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Some *m*-polar fuzzy operators and their application in multiple-attribute decision-making process

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Abstract. In this study, Dombi operations are introduced on two *m*-polar fuzzy sets (mFSs). Here, Dombi operation on *m*-polar fuzzy numbers (mFNs), some new averaging and geometric averaging operators, namely mF Dombi weighted averaging (mFDWA) operator, mF Dombi ordered weighted averaging (mFDOWA) operator, mF Dombi hybrid weighted averaging (mFDHWA) operator, mF Dombi weighted geometric (mFDWG) operator, mF Dombi ordered weighted geometric (mFDOWG) operator, and mF Dombi hybrid weighted geometric (mFDHWGA) operator, have been proposed. Further, some properties like idempotency, boundedness, monotonicity, and commutativity are established. Next, a multi-attribute decision-making (MADM) method in mFNs environment based on mFDWA and mFDWG operators is constructed. Finally, an application of the present MADM method for selecting the best location for the construction of thermal power stations is presented. The present approach with the existing procedure is also compared and a sensitivity analysis of the proposed plan is given.

Keywords. *m*-polar fuzzy sets; *m*-polar fuzzy Dombi weighted averaging operator; *m*-polar fuzzy Dombi weighted geometric operator; MADM approach.

1. Introduction

It is unfavorable to consider real attribute values as complexity appears at a significant level in decision science. In 1965 the theory of fuzzy sets (FSs) was published by Zadeh [1], a new logic of mathematical systems that successfully handle multi-attribute decision-making (MADM) and multi-attribute group decision-making (MADGM) problems. Although FS is a robust framework it has a deficiency in a general mathematical structure. For this cause Atanassov [2] launched theory of intuitionistic fuzzy sets (IFSs), which easily handle complex, vague information. IFS addresses an object in the universe by expressing membership as well as non-membership functions. The topic of aggregation of information process attracted the researchers to this area, and some tremendous works in IFS environment (see [3–7]). Some traditional works [8–11] have been developing on the basis of aggregation operators to aggregate a set of real values. Although IFS and IVIFS can solve the uncertainty of the real-world problems they cannot signify the information of an object that corresponds to each property, and there exists a counter-property. On this issue, [12, 13] proposed another concept of FS called bipolar fuzzy set (BFS) whose membership degrees belong to $[-1, 1]$. The BFS is connected with two functions: positive membership to [0, 1] and another is negative membership to $[-1, 0]$. Later many MADM problems based on aggregation operators in BF environment have been developed such as [14, 15] MADM problems based on Dombi norms in BF structures; Wei *et al* [16] studied MADM problems based on BF Hamacher operator, and Gao *et al* [17] developed MADM model dual hesitant bipolar Hamacher aggregation. Xu and Wei [18] constructed dual hesitant BF arithmetic and decision making based on geometric operators. Jana *et al* [19] proposed an MCDM method based on bipolar soft aggregation operator. Simultaneously, researchers have seen that developing projects like a petrol pump, diesel power plant, etc. requires multi-polar information to analyze such projects. For this, *m*-polar fuzzy set (mFS) has been developed. The concept of mFS was first introduced by Chen *et al* [20] as a generalization of BFS. Later, several mFSs are in many areas such as Graph theory, Group theory, Lie algebras, and BCK/BCI-algebras [21–23]. Khameneh and Kilicman [24] have developed the *m*-polar fuzzy soft weighted aggregation operators. Akram and co-workers [25, 26] have designed many applications of mFSs with different notions. Recently, Waseem *et al* [27] utilized Hamacher operators to aggregate *m*-polar fuzzy numbers (mFNs) information and developed an MADM problem. There are other strategies

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apart from m-polar fuzzy aggregation operators. For these issues, Akram *et al* [28] used an m-polar hesitant fuzzy TOPSIS approach to solve MAGDM problems. After this, Akram *et al* [29] applied PROMETHEE approach based on AHP for developing group decision-making method in m-polar fuzzy environment. The notion of Dombi operator has been proposed by Dombi [30], which has an excellent advantage of parameter flexibility for aggregation of various ambiguous information other than some limitation in direct aggregate linguistic details. Later researches have drawn attention to develop Dombi-operator-based MADM models as follows: Liu *et al* [31] studied Dombi Bonferroni mean operators, applying Dombi operators to develop MADM models to aggregate BF numbers [14], picture fuzzy numbers [32], Pythagorean fuzzy numbers [33], and q-rung orthopair fuzzy numbers [34]. There are no more MADM models that aggregate mF information based on Dombi norms to the best of our knowledge. Therefore, the issue is that mFSs permit a distinct ability to model incomplete information that arises in real-world problems. This apart, legal issues [8, 9, 11] and MADM problems using Dombi norms [15, 30–33] in various fuzzy structures provide us enough motivation to develop the present paper. The objective of this paper is to aggregate mFNs using Dombi models in the environment of mFS and utilize these operators to form an MADM approach.

This paper is organized as follows. In the next section, some basic concepts of the mFS and Dombi operations on mFNs are briefly reviewed. In Section 3, mF Dombi weighted averaging (mFDWA), ordered weighted averaging (mFDOWA), and hybrid weighted averaging (mFDHWA) operators are defined. In Section 4, mF Dombi weighted geometric (mFDWG), order weighted geometric (mFDOWG), and hybrid weighted geometric (mFDHWG) operators are proposed. In Section 5, using these operators, an MADM approach is developed. An illustrative example for selecting a suitable location to set up a thermal power station is given in Section 6. In Section 7, some remarks are made.

2. Preliminaries

In this section, we review some notion about mFSs.

Definition 2.1 [27] An mFS over the universe U is a mapping $\Omega : U \to [0,1]^m$. The membership grade of each object is given as $\Omega(u) = (p_1 * \pi(u), p_2 * \pi(u) \ldots, p_m * \pi(u))$ where $p_l * \pi : [0,1]^m \to [0,1]$ is the l-th projection mapping. Let $\Omega = (p_1 * \pi, \ldots, p_m * \pi)$ be an mFS and $p_l * \pi \in [0,1]$ for $l = 1, 2, \ldots, m$. The subset of all mFSs of U is denoted as $mFS(u)$

If $\{\Omega_k\}_k$ is a family of mFSs [24] over the universe U, then for any $u \in U$:

(i) $\Omega_i^s \leq \Omega_j^s$ for all $s = 1, 2, \ldots, m$, then $\Omega_i \leq \Omega_j$

(ii) $\left(\bigvee_k \Omega_k\right)(u) = \sup_k\{\Omega_k(u)\} = (\sup_k\{\Omega_k^1(u)\}, \ldots, \sup_k\{\Omega_k^m(u)\})$

(iii) $\left(\bigwedge_k \Omega_k\right)(u) = \inf_k\{\Omega_k(u)\} = (\inf_k\{\Omega_k^1(u)\}, \ldots, \inf_k\{\Omega_k^m(u)\})$.

