($X = X_i$) within the feasible design space. At $X = X_i$, determine the objective function ($f(X)$) and penalty parameter (r_k) such that two terms on the right-hand side of equation (10) are almost equal at starting point.

Step 2: ϕ function is constructed at the initial design point.

Step 3: Using the DFP method, the search direction (S_i) is determined from the starting point ($X = X_i$).

Step 4: Optimum step length (λ_i^*) along the direction of search (S_i) is evaluated in this step by QIM.

Step 5: New design point is obtained as

$$X_{i+1} = X_i + \lambda_i^* S_i$$

The objective function ($f(X)$) and ϕ function are evaluated at this new point.

Step 6: Repeat step 3 to step 5 until the relative change in objective function value and or the change in all the design variables between two consecutive cycles are less than the tolerance limit, i.e.

$$\left| \frac{f(X_{i+1}) - f(X_i)}{f(X_i)} \right| \le \varepsilon_1 \tag{13}$$

$$|X_{i+1} - X_i| \le \varepsilon_2 \tag{14}$$

Step 7: When equations (13) and (14) are satisfied, consider the latest obtained point in step 5 as a new starting point and use a reduction factor C, the penalty parameter (r_k) is made successively smaller such that

$$r_{k+1} = C.r_k \tag{15}$$

The objective function ($f(X)$) and ϕ function are determined at the new starting point.

Step 8: Repeat step 3 to step 7. The procedure is continued and stops when ϕ and $f(X)$ are almost equal and two consecutive values of $f(X)$ are also almost equal. It should be mentioned here that since $f(X)$ represents an error, the optimized value $f(X)$ should be very small and within tolerance or acceptable limit. The latest design point thus obtained is the optimum solution of the original constrained problem representing the values of sandy soil parameters.

6. Illustrative example

The proposed methodology is demonstrated through two numerical examples that have been presented in this section.

The drilled shaft is considered as an axially loaded pile in the present investigation is designed based on the working stress method. In the first example single layer sandy soil medium (type 2 soil) is considered and two-layered sandy soil deposits (type 1 and type 2 soil) are taken in the second example. One of the important considerations in selecting the soil profiles is that the construction of piles should be technically feasible. In this study, the water table is taken at the ground surface. Numerical data that are considered in the present study are stated below:

6.1 *Input data*

i. The relative density of type 1 sandy soil (at reference point) is 50%
ii. Critical state friction angle of type 1 sandy soil (at reference point) is 33°
iii. The relative density of type 2 sandy soil (at reference point) is 55%
iv. Critical state friction angle of type 2 sandy soil (at reference point) is 35°
v. Coefficient of earth pressure at rest = 0.4
vi. Bulk unit weight of type 1 sandy soil = 16.93 kN/m^3
vii. Bulk unit weight of type 2 sandy soil = 19.93 kN/m^3
viii. Unit weight of water is taken as $\gamma_w = 9.81$ kN/m^3
ix. Reference stress p_A = 100 kPa
x. Working load is taken as 2000 kN
xi. The factor of safety (F.O.S.) = 2.5
xii. The length of the pile is 12 m

Data related to optimization

xiii. Reduction factor (C) = 0.1

6.2 *Example 1*

The axially loaded pile is installed in single layer sandy soil (type 2 soil) medium.

6.2a *Results of shaft resistance and base resistance*: Considering the reference values, a forward analysis is carried out to evaluate the system response. The design calculations (as discussed in section 2.1) of shaft resistance and base resistance for the drilled shaft in sandy soil medium are presented in table 1.

Using equation (4) and results of table 1 one can determine the pile diameter and skin resistance of pile (foundation response) for specified values (reference value) of relative density and critical state friction angle of sand. The foundation response results are presented in the following section.

6.2b *Results of foundation response*: Through a forward analysis obtained foundation response results are shown in table 2.

For this given axially loaded pile system, assume pile diameter and skin resistance are known. The problem is to back predict the proper value of the soil parameters (relative density and critical state friction angle of sand) from these known quantities.

6.2c *Optimization problem related to the inverse analysis of parameter estimation*: The soil parameters are treated as design variables (X) in the problem and it can be expressed as

$$X = \{x_1, x_2\}^T \tag{16}$$

Where, x_1 and x_2 represent the relative density and critical state friction angle of sand respectively.

The objective function is chosen as the sum of the absolute difference of responses at the reference point and design point. Thus, the objective function is written as

$$f(X) = \alpha_1 f_1(X) + \alpha_2 f_2(X) \tag{17}$$

Table 1. Design calculations for the drilled shaft in sand.