Definition 2.2 [27] The score function Φ of an mFS $\Omega = (p_1 * \pi, \ldots, p_m * \pi)$ is evaluated by the following calculation:

$$\Phi(\Omega) = \frac{1}{m}\left(\sum_{l=1}^{m}(p_l * \pi)\right), \Phi(\Omega) \in [0,1]. \tag{1}$$

Definition 2.3 [27] The accuracy function Ψ of an mFS $\Omega = (p_1 * \pi, \ldots, p_m * \pi)$ is evaluated by the following rule:

$$\Psi(\Omega) = \frac{1}{m}\left(\sum_{l=1}^{m}(-1)^l(p_l * \pi - 1)\right), \Phi(\Omega) \in [-1,1]. \tag{2}$$

Using the definitions of score and accuracy functions, we introduce prioritized relation between two mFSs.

Definition 2.4 Let $\Omega_1 = (p_1 * \pi_1, \ldots, p_m * \pi_1)$ and $\Omega_2 = (p_1 * \pi_2, \ldots, p_m * \pi_2)$ be two mFNs. Then

(i) If $\Phi(\Omega_1) < \Phi(\Omega_2)$, it indicates $\Omega_1 \prec \Omega_2$
(ii) If $\Phi(\Omega_1) > \Phi(\Omega_2)$, it indicates $\Omega_1 \succ \Omega_2$
(iii) If $\Phi(\Omega_1) = \Phi(\Omega_2)$, then

(1) If $\Psi(\Omega_1) < \Psi(\Omega_2)$, it indicates $\Omega_1 \prec \Omega_2$.
(2) If $\Psi(\Omega_1) > \Psi(\Omega_2)$, it indicates $\Omega_1 \succ \Omega_2$.
(3) If $\Psi(\Omega_1) = \Psi(\Omega_2)$, it indicates $\Omega_1 \sim \Omega_2$.

Some basic operations on mFNs are defined as follows.

Definition 2.5 [27] Let $\Omega_1 = (p_1 * \pi_1, \ldots, p_m * \pi_1)$ and $\Omega_2 = (p_1 * \pi_2, \ldots, p_m * \pi_2)$ be two mFSs, and $\tau > 0$. Then

(1) $$\Omega_1 \oplus \Omega_2 = \Big(p_1 * \pi_1 + p_2 * \pi_2 - p_1 * \pi_1 p_2 * \pi_2, \ldots, p_m * \pi_1 + p_m * \pi_2 - p_m * \pi_1 p_m * \pi_2\Big)$$

(2) $$\Omega_1 \otimes \Omega_2 = \left(p_1 * \pi_1 p_1 * \pi_2, \ldots, p_m * \pi_1 p_m * \pi_2\right)$$

(3) $$\tau\Omega_1 = \left(1 - (1 - p_1 * \pi_1)^\tau, \ldots, 1 - (1 - p_m * \pi_1)^\tau\right)$$

(4) $$(\Omega_1)^\tau = \left((p_1 * \pi_1)^\tau, \ldots, (p_m * \pi_1)^\tau\right)$$

(5) $$(\Omega_1)^c = \left((p_1 * \pi_1)^c, \ldots, (p_m * \pi_1)^c\right)$$

(6) $\Omega_1 \subseteq \Omega_2$ if and only if $p_1 * \pi_1 \leq p_1 * \pi_2, \ldots, p_m * \pi_1 \leq p_m * \pi_2$

(7) $$\Omega_1 \bigcup \Omega_2 = \left\{ \max\left(p_1 * \pi_1, p_1 * \pi_2\right), \ldots, \max\left(p_m * \pi_1, p_m * \pi_2\right) \right\}$$

(8) $$\Omega_1 \bigcap \Omega_2 = \left\{ \min\left(p_1 * \pi_1, p_1 * \pi_2\right), \ldots, \min\left(p_m * \pi_1, p_m * \pi_2\right) \right\}.$$

2.1 *Dombi operations on mFNs*

Dombi recommended operations Dombi product and sum, which are defined here.

Definition 2.6 [30] Let p and q be any two real numbers. Then, Dombi norm and Dombi conorm are defined by the following expressions:

$$Dom(p,q) = \frac{1}{1 + \{(\frac{1-p}{p})^{\varrho} + (\frac{1-q}{q})^{\varrho}\}^{1/\varrho}} \tag{3}$$

$$Dom^*(p,q) = 1 - \frac{1}{1 + \{(\frac{p}{1-p})^{\varrho} + (\frac{q}{1-q})^{\varrho}\}^{1/\varrho}} \tag{4}$$

where $\varrho \geq 1$ and $(p,q) \in [0,1] \times [0,1]$.

Using Dombi norms and Dombi conorms, we explain Dombi operations with respect to mFSs.

Definition 2.7 Let $\Omega_1 = (p_1 * \pi_1, \ldots, p_m * \pi_1)$ and $\Omega_2 = (p_1 * \pi_2, \ldots, p_m * \pi_2)$ be two mFSs, and $\tau > 0$. Now, we define Dombi operations on mFSs:

(1) $$\Omega_1 \oplus_D \Omega_2 = \left\langle 1 - \frac{1}{1+\left\{\left(\frac{p_1*\pi_1}{1-p_1*\pi_1}\right)^{\varrho} + \left(\frac{p_1*\pi_2}{1-p_1*\pi_2}\right)^{\varrho}\right\}^{1/\varrho}}, \ldots, 1 - \frac{1}{1+\left\{\left(\frac{p_m*\pi_1}{1-p_m*\pi_1}\right)^{\varrho} + \left(\frac{p_m*\pi_2}{1-p_m*\pi_2}\right)^{\varrho}\right\}^{1/\varrho}} \right\rangle$$

(2) $$\Omega_1 \otimes_D \Omega_2 = \left\langle \frac{1}{1+\left\{\left(\frac{1-p_1*\pi_1}{p_1*\pi_1}\right)^{\varrho} + \left(\frac{1-p_1*\pi_2}{p_1*\pi_2}\right)^{\varrho}\right\}^{1/\varrho}}, \ldots, \frac{1}{1+\left\{\left(\frac{1-p_m*\pi_1}{p_m*\pi_1}\right)^{\varrho} + \left(\frac{1-p_m*\pi_2}{p_m*\pi_2}\right)^{\varrho}\right\}^{1/\varrho}} \right\rangle$$

(3) $$\tau.\,\Omega_1 = \left\langle 1 - \frac{1}{1+\left\{\tau\left(\frac{p_1*\pi_1}{1-p_1*\pi_1}\right)^{\varrho}\right\}^{1/\varrho}}, \ldots, 1 - \frac{1}{1+\left\{\tau\left(\frac{p_m*\pi_1}{1-p_m*\pi_1}\right)^{\varrho}\right\}^{1/\varrho}} \right\rangle$$

(4) $$(\Omega_1)_1^{\tau} = \left\langle \frac{1}{1+\left\{\tau\left(\frac{1-p_1*\pi_1}{p_1*\pi_1}\right)^{\varrho}\right\}^{1/\varrho}}, \ldots, \frac{1}{1+\left\{\tau\left(\frac{1-p_m*\pi_1}{p_m*\pi_1}\right)^{\varrho}\right\}^{1/\varrho}} \right\rangle.$$

3. mF Dombi arithmetic aggregation operators

Here, the mFDWA operator, mFDOWA operator, and mFDHWA operator are defined and their properties established.

Definition 3.1 Let $\Omega_b = (p_1 * \pi_b, \ldots, p_m * \pi_b)$ $(b = 1,2,\ldots,t)$ be t mFSs. Then an mFDWA operator is a function $mFDWA : \Theta^t \to \Theta$ defined as follows:

$$mFDWA_{\delta}(\Omega_1, \Omega_2, \ldots, \Omega_t) = \bigoplus_{b=1}^{t}\left(\delta_b \Omega_b\right) \tag{5}$$

where $\delta = (\delta_1, \delta_2, \ldots, \delta_t)^T$ is a set of weight vector on Ω_b such that $\delta_b > 0$ and $\sum_{b=1}^{t} \delta_b = 1$.

Theorem 3.2 *Let* $\Omega_b = (p_1 * \pi_b, \ldots, p_m * \pi_b)$ $(b = 1,2,\ldots,t)$ *be t mFSs. Then collection of values of all mFSs under mFDWA operator is also an mFS, which follows as*

$$mFDWA_{\delta}(\Omega_1, \Omega_2, \ldots, \Omega_t) = \bigoplus_{b=1}^{t}\left(\delta_b \Omega_b\right) = \left(1 - \frac{1}{1 + \left\{\sum_{b=1}^{t} \delta_b \left(\frac{p_1*\pi_b}{1-p_1*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}}, \ldots, 1 - \frac{1}{1 + \left\{\sum_{b=1}^{t} \delta_b \left(\frac{p_m*\pi_b}{1-p_m*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}} \right) \tag{6}$$

where $\delta = (\delta_1, \delta_2, \ldots, \delta_t)$ *is the set of weight vectors of* Ω_b $(b = 1,2,\ldots,t)$ *such that* $\delta_b > 0$, *and* $\sum_{b=1}^{t} \delta_b = 1$.

Now, this theorem can be proved by mathematical induction technique.

Proof If we take $b = 2$ and $\delta = 2$, then left side of theorem 3.2 becomes $mFDWA_{\delta}(\Omega_1, \Omega_2, \ldots, \Omega_t) = \bigoplus_{b=1}^{t}\left(\delta_b \Omega_b\right) = \delta_1 \Omega_1 \oplus \delta_2 \Omega_2$ and then right hand side of theorem 3.2 becomes

$$\delta_1\Omega_1 \oplus \delta_2\Omega_2 = \left(\begin{array}{c} 1 - \dfrac{1}{1+\left\{\delta_1\left(\frac{p_1*\pi_1}{1-p_1*\pi_1}\right)^{\varrho} + \delta_2\left(\frac{p_1*\pi_2}{1-p_1*\pi_2}\right)^{\varrho}\right\}^{1/\varrho}}, \ldots, \\ 1 - \dfrac{1}{1+\left\{\delta_1\left(\frac{p_m*\pi_1}{1-p_m*\pi_1}\right)^{\varrho} + \delta_2\left(\frac{p_m*\pi_2}{1-p_m*\pi_2}\right)^{\varrho}\right\}^{1/\varrho}} \end{array} \right) = \left(\begin{array}{c} 1 - \dfrac{1}{1+\left\{\sum_{b=1}^{2}\delta_b\left(\frac{p_1*\pi_b}{1-p_1*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}}, \ldots, \\ 1 - \dfrac{1}{1+\left\{\sum_{b=1}^{2}\delta_b\left(\frac{p_m*\pi_b}{1-p_m*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}} \end{array} \right).$$

Thus, this technique is true for $b = 2$.

Let us assume that theorem 3.2 holds for $b \geq r$, where $r \in \mathbb{N}$ (the set of natural numbers), which gives

$$mFDWA_{\delta}(\Omega_1, \Omega_2, \ldots, \Omega_r) = \bigoplus_{b=1}^{r}\left(\delta_b\Omega_b\right) = \left(\begin{array}{c} 1 - \dfrac{1}{1+\left\{\sum\limits_{b=1}^{r}\delta_b\left(\frac{p_1*\pi_b}{1-p_1*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, 1 - \dfrac{1}{1+\left\{\sum\limits_{b=1}^{r}\delta_b\left(\frac{p_m*\pi_b}{1-p_m*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}} \end{array} \right). \quad (7)$$

Now, for $b = r + 1$

$$mFDWA_{\delta}(\Omega_1, \Omega_2, \Omega_r \ldots, \Omega_{r+1}) = \bigoplus_{b=1}^{r}(\delta_b\Omega_b) \oplus \delta_{r+1}\Omega_{r+1} = \left(\begin{array}{c} 1 - \dfrac{1}{1+\left\{\sum\limits_{b=1}^{r}\delta_b\left(\frac{p_1*\pi_b}{1-p_1*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, 1 - \dfrac{1}{1+\left\{\sum\limits_{b=1}^{r}\delta_b\left(\frac{p_m*\pi_b}{1-p_m*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}} \end{array} \right) \oplus \left(\begin{array}{c} 1 - \dfrac{1}{1+\left\{\delta_{r+1}\left(\frac{p_1*\pi_{r+1}}{1-p_1*\pi_{r+1}}\right)^{\varrho}\right\}^{1/\varrho}}, \ldots, \\ 1 - \dfrac{1}{1+\left\{\delta_{r+1}\left(\frac{p_m*\pi_{r+1}}{1-p_m*\pi_{r+1}}\right)^{\varrho}\right\}^{1/\varrho}} \end{array} \right) = \left(\begin{array}{c} 1 - \dfrac{1}{1+\left\{\sum\limits_{b=1}^{r+1}\delta_b\left(\frac{p_1*\pi_b}{1-p_1*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, 1 - \dfrac{1}{1+\left\{\sum\limits_{b=1}^{r+1}\delta_b\left(\frac{p_m*\pi_b}{1-p_m*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}} \end{array} \right). \quad (8)$$

Hence, theorem 3.2 is true for all natural numbers. □

Example 3.3 Let $\Omega_1 = (0.3, 0.4, 0.6, 0.7)$, $\Omega_2 = (0.2, 0.5, 0.4, 0.3)$, $\Omega_3 = (0.4, 0.6, 0.5, 0.4)$ be 4FSs with a weight vector $\delta = (0.3, 0.4, 0.3)$ for 4FSs. Then for $\varrho = 3$

$$mFDWA_{\delta}(\Omega_1, \Omega_2, \Omega_3) = \bigoplus_{b=1}^{3}\left(\delta_b\Omega_b\right) = \left(\begin{array}{c} 1 - \dfrac{1}{1+\left\{\sum_{b=1}^{3}\delta_b\left(\frac{p_1*\pi_b}{1-p_1*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, 1 - \dfrac{1}{1+\left\{\sum_{b=1}^{3}\delta_b\left(\frac{p_m*\pi_b}{1-p_m*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}} \end{array} \right)$$

$$= \left(\begin{array}{c} 1 - \dfrac{1}{1+\left\{0.3\left(\frac{0.3}{1-0.3}\right)^3 + 0.4\left(\frac{0.2}{1-0.2}\right)^3 + 0.3\left(\frac{0.4}{1-0.4}\right)^3\right\}^{1/3}}, \\ 1 - \dfrac{1}{1+\left\{0.3\left(\frac{0.4}{1-0.4}\right)^3 + 0.4\left(\frac{0.5}{1-0.5}\right)^3 + 0.3\left(\frac{0.6}{1-0.6}\right)^3\right\}^{1/3}}, \\ 1 - \dfrac{1}{1+\left\{0.3\left(\frac{0.6}{1-0.6}\right)^3 + 0.4\left(\frac{0.4}{1-0.4}\right)^3 + 0.3\left(\frac{0.5}{1-0.5}\right)^3\right\}^{1/3}}, \\ 1 - \dfrac{1}{1+\left\{0.3\left(\frac{0.7}{1-0.7}\right)^3 + 0.4\left(\frac{0.3}{1-0.3}\right)^3 + 0.3\left(\frac{0.4}{1-0.4}\right)^3\right\}^{1/3}} \end{array} \right) = \left\langle 0.3295, 0.5338, 0.5298, 0.6121 \right\rangle.$$

The mFDWA operators satisfy the following properties.