Limit Unit Shaft Resistance q_{sL} (kPa)	Limit Unit Base Resistance q_{bL} (kPa)	Ultimate Unit Base Resistance $q_{b,10\%}$ (kPa)
23.67	11965.44	1914.28

Table 2. Evaluated foundation response for reference values of soil parameters.

Input (Soil parameter)	Output (Foundation response)
Relative density of sand = 55%	Pile diameter = 1.55 m
Critical state friction angle of sand = 35°	Skin resistance of pile = 1383.996 kN

where, $f_1(X)$ represents the absolute value of error between the reference value of the pile diameter (d_{ref}) and model-predicted value (d_{pre}) at any design vector X.

Similarly $f_2(X)$ indicates the absolute value of mismatch between the reference value of skin resistance (SR_{ref}) and its model predicted value (SR_{pre}) at any design vector X. α_1 and α_2 are two scale factors that are chosen in such a way that two terms in equation (17) are of the same order at the starting design vector. In this case $\alpha_1 = 1000$ and $\alpha_2 = 1$.

The constraint set may be developed by implementing a lower and an upper bound on design variables and can be represented as

$$\begin{aligned} 35 \le x_1 \le 75 \\ 30 \le x_2 \le 40 \end{aligned} \tag{18}$$

The task is now to find X that minimize $f(X)$ subject to constraints given in equation (18).

6.2d *Optimization results using back analysis*: Using the computational procedure presented in section 5, the numerical results are obtained with two different initial points in the feasible region. These are described below.

Case 1: Initial design vector (X_i) is taken as $X_i = \{40 \quad 32\}^T$

Starting value of the penalty parameter in IPF (r_0) = 600. The results obtained for Case 1 are given in table 3.

From the results of table 3, it can be observed that the back analyzed values matches very closely with their respective reference values. The predicted pile diameter and skin resistance of the pile are almost identical with their respective reference value. To appreciate the success of the adopted optimization scheme, the iteration history for x_1 (figure 2) and x_2 (figure 3) are drawn.

Even though the initial points are away from the actual (reference) values, both the design variables converge to their respective actual values.

Case 2: Initial design vector (X_i) is taken as $X_i = \{70 \quad 38\}^T$

Starting value of the penalty parameter in IPF (r_0) = 500. The results evaluated for Case 2 are presented in table 4.

From the results of table 4, it can be seen that the back analyzed values excellently agree with their actual values. The effectiveness of the proposed optimization scheme is studied and presented in figures 4 and 5.

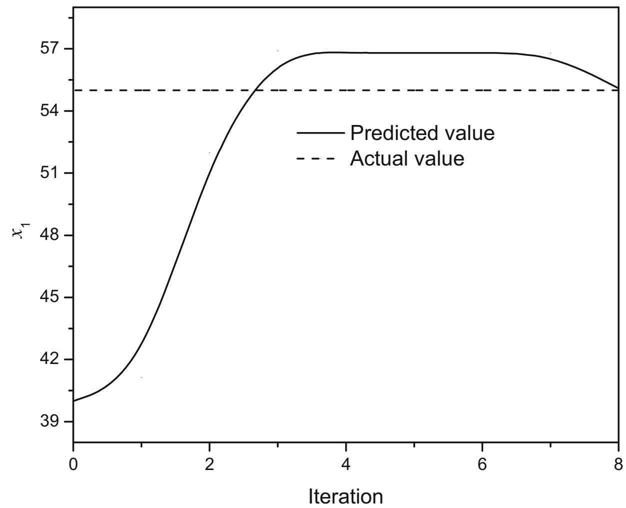

Figure 2. Iteration history for x_1.

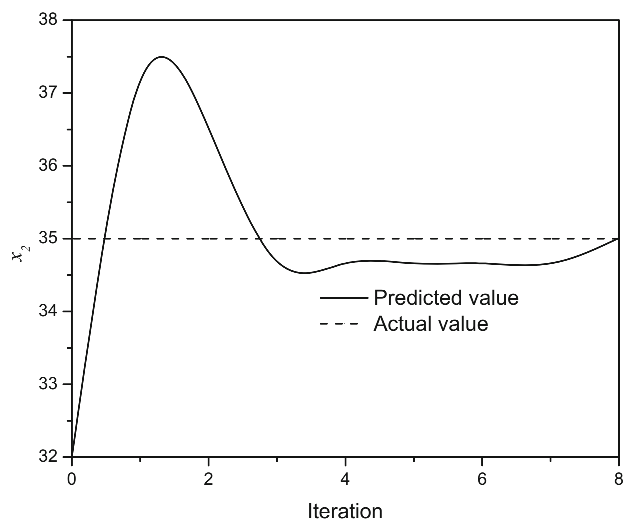

Figure 3. Iteration history for x_2.