Theorem 3.4 (Idempotency property) *Let* $\Omega_b = (p_1 * \pi_b, \ldots, p_m * \pi_b)$ $(b = 1, 2, \ldots, t)$ *be t mFSs. If all these t mFSs are equal, i.e.* $\Omega_b = \Omega$ *for all* $(b = 1, 2, \ldots, t)$, *then*

$$mFDWA_{\delta}(\Omega_1, \Omega_2, \ldots, \Omega_t) = \Omega. \quad (9)$$

Proof Let $\Omega_b = (p_1 * \pi_b, \ldots, p_m * \pi_b)$ $(b = 1, 2, \ldots, t)$ be t mFSs. If all these mFNs are equal, $\Omega_b = \Omega$ for all $(b = 1, 2, \ldots, t)$. Then, from equation (6), we have

$$mFDWA_{\delta}(\Omega_1, \Omega_2, \ldots, \Omega_t) = \bigoplus_{b=1}^{t}\left(\delta_b\Omega_b\right) = \left(\begin{array}{c} 1 - \dfrac{1}{1+\left\{\sum_{b=1}^{t}\delta_b\left(\frac{p_1*\pi_b}{1-p_1*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, 1 - \dfrac{1}{1+\left\{\sum_{b=1}^{t}\delta_b\left(\frac{p_m*\pi_b}{1-p_m*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}} \end{array} \right) = \left(\begin{array}{c} 1 - \dfrac{1}{1+\left\{\left(\frac{p_1*\pi}{1-p_1*\pi}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, 1 - \dfrac{1}{1+\left\{\left(\frac{p_m*\pi}{1-p_m*\pi}\right)^{\varrho}\right\}^{1/\varrho}} \end{array} \right) = \left(\begin{array}{c} 1 - \dfrac{1}{1+\left\{\left(\frac{p_1*\pi}{1-p_1*\pi}\right)\right.}, \\ \ldots, 1 - \dfrac{1}{1+\left\{\left(\frac{p_m*\pi}{1-p_m*\pi}\right)\right.} \end{array} \right) = \left\langle p_1 * \pi, \ldots, p_m * \pi \right\rangle = \Omega.$$

Hence, the proof is completed. □

Theorem 3.5 (Boundedness property) *Let* $\Omega_b = (p_1 * \pi_b, \ldots, p_m * \pi_b)$ $(b = 1, 2, \ldots, t)$ *be t mFSs. If* $\Omega^- = \min\{\Omega_1, \Omega_2, \ldots, \Omega_t\}$ *and*
$\Omega^+ = \max\{\Omega_1, \Omega_2, \ldots, \Omega_t\}$ *then*

$$\Omega^- \leq mFDWA_\delta(\Omega_1, \Omega_2, \ldots, \Omega_t) \leq \Omega^+. \tag{10}$$

Proof Let $\Omega^- = \min\{\Omega_1, \Omega_2, \ldots, \Omega_t\}$ and $\Omega^+ = \max\{\Omega_1, \Omega_2, \ldots, \Omega_t\}$. Then, we have the inequality

$$1 - \frac{1}{1 + \left\{ \sum_{b=1}^t \delta_b \left(\frac{p_1 * \pi_b^-}{1 - p_1 * \pi_b^-} \right)^\varrho \right\}^{1/\varrho}}, \ldots, 1 - \frac{1}{1 + \left\{ \sum_{b=1}^t \delta_b \left(\frac{p_m * \pi_b^-}{1 - p_m * \pi_b^-} \right)^\varrho \right\}^{1/\varrho}}$$
$$\leq 1 - \frac{1}{1 + \left\{ \sum_{b=1}^t \delta_b \left(\frac{p_1 * \pi_b}{1 - p_1 * \pi_b} \right)^\varrho \right\}^{1/\varrho}}, \ldots, 1 - \frac{1}{1 + \left\{ \sum_{b=1}^t \delta_b \left(\frac{p_m * \pi_b}{1 - p_m * \pi_b} \right)^\varrho \right\}^{1/\varrho}}$$
$$\leq 1 - \frac{1}{1 + \left\{ \sum_{b=1}^t \delta_b \left(\frac{p_1 * \pi_b^+}{1 - p_1 * \pi_b^+} \right)^\varrho \right\}^{1/\varrho}}, \ldots, 1 - \frac{1}{1 + \left\{ \sum_{b=1}^t \delta_b \left(\frac{p_m * \pi_b^+}{1 - p_m * \pi_b^+} \right)^\varrho \right\}^{1/\varrho}}.$$

Hence

$$\Omega^- \leq mFDWA_\delta(\Omega_1, \Omega_2, \ldots, \Omega_t) \leq \Omega^+.$$

□

Theorem 3.6 (Monotonicity property) *Let* $\Omega_b = (p_1 * \pi_b, \ldots, p_m * \pi_b)$ *and* $\Omega_{\sigma(b)} = (p_{\sigma(1)} * \pi_{\sigma(b)}, \ldots, p_{\sigma(m)} * \pi_{\sigma(b)})$ $(b = 1, 2, \ldots, t)$ *be two sets of t mFSs such that* $\Omega_b \leq \Omega_{\sigma(b)}$ *for all t ; then*

$$mFDWA_\delta(\Omega_1, \Omega_2, \ldots, \Omega_t) \leq mFDWA_\delta(\Omega_{\sigma(1)}, \Omega_{\sigma(2)}, \ldots, \Omega_{\sigma(t)}) \tag{11}$$

where $\Omega_{\sigma(b)}$ *is any arbitrary permutation of* Ω_b *for all* $(b = 1, 2, \ldots, t)$.

Proof $\Omega_b \leq \Omega_{\sigma(b)}$ for $b = 1, 2, \ldots, t$.

Thus, for $s = 1, 2, \ldots, m$ the relation holds since $\min\{\Omega_1(u), \Omega_2(u), \ldots, \Omega_t(u)\} \leq \min\{\Omega_{\sigma(1)}(u), \Omega_{\sigma(2)}(u), \ldots, \Omega_{\sigma(t)}(u)\}$ for all $u \in m(u)$ and $\Omega_{\sigma(b)}$ is any permutation of Ω_b. □

Next, proceed with the mFDOWA operator.