Like the previous case, both design variables have well converged to their respective actual values.

6.2e *Global optimum solution*: The objective function considered in the presented study is associated with two

Table 3. Back analysis results for Case 1.

Parameter/Function	Initial value	Back analyzed value	Actual (reference) value
x_1	40.0	55.088	55.0
x_2	32.0	35.006	35.0
Pile diameter (m)	–	1.549	1.55
Skin resistance (kN)	–	1384.375	1383.996
$f(X)$	525.037	1.209	–

Percentage error in x_1 = 0.16%; Percentage error in x_2 = 0.01%

terms where the first one is related to pile diameter and the second with skin resistance. Instead of these two terms, if only one is taken, there may exist a set of solutions instead of a unique solution i.e. depending upon the starting point in feasible region, several local optimum solutions may be found.

To overcome this phenomenon, the objective function is generated with two terms. So whatever the starting point in feasible space, the final (optimum) solution hits almost the same point (x_1 = 55.0; x_2 = 35.0). This indicates that irrespective of the starting point, the results converge to the same optimum point.

In the present study, the objective function represents the sum of the absolute value of error between the reference value and model-predicted value, of response parameters. The minimum value of the objective function $f(X)$ is zero and this would result $f_1(X) = 0$ and $f_2(X) = 0$. In other words, mismatch or error between reference and model value for both response parameters have become zero. At the optimum design point, where the objective function $f(X)$ is zero (or close to zero) it represents the global optimum point because two individual error terms have become zero.

6.3 *Example 2*

Axially loaded pile driven in sandy soil deposits composed of two strata. It is assumed that the type 1 soil layer exists up to 6 m from the ground surface and the next 6 m is of type 2 soil.

6.3a *Results of shaft resistance and base resistance*: The design calculations (as mentioned in section 2.1) of shaft resistance and base resistance for the drilled shaft in sandy soil deposits are described in table 5.

From the results of table 5 and using equation (4) one can evaluate the pile diameter and skin resistances of the pile (foundation response) for specified values (reference values) of relative density and critical state friction angle of sandy soil deposits with two layers. The foundation response results are given in the following section.

6.3b *Results of foundation response*: Using forward analysis obtained foundation response results are given in table 6.

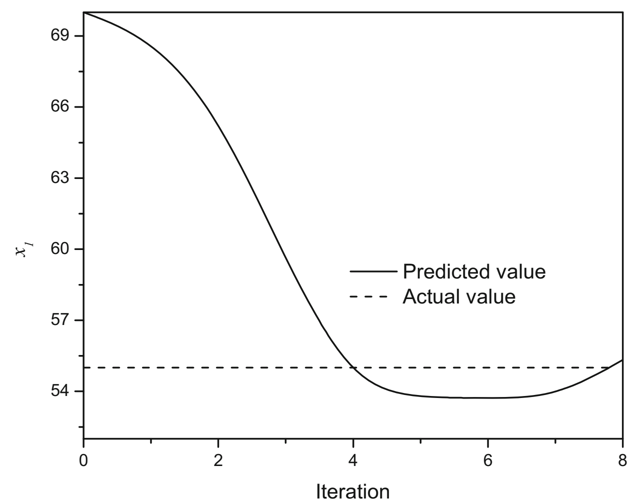

Figure 4. Iteration history for x_1.

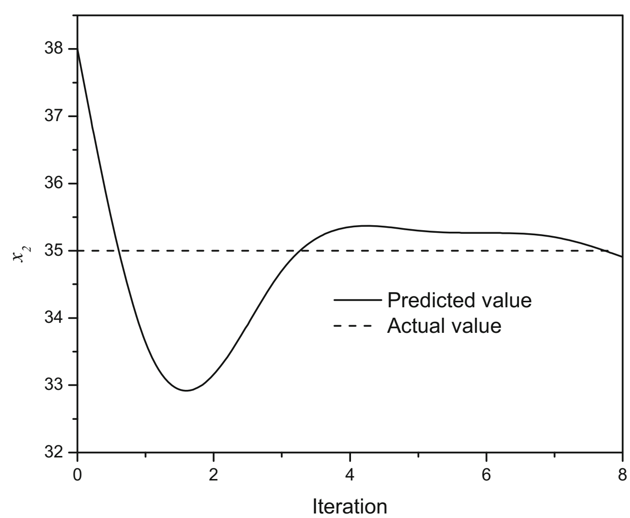

Figure 5. Iteration history for x_2.