Definition 3.7 Let $\Omega_b = (p_1 * \pi_b, \ldots, p_m * \pi_b)$ $(b = 1, 2, \ldots, t)$ be t mFSs. Then an mFSs Dombi ordered weighted averaging (mFDOWA) operator is a function $mFDOWA : \Theta^t \to \Theta$ defined as follows:

$$mFDOWA_\theta(\Omega_1, \Omega_2, \ldots, \Omega_t) = \bigoplus_{b=1}^t \left(\theta_b \Omega_{\sigma(b)} \right) \tag{12}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_t)^T$ is a set of weight vector on π_b such that $\theta_b > 0$ and $\sum_{b=1}^t \theta_b = 1$. Also, $\sigma(1), \sigma(2), \ldots, \sigma(t)$ is the permutation of $(1, 2, \ldots, t)$ for which $\Omega_{\sigma(t-1)} \geq \Omega_{\sigma(t)}$ for all $b = 1, 2, \ldots, t$.

Theorem 3.8 *Let* $\Omega_b = (p_1 * \pi_b, \ldots, p_m * \pi_b)$ $(b = 1, 2, \ldots, t)$ *be mFSs. Then collection of values of all mFSs under mFDOWA operator is also an mFS, which follows as*

$$mFDOWA_\theta(\Omega_1, \Omega_2, \ldots, \Omega_t) = \bigoplus_{b=1}^t \left(\theta_b \Omega_{\sigma(b)} \right)$$
$$= \begin{pmatrix} 1 - \dfrac{1}{1 + \left\{ \sum\limits_{b=1}^t \theta_b \left(\frac{p_1 * \pi_{\sigma(b)}}{1 - p_1 * \pi_{\sigma(b)}} \right)^\varrho \right\}^{1/\varrho}}, \\ \ldots, \quad 1 - \dfrac{1}{1 + \left\{ \sum\limits_{b=1}^t \theta_b \left(\frac{p_m * \pi_{\sigma(b)}}{1 - p_m * \pi_{\sigma(b)}} \right)^\varrho \right\}^{1/\varrho}} \end{pmatrix} \tag{13}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_t)$ *is the set of weight vector of* Ω_b $(b = 1, 2, \ldots, t)$ *such that* $\theta_b > 0$ *and* $\sum_{b=1}^t \theta_b = 1$. *Also,* $\sigma(1), \sigma(2), \ldots, \sigma(t)$ *is the permutation of* $(1, 2, \ldots, t)$ *for which* $\Omega_{\sigma(t-1)} \geq \Omega_{\sigma(t)}$ *for all* $b = 1, 2, \ldots, t$.

Proof This theorem can be proved easily with the help of theorem 3.2. □

Example 3.9 Let $\Omega_1 = (0.2, 0.5, 0.7, 0.6)$,
$\Omega_2 = (0.7, 0.6, 0.4, 0.5)$,
$\Omega_3 = (0.4, 0.8, 0.6, 0.7)$ be 4FSs with associated weight vector $\theta = (0.3, 0.5, 0.2)$ for 4FSs. Then, as per score function given in definition 2.2, we have
$\Phi(\Omega_1) = \frac{0.2+0.5+0.7+0.6}{4} = 0.5000$,
$\Phi(\Omega_2) = \frac{0.7+0.6+0.4+0.5}{4} = 0.5500$, and $\Phi(\Omega_1) = \frac{0.4+0.8+0.6+0.4}{4} = 0.6250$.
Here, $\Phi(\Omega_3) > \Phi(\Omega_2) > \Phi(\Omega_1)$, which imply that $\Omega_{\sigma(1)} = \Omega_3 = (0.4, 0.8, 0.6, 0.7)$,
$\Omega_{\sigma(2)} = \Omega_2 = (0.7, 0.6, 0.4, 0.5)$, and $\Omega_{\sigma(3)} = \Omega_1 = (0.2, 0.5, 0.7, 0.6)$. Then, theorem 3.8 for $\varrho = 3$ gives

$$mFDOWA_\theta(\Omega_1,\Omega_2,\Omega_3)=\bigoplus_{b=1}^{3}\left(\theta_b\Omega_{\sigma(b)}\right)$$

$$=\left(\begin{array}{c}1-\dfrac{1}{1+\left\{\sum_{b=1}^{3}\theta_b\left(\frac{p_1*\pi_{\sigma(b)}}{1-p_1*\pi_{\sigma(b)}}\right)^{\varrho}\right\}^{1/\varrho}},\\ \ldots,1-\dfrac{1}{1+\left\{\sum_{b=1}^{3}\theta_b\left(\frac{p_m*\pi_{\sigma(b)}}{1-p_m*\pi_{\sigma(b)}}\right)^{\varrho}\right\}^{1/\varrho}}\end{array}\right)$$

$$=\left(\begin{array}{c}1-\dfrac{1}{1+\left\{0.3\left(\frac{0.4}{1-0.4}\right)^3+0.5\left(\frac{0.7}{1-0.7}\right)^3+0.2\left(\frac{0.2}{1-0.2}\right)^3\right\}^{1/3}},\\ 1-\dfrac{1}{1+\left\{0.3\left(\frac{0.8}{1-0.8}\right)^3+0.5\left(\frac{0.6}{1-0.6}\right)^3+0.2\left(\frac{0.5}{1-0.5}\right)^3\right\}^{1/3}},\\ 1-\dfrac{1}{1+\left\{0.3\left(\frac{0.6}{1-0.6}\right)^3+0.5\left(\frac{0.4}{1-0.4}\right)^3+0.2\left(\frac{0.7}{1-0.7}\right)^3\right\}^{1/3}},\\ 1-\dfrac{1}{1+\left\{0.3\left(\frac{0.7}{1-0.7}\right)^3+0.5\left(\frac{0.5}{1-0.5}\right)^3+0.2\left(\frac{0.6}{1-0.6}\right)^3\right\}^{1/3}}\end{array}\right)$$

$$=\left\langle 0.6505,0.7342,0.6074,0.6308\right\rangle.$$

The following properties of mFDOWA can be proved easily.

Theorem 3.10 (Idempotency property) *Let* $\Omega_b=(p_1*\pi_b,\ldots,p_m*\pi_b)$ $(b=1,2,\ldots,t)$ *be t mFSs.*

If all these t mFSs are equal, i.e. $\Omega_b=\Omega$ for all $(b=1,2,\ldots,t)$ then

$$mFDOWA_\theta(\Omega_1,\Omega_2,\ldots,\Omega_t)=\Omega. \tag{14}$$

Proof The proof of the theorem is obvious. □

Theorem 3.11 (Boundedness property) *Let* $\Omega_b=(p_1*\pi_b,\ldots,p_m*\pi_b)$ $(b=1,2,\ldots,t)$ *be t mFSs. If* $\Omega^-=\min\{\Omega_1,\Omega_2,\ldots,\Omega_t\}$ *and* $\Omega^+=\max\{\Omega_1,\Omega_2,\ldots,\Omega_t\}$ *then*

$$\Omega^-\le mFDOWA_\theta(\Omega_1,\Omega_2,\ldots,\Omega_t)\le\Omega^+. \tag{15}$$

Theorem 3.12 (Monotonicity property) *Let* $\Omega_b=(p_1*\pi_b,\ldots,p_m*\pi_b)$ *and* $\Omega_{\sigma(b)}=(p_{\sigma(1)}*\pi_{\sigma(b)},\ldots,p_{\sigma(m)}*\pi_{\sigma(b)})$ $(b=1,2,\ldots,t)$ *be two sets of t mFSs such that* $\Omega_b\le\Omega_{\sigma(b)}$ *for all b ; then*

$$mFDOWA_\theta(\Omega_1,\Omega_2,\ldots,\Omega_t)\le mFDOWA_\theta(\Omega_{\sigma(1)},\Omega_{\sigma(2)},\ldots,\Omega_{\sigma(t)}) \tag{16}$$

where $\Omega_{\sigma(b)}$ *is any arbitrary permutation of* Ω_b *for all* $(b=1,2,\ldots,t)$.

Proof The proof of the theorem is obvious. □

Theorem 3.13 (Commutative property) *Let* $\Omega_b=(p_1*\pi_b,\ldots,p_m*\pi_b)$ *and* $\Omega_{\sigma(b)}=(p_{\sigma(1)}*\pi_{\sigma(b)},\ldots,p_{\sigma(m)}*\pi_{\sigma(b)})$ $(b=1,2,\ldots,t)$ *be two sets of t mFSs; then*

$$mFDOWA_\theta(\Omega_1,\Omega_2,\ldots,\Omega_t)=mFDOWA_\theta(\Omega_{\sigma(1)},\Omega_{\sigma(2)},\ldots,\Omega_{\sigma(t)}) \tag{17}$$

where $\Omega_{\sigma(b)}$ *is any arbitrary permutation of* Ω_b *for all* $(b=1,2,\ldots,t)$.

Proof The proof of the theorem is obvious. □

In definitions 3.1 and 3.7, mFDWA operator takes weights of mFv; again mFDOWA operator's weights imply the ordered position of mFv instead of the weights of mFv themselves. Hence we define another operator, namely mF Dombi hybrid averaging (mFDHWA) operator, with the qualitative use of both the mFDWA and mFDOWA operators.

Definition 3.14 Let $\Omega_b=(p_1*\pi_b,\ldots,p_m*\pi_b)$ $(b=1,2,\ldots,t)$ be t mFSs. Then an mFNs Dombi hybrid weighted averaging (mFDHWA) operator is a function $mFDHWA:\Theta^t\to\Theta$ defined as follows:

$$mFDHWA_{\delta,\theta}(\Omega_1,\Omega_2,\ldots,\Omega_t)=\bigoplus_{b=1}^{t}\left(\theta_b\dot{\Omega}_{\sigma(b)}\right). \tag{18}$$

Also $\sigma(1),\sigma(2),\ldots,\sigma(t)$ is the permutation of $(1,2,\ldots,t)$ for which $\Omega_{\sigma(t-1)}\ge\Omega_{\sigma(t)}$ for all $b=1,2,\ldots,t$ for mFNs π_b, and $\theta=(\theta_1,\theta_2,\ldots,\theta_t)^T$ is the associated weighted vector of the mFSs $(\Omega_1,\Omega_2,\ldots,\Omega_t)$ such that $\theta_b>0$ and $\sum_{b=1}^{t}\theta_b=1$. $\dot{\Omega}_b$ are biggest mFSs, where $\dot{\Omega}_b=(t\delta)\Omega_t$, $(b=1,2,\ldots,t)$ for which $\delta=(\delta_1,\delta_2,\ldots,\delta_t)^T$ is the weight vector such that $\delta_b>0$ and $\sum_{b=1}^{t}\delta_b=1$.

Theorem 3.15 *Let* $\Omega_b=(p_1*\pi_b,\ldots,p_m*\pi_b)$ $(b=1,2,\ldots,t)$ *be t mFSs. Then collection of values of mFSs* Ω_b *using mFDHWA operator is also an mFS. Further, we get*

$$mFDHWA_{\delta,\theta}(\Omega_1,\Omega_2,\ldots,\Omega_t)=\bigoplus_{b=1}^{t}\left(\theta_b\dot{\Omega}_{\sigma(b)}\right)$$

$$=\left(\begin{array}{c}1-\dfrac{1}{1+\left\{\sum\limits_{b=1}^{t}\theta_b\left(\frac{p_1*\dot{\pi}_{\sigma(b)}}{1-p_1*\dot{\pi}_{\sigma(b)}}\right)^{\varrho}\right\}^{1/\varrho}},\\ \ldots,1-\dfrac{1}{1+\left\{\sum\limits_{b=1}^{t}\theta_b\left(\frac{p_m*\dot{\pi}_{\sigma(b)}}{1-p_m*\dot{\pi}_{\sigma(b)}}\right)^{\varrho}\right\}^{1/\varrho}}\end{array}\right). \tag{19}$$

Proof It can be proved obviously. □

Example 3.16 Let $\Omega_1=(0.4,0.6,0.7)$, $\Omega_2=(0.7,0.5,0.8)$, $\Omega_3=(0.3,0.4,0.2)$ be 3FSs with associated weight vector and $\delta=(0.4,0.2,0.4)^T$ be weight vector of these 3FSs, and associated weighted vector be $\theta=(0.3,0.2,0.5)^T$. Then, by theorem 3.15, aggregated mFSs for $(\varrho=3)$

$$\dot{\Omega}_1 = \left\langle \left(1 - \frac{1}{1+\left\{3\times 0.3\times\left(\frac{0.4}{1-0.4}\right)^3\right\}^{1/3}}, 1 - \frac{1}{1+\left\{3\times 0.3\times\left(\frac{0.6}{1-0.6}\right)^3\right\}^{1/3}}, 1 - \frac{1}{1+\left\{3\times 0.3\times\left(\frac{0.7}{1-0.7}\right)^3\right\}^{1/3}}\right)\right\rangle = \left\langle 0.3916, 0.5915, 0.6926 \right\rangle$$

$$\dot{\Omega}_2 = \left\langle \left(1 - \frac{1}{1+\left\{3\times 0.2\times\left(\frac{0.7}{1-0.7}\right)^3\right\}^{1/3}}, 1 - \frac{1}{1+\left\{3\times 0.2\times\left(\frac{0.5}{1-0.5}\right)^3\right\}^{1/3}}, 1 - \frac{1}{1+\left\{3\times 0.2\times\left(\frac{0.8}{1-0.8}\right)^3\right\}^{1/3}}\right)\right\rangle = \left\langle 0.6631, 0.4575, 0.7714 \right\rangle$$

$$\dot{\Omega}_3 = \left\langle \left(1 - \frac{1}{1+\left\{3\times 0.5\times\left(\frac{0.3}{1-0.3}\right)^3\right\}^{1/3}}, 1 - \frac{1}{1+\left\{3\times 0.5\times\left(\frac{0.4}{1-0.4}\right)^3\right\}^{1/3}}, 1 - \frac{1}{1+\left\{3\times 0.5\times\left(\frac{0.2}{1-0.2}\right)^3\right\}^{1/3}}\right)\right\rangle = \left\langle 0.3291, 0.4328, 0.2225 \right\rangle.$$

Scores of Ω_t $(t = 1, 2, 3)$ are calculated as follows:

$\Phi(\dot{\Omega}_1) = \frac{0.3916+0.5915+0.6926}{3} = 0.5586$,

$\Phi(\dot{\Omega}_2) = \frac{0.6631+0.4575+0.7714}{3} = 0.6307$,

$\Phi(\dot{\Omega}_3) = \frac{0.3291+0.4328+0.2225}{3} = 0.3281$.