Assume pile diameter and skin resistances are known for a given axially loaded pile system. The problem is now to back predict the proper values of the soil parameters

Table 4. Back analysis results for Case 2.

Parameter/Function	Initial value	Back analyzed value	Actual (reference) value
x_1	70.0	55.332	55.0
x_2	38.0	34.908	35.0
Pile diameter (m)	–	1.552	1.55
Skin resistance (kN)	–	1386.693	1383.996
$f(X)$	405.659	5.385	–

Percentage error in x_1 = 0.6%; Percentage error in x_2 = 0.26%

Table 5. Design calculations for the drilled shaft in sandy soil deposits

Limit Unit Shaft Resistance in type 1 soil q_{sL1} (kPa)	Limit Unit Shaft Resistance in type 2 soil q_{sL2} (kPa)	Limit Unit Base Resistance q_{bL} (kPa)	Ultimate Unit Base Resistance $q_{b,10\%}$ (kPa)
8.13	27.78	10823.66	1731.62

(relative density and critical state friction angle of sandy soil deposits with two layers) from these known quantities.

6.3c *Optimization problem related to the inverse analysis of parameter estimation*: The soil parameters are taken as design variables (X) in the problem and it can be expressed as

$$X = \{x_1, x_2, x_3, x_4\}^T \tag{19}$$

where, x_1, x_2 represent the relative density and critical state friction angle of type 1 sandy soil respectively, and x_3, x_4 represent the same for type 2 sandy soil respectively.

The objective function is defined as the sum of absolute differences of responses at the reference point and design point. Thus, the objective function is written as

$$f(X) = \alpha_1 f_1(X) + \alpha_2 f_2(X) + \alpha_3 f_3(X) \tag{20}$$

where, $f_1(X)$ represents the absolute value of error between the reference value of the pile diameter (d_{ref}) and model-predicted value (d_{pre}) at any design vector X and $f_2(X)$ indicates the absolute value of mismatch between reference value of skin resistance due to type 1 sandy soil ($SR1_{ref}$) and its model predicted value ($SR1_{pre}$) at any design vector X. Similarly,$f_3(X)$ represents the absolute value of mismatch between reference value of skin resistance due to type 2 sandy soil ($SR2_{ref}$) and its model predicted value ($SR2_{pre}$) at any design vector X. α_1, α_2 and α_3 are scale factors that are selected in such a way that three terms in equation (20) are of the same order at the starting design vector. In this case $\alpha_1 = 1000$, $\alpha_2 = 1$ and $\alpha_3 = 10$.

The constraint set may be formed by implementing a lower and an upper bound on design variables and can be written as

$$30 \le x_1 \le 70; \quad 28 \le x_2 \le 38; \quad 35 \le x_3 \le 75; \quad 30 \le x_4 \le 40 \tag{21}$$

The task is now to evaluate X that minimize $f(X)$ subject to constraints given in equation (21).

6.3d *Optimization results using back analysis*: Using the computational procedure described in section 5, the numerical results are obtained by considering an initial point within the feasible domain. These are presented below:

The initial design vector (X_i) is taken as $X_i = \{35 \quad 30 \quad 70 \quad 38\}^T$

Starting value of the penalty parameter in IPF (r_0) = 500. The optimization results are shown in table 7.

It is observed from the results of table 7 that the final optimized values are matching almost identically with the actual values.