Since $\Phi(\dot{\Omega}_2) > \Phi(\dot{\Omega}_1) > \Phi(\dot{\Omega}_3)$

$\dot{\Omega}_{\sigma(1)} = \dot{\Omega}_2 = \left\langle 0.3916, 0.5915, 0.6926 \right\rangle$,

$\dot{\Omega}_{\sigma(2)} = \dot{\Omega}_1 = \left\langle 0.6631, 0.4575, 0.7714 \right\rangle$,

$\dot{\Omega}_{\sigma(3)} = \dot{\Omega}_3 = \left\langle 0.3291, 0.4328, 0.2225 \right\rangle$. Therefore, aggregated value of mFSs $(\varrho = 3)$ by the definition of mFDHWA operator is

$$mFDHWA_{\delta,\theta}(\Omega_1, \Omega_2, \Omega_3) = \bigoplus_{t=1}^{3}(\theta_t \dot{\Omega}_{\sigma(t)}) = \begin{pmatrix} 1 - \frac{1}{1+\left\{\sum_{b=1}^{t}\theta_b\left(\frac{p_1 * \dot{\pi}_{\sigma(b)}}{1-p_1 * \dot{\pi}_{\sigma(b)}}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, \quad 1 - \frac{1}{1+\left\{\sum_{b=1}^{t}\theta_b\left(\frac{p_m * \dot{\pi}_{\sigma(b)}}{1-p_m * \dot{\pi}_{\sigma(b)}}\right)^{\varrho}\right\}^{1/\varrho}} \end{pmatrix}$$
$$= \begin{pmatrix} 1 - \frac{1}{1+\left\{0.4\left(\frac{0.3916}{1-0.3916}\right)^3 + 0.2\left(\frac{0.6631}{1-0.6631}\right)^3 + 0.4\left(\frac{0.3291}{1-0.3291}\right)^3\right\}^{1/3}}, \\ 1 - \frac{1}{1+\left\{0.4\left(\frac{0.5915}{1-0.5915}\right)^3 + 0.2\left(\frac{0.4575}{1-0.4575}\right)^3 + 0.4\left(\frac{0.4328}{1-0.4328}\right)^3\right\}^{1/3}}, \\ 1 - \frac{1}{1+\left\{0.4\left(\frac{0.6926}{1-0.6926}\right)^3 + 0.2\left(\frac{0.7714}{1-0.7714}\right)^3 + 0.4\left(\frac{0.2225}{1-0.2225}\right)^3\right\}^{1/3}} \end{pmatrix}$$
$$= \left\langle 0.6476, 0.7188, 0.3446 \right\rangle. \quad (20)$$

4. mF Dombi geometric aggregation operators

Here, we introduce the definition of mFDWG operator, mFDOWG operator, and mFDHWG operator and establish their properties.

Definition 4.1 Let $\Omega_b = (p_1 * \pi_b, \ldots, p_m * \pi_b)$ $(b = 1, 2, \ldots, t)$ be t mFSs. Then an mFSs Dombi weighted geometric (mFDWG) operator is a function $mFDWG : \Theta^t \to \Theta$ defined as follows:

$$mFDWG_{\delta}(\Omega_1, \Omega_2, \ldots, \Omega_t) = \bigotimes_{b=1}^{t}\left(\Omega_b\right)^{\delta_b} \quad (21)$$

where $\delta = (\delta_1, \delta_2, \ldots, \delta_t)^T$ is a set of weight vector of Ω_b such that $\delta_b > 0$ and $\sum_{b=1}^{t}\delta_b = 1$.

Theorem 4.2 *Let* $\Omega_a = (p_1 * \pi_a, \ldots, p_m * \pi_a)$ $(b = 1, 2, \ldots, t)$ *be t mFSs. Then collection of values of all mFSs under mFDWG operator is also an mFS, which follows as*

$$mFDWG_{\delta}(\Omega_1, \Omega_2, \ldots, \Omega_t) = \bigotimes_{b=1}^{t}\left(\Omega_b\right)^{\delta_b} = \begin{pmatrix} \frac{1}{1+\left\{\sum_{b=1}^{t}\delta_b\left(\frac{1-p_1 * \pi_b}{p_1 * \pi_b}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, \frac{1}{1+\left\{\sum_{b=1}^{t}\delta_b\left(\frac{1-p_m * \pi_b}{p_m * \pi_b}\right)^{\varrho}\right\}^{1/\varrho}} \end{pmatrix} \quad (22)$$

where $\delta = (\delta_1, \delta_2, \ldots, \delta_t)$ *be the set of weight vector of* Ω_b $(b = 1, 2, \ldots, t)$ *such that* $\delta_b > 0$ *and* $\sum_{b=1}^{t}\delta_b = 1$.

This theorem can be proved by mathematical induction technique.

Proof If we take $b = 2$ and $\delta = 2$, then the left side of theorem 4.2 becomes

$$
\begin{aligned}
mFDWG_{\delta}(\Omega_1, \Omega_2, \ldots, \Omega_t) &= \bigotimes_{b=1}^{t}\left(\delta_b \Omega_b\right) \\
&= \delta_1 \Omega_1 \oplus \delta_2 \Omega_2
\end{aligned}
$$

and then the right hand side of theorem 4.2 becomes

$$
(\Omega_1)^{\delta_1} \otimes (\Omega_2)^{\delta_2} = \begin{pmatrix} \dfrac{1}{1+\left\{\delta_1\left(1-\frac{p_1*\pi_1}{p_1*\pi_1}\right)^{\varrho}+\delta_2\left(\frac{1-p_1*\pi_2}{p_1*\pi_2}\right)^{\varrho}\right\}^{1/\varrho}}, \ldots, \\ \dfrac{1}{1+\left\{\delta_1\left(\frac{1-p_m*\pi_1}{p_m*\pi_1}\right)^{\varrho}+\delta_2\left(\frac{1-p_m*\pi_2}{p_m*\pi_2}\right)^{\varrho}\right\}^{1/\varrho}} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{1+\left\{\sum_{b=1}^{2}\delta_b\left(\frac{1-p_1*\pi_b}{p_1*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}}, \ldots, \\ \dfrac{1}{1+\left\{\sum_{b=1}^{2}\delta_b\left(\frac{1-p_m*\pi_b}{p_m*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}} \end{pmatrix}.
$$

Thus, this technique is true for $b = 2$.