Table 6. Evaluated foundation response results for reference values of soil parameters

Input (Soil parameter)	Output (Foundation response)
Relative density of type 1 sandy soil = 50%	Pile diameter = 1.679 m
Critical state friction angle of type 1 sandy soil = 33°	Skin resistance of pile due to type 1 sandy soil = 257.599 kN
Relative density of type 2 sandy soil = 55%	Skin resistance of pile due to type 2 sandy soil = 881.745 kN
Critical state friction angle of type 2 sandy soil = 35°	

Table 7. Back analysis result

Parameter/Function	Initial value	Back analyzed value	Actual (reference) value
x_1	35.0	50.026	50.0
x_2	30.0	33.024	33.0
x_3	70.0	55.126	55.0
x_4	38.0	34.95	35.0
Pile diameter (m)	–	1.68	1.679
Skin resistance due to type 1 soil (kN)	–	258.217	257.599
Skin resistance due to type 2 soil (kN)	–	882.414	881.745
$f(X)$	838.872	9.235	–

Percentage error in x_1 = 0.052%; Percentage error in x_2 = 0.072%; Percentage error in x_3 = 0.229%; Percentage error in x_4 = 0.142%

7. Conclusion

The estimation of parameters associated with axially loaded pile and sandy soil interaction problem has been formulated as an inverse problem. The backward problem is treated as a constrained optimization problem and solved using an optimization approach namely interior penalty function method coupled with sequential unconstrained minimization technique with quadratic interpolation method for one-dimensional optimum step length along the search direction. The objective function has been chosen to be the sum of absolute difference of foundation responses at the reference point and predicted value at any design vector. The sandy soil parameters are taken as design variables and the constraint set is developed by implementing a lower and an upper bound on design variables. To demonstrate the methodology, two illustrative examples have been considered. In both cases, optimum results converge to reference values showing the effectiveness of the procedure. To summarize, the following important points may be noted from this study:

- The important objective of this study is for a given output of the system performance, development of a mathematical procedure to evaluate the system input using nonlinear programming.
- Demonstration of the proposed methodology through illustrative examples related to axially loaded pile installed in sandy soil.
- Numerical results indicate that the predicted values of the design vector are in excellent agreement with their respective reference values.
- Foundation response (pile diameter and skin resistance) closely matches with their respective reference values at the optimum point.
- At optimum solution, the value of objective function which indicates the total error is very small compared to its initial value.
- The proposed methodology for parameters estimation using the interior penalty function method in conjunction with the Davidon-Fletcher-Powell technique and quadratic interpolation scheme has been proved to be effective and efficient.
- The methodology presented in this study is quite general and can be applied to a wide spectrum of geotechnical problems related to parameter estimation.

Appendix A

Mathematical method to solve a nonlinear programming (NLP) problem

A standard NLP problem may be defined as:

Determine a vector of design variables $\vec{X}$ that minimizes the objective function $f(\vec{X})$ satisfying the set of constraint

$$g_j(\vec{X}) \leq 0, j = 1, 2, \ldots, m \tag{A1}$$

To solve the NLP problem, IPF based method coupled with the DFP method and QIM is used in this study. These three methods are used for the following purposes.

IPF: constrained optimization problem is converted into an equivalent unconstrained optimization problem.

DFP: minimizes the unconstrained problem.

QIM: calculation of optimum step length along the direction of search from a given point.

IPF method

Generate a new function $\phi(\vec{X}, r_k)$ as

$$\phi(\vec{X}, r_k) = f(\vec{X}) - r_k \sum_{j=1}^{m} \frac{1}{g_j(\vec{X})} \tag{A2}$$

where, r_k is the positive constant (also called as penalty parameter). In the general initial value of r_k is taken as such two terms in the right-hand side of equation (A2) are the same at the initial feasible design point $\vec{X}_0$.

Starting value of penalty parameter

$$r = r_1 = -\frac{f(\vec{X}_0)}{\sum_j \frac{1}{g_j(\vec{X}_0)}} \tag{A3}$$

DFP method

For $r = r_1$, ϕ function is minimized beginning from the design point $\vec{X}_0$ according to the DFP method. This involves the following steps:

(1) At a point $\vec{X}_0$, generate a direction of movement $\vec{S}_0$ as

$$S_0 = -[H_0]\nabla\phi_0 \tag{A4}$$

$[H_0]$ is an initial symmetric positive definite matrix. Initially, it may be assumed as an identity matrix. $\nabla\phi_0$ is gradient vector of ϕ function calculated at $\vec{X}_0$. S_0 is known as the direction of search vector evaluated at $\vec{X}_0$.

(2) Determine a new vector $\vec{X}_{i+1}$ as

$$\vec{X}_{i+1} = \vec{X}_i + \lambda_i^* \vec{S}_i \tag{A5}$$

here, λ_i^* minimizes $\phi(X_i + \lambda_i^* S_i)$, it is known as one-dimensional minimization. λ_i^* is known as optimum step length along the direction of search $\vec{S}_i$ and the quadratic interpolation method [28] is used to evaluate the optimum step length.

(3) Evaluate $\nabla\phi_{i+1}$, that indicates the gradient of ϕ at $\vec{X}_{i+1}$.

At $\vec{X}_{i+1}$, the new direction of movement is obtained as

$$\vec{S}_{i+1} = -[H_{i+1}]\nabla\phi_{i+1} \quad (A6)$$

where

$$[H_{i+1}] = [H_i] + [M_i] + [N_i] \quad (A7)$$

with

$$[M_i] = \lambda_i^* \frac{S_i S_i^T}{S_i^T Y_i} \quad (A8)$$

$$\vec{Y}_i = \nabla\phi_{i+1} - \nabla\phi_i \quad (A9)$$

and

$$[N_i] = -\frac{P_i P_i^T}{Y_i^T P_i} \quad (A10)$$

$$P_i = [H_i]\vec{Y}_i \quad (A11)$$

(4) Once the new direction of movement $\vec{S}_{i+1}$ is evaluated, continue steps (2) through (4) until the function ϕ is minimized at $X = X_{i+1}^*$.
(5) Set a new value of r that is equal to 1/10 of its present value and generate a new function ϕ with the new value of r. Taking the end point of step (4) as a new starting point and repeating the above steps $\phi(\vec{X}, r_k)$ is minimized. This is continued until the specified convergence condition is satisfied.

QIM

The optimum step length (λ_i^*) along the search direction $\vec{S}_i$ from the present point $\vec{X}_i$ is evaluated by the quadratic interpolation method. In this method, three values of step length (λ) are chosen in a specific manner. A quadratic equation $h(\lambda)$ is fit between these three points as an approximation of the actual curve $\phi(\lambda)$. The value of optimum step length along the search direction is obtained by differentiating $h(\lambda)$ with respect to λ and its closed-form expression can be obtained in terms of the ϕ function value at three chosen points. Major steps are given below.

The function $\phi(\lambda)$ is approximated by a quadratic function $h(\lambda)$ which has an easily determinable minimum point. $h(\lambda)$ is expressed as:

$$h(\lambda) = a + b\lambda + c\lambda^2 \quad (A12)$$

the minimum of which occurs when

$$\frac{dh}{d\lambda} = b + 2c\lambda^* = 0.0 \quad (A13)$$

$$\lambda^* = \frac{-b}{2c} \quad (A14)$$

The constants a, b and c of the approximating quadratic can be determined by sampling the function at three different λ values, λ_1, λ_2 and λ_3 and solving the equations:

$$\phi_1 = a + b\lambda_1 + c\lambda_1^2 \quad (A15)$$

$$\phi_2 = a + b\lambda_2 + c\lambda_2^2 \quad (A16)$$

$$\phi_3 = a + b\lambda_3 + c\lambda_3^2 \quad (A17)$$

where, ϕ_1 denotes $\phi_1(\lambda_1)$ and so on.

The constants a, b and c can be determined as follows.

$$c = \frac{[(\phi_3 - \phi_1)(\lambda_2 - \lambda_1) - (\phi_2 - \phi_1)(\lambda_3 - \lambda_1)]}{[(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)]} \quad (A18)$$

$$b = [(\phi_2 - \phi_1)/(\lambda_2 - \lambda_1)] - c(\lambda_2 + \lambda_1) \quad (A19)$$

$$a = \phi_1 - b\lambda_1 - c\lambda_1^2 \quad (A20)$$

Further details of this method have been elaborated in standard textbooks on optimization techniques [28].

List of symbols

$\vec{X}$	Vector of design variables
$f(\vec{X})$	Objective function
$g_j(\vec{X}) \leq 0$	System constraint
Q_{ult}	Ultimate load
q_{sL}	Limit unit shaft resistance
q_{bL}	Limit unit base resistance
$q_{b,10\%}$	Ultimate unit base resistance
σ_v'	Vertical effective stress
K_0	Coefficient of earth pressure at rest
ϕ_c	Critical state friction angle
D_R	Relative density
p_A	Reference stress
σ_h'	Horizontal effective stress
r_k	Penalty parameter
C	Reduction factor
S_i	Search direction
λ^*	Optimum step length along the search direction
ε	Tolerance value
α	Scale factor
γ_{sat}	Bulk unit weight of sandy soil
γ_w	Unit weight of water

Data Availability

Some or all data, models, or codes that support the findings of this study are available from the corresponding author upon reasonable request.

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