Let us assume that theorem 4.2 holds for $b \geq r$, where $r \in \mathbb{N}$ (the set of natural numbers), which gives

$$
mFDWG_{\delta}(\Omega_1, \Omega_2, \ldots, \Omega_r) = \bigotimes_{b=1}^{r}\left(\Omega_b\right)^{\delta_b} = \begin{pmatrix} \dfrac{1}{1+\left\{\sum\limits_{b=1}^{r}\delta_b\left(\frac{1-p_1*\pi_b}{p_1*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, \dfrac{1}{1+\left\{\sum\limits_{b=1}^{r}\delta_b\left(\frac{1-p_m*\pi_b}{p_m*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}} \end{pmatrix}. \tag{23}
$$

Now for $b = r + 1$

$$
\begin{aligned}
mFDWG_{\delta}(\pi_1, \pi_2, \pi_r, \ldots, \pi_{r+1}) &= \bigotimes_{b=1}^{r}(\pi_b)^{\delta_b} \otimes (\pi_{r+1})^{\delta_{r+1}} \\
&= \begin{pmatrix} \dfrac{1}{1+\left\{\sum_{b=1}^{r}\delta_b\left(\frac{1-p_1*\pi_b}{p_1*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, \dfrac{1}{1+\left\{\sum_{b=1}^{r}\delta_b\left(\frac{1-p_m*\pi_b}{p_m*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}} \end{pmatrix} \otimes \begin{pmatrix} \dfrac{1}{1+\left\{\delta_{r+1}\left(\frac{1-p_1*\pi_{r+1}}{p_1*\pi_{r+1}}\right)^{\varrho}\right\}^{1/\varrho}}, \ldots, \\ \dfrac{1}{1+\left\{\delta_{r+1}\left(\frac{1-p_m*\pi_{r+1}}{p_m*\pi_{r+1}}\right)^{\varrho}\right\}^{1/\varrho}} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{1}{1+\left\{\sum_{b=1}^{r+1}\delta_b\left(\frac{1-p_1*\pi_b}{p_1*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, \dfrac{1}{1+\left\{\sum_{b=1}^{r+1}\delta_b\left(\frac{1-p_m*\pi_b}{p_m*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}} \end{pmatrix}.
\end{aligned}
$$

Hence, theorem 4.2 is true for all natural numbers. □

Example 4.3 Let $\Omega_1 = (0.5, 0.6, 0.4, 0.3)$, $\Omega_2 = (0.3, 0.7, 0.5, 0.4)$, $\Omega_3 = (0.4, 0.8, 0.6, 0.2)$ be 4FSs with a weight vector $\delta = (0.3, 0.4, 0.3)$ for 4FSs. Then for $\varrho = 3$

$$
\begin{aligned}
mFDWG_{\delta}(\Omega_1, \Omega_2, \Omega_3) &= \bigotimes_{b=1}^{3}\left(\Omega_b\right)^{\delta_b} \\
&\begin{pmatrix} \dfrac{1}{1+\left\{\sum_{b=1}^{3}\delta_b\left(\frac{1-p_1*\pi_b}{p_1*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, \dfrac{1}{1+\left\{\sum_{b=1}^{3}\delta_b\left(\frac{1-p_m*\pi_b}{p_m*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{1}{1+\left\{0.3\left(\frac{1-0.5}{0.5}\right)^{3}+0.4\left(\frac{1-0.3}{0.3}\right)^{3}+0.3\left(\frac{1-0.4}{0.4}\right)^{3}\right\}^{1/3}}, \\ \dfrac{1}{1+\left\{0.3\left(\frac{1-0.6}{0.6}\right)^{3}+0.4\left(\frac{1-0.7}{0.7}\right)^{3}+0.3\left(\frac{1-0.8}{0.8}\right)^{3}\right\}^{1/3}}, \\ \dfrac{1}{1+\left\{0.3\left(\frac{1-0.4}{0.4}\right)^{3}+0.4\left(\frac{1-0.5}{0.5}\right)^{3}+0.3\left(\frac{1-0.6}{0.6}\right)^{3}\right\}^{1/3}}, \\ \dfrac{1}{1+\left\{0.3\left(\frac{1-0.3}{0.3}\right)^{3}+0.4\left(\frac{1-0.4}{0.4}\right)^{3}+0.3\left(\frac{1-0.2}{0.2}\right)^{3}\right\}^{1/3}} \end{pmatrix} \\
&= \left\langle 0.3501, 0.6666, 0.4662, 0.2565 \right\rangle.
\end{aligned}
$$

The mFDWG operators obeys the following properties.

Theorem 4.4 (Idempotency property) *Let $\Omega_b = (p_1 * \pi_b, \ldots, p_m * \pi_b)$ $(b = 1, 2, \ldots, t)$ be t mFSs. If all these t mFSs are equal, i.e. $\Omega_b = \Omega$ for all $(b = 1, 2, \ldots, t)$, then*

$$
mFDWG_{\delta}(\Omega_1, \Omega_2, \ldots, \Omega_t) = \Omega. \tag{24}
$$

Proof Let $\Omega_b = (p_1 * \pi_b, \ldots, p_m * \pi_b)$ $(b = 1, 2, \ldots, t)$ be mFSs. If all these mFSs are equal, i.e. $\Omega_b = \Omega$ for all $(b = 1, 2, \ldots, t)$, from equation (22) we have

$$
\begin{aligned}
mFDWG_{\delta}(\Omega_1, \Omega_2, \ldots, \Omega_t) &= \bigotimes_{b=1}^{t}\left(\Omega_b\right)^{\delta_b} \\
&= \begin{pmatrix} \dfrac{1}{1+\left\{\sum\limits_{b=1}^{t}\delta_b\left(\frac{1-p_1*\pi_b}{p_1*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, \dfrac{1}{1+\left\{\sum\limits_{b=1}^{t}\delta_b\left(\frac{1-p_m*\pi_b}{p_m*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{1}{1+\left\{\left(\frac{1-p_1*\pi}{p_1*\pi}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, \dfrac{1}{1+\left\{\left(\frac{1-p_m*\pi}{p_m*\pi}\right)^{\varrho}\right\}^{1/\varrho}} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{1}{1+\left\{\left(\frac{1-p_1*\pi}{p_1*\pi}\right)}, \\ \ldots, \dfrac{1}{1+\left\{\left(\frac{1-p_m*\pi}{p_m*\pi}\right)} \end{pmatrix} \\
&= \left\langle p_1 * \pi, \ldots, p_m * \pi \right\rangle = \Omega.
\end{aligned} \tag{25}
$$

Hence, proof of the theorem is completed. □

Theorem 4.5 (Boundedness property)
Let $\Omega_b = (p_1 * \pi_b, \ldots, p_m * \pi_b)$ $(b = 1, 2, \ldots, t)$ *be* t *mFSs. If* $\Omega^- = \min\{\Omega_1, \Omega_2, \ldots, \Omega_t\}$ *and* $\Omega^+ = \max\{\Omega_1, \Omega_2, \ldots, \Omega_t\}$ *then*

$$\Omega^- \leq mFDWG_\delta(\Omega_1, \Omega_2, \ldots, \Omega_t) \leq \Omega^+. \tag{26}$$

Proof Let $\Omega^- = \min\{\Omega_1, \Omega_2, \ldots, \Omega_t\}$ and $\Omega^+ = \max\{\Omega_1, \Omega_2, \ldots, \Omega_t\}$. Then, we have the inequality

$$\frac{1}{1+\left\{\sum_{b=1}^t \delta_b\left(\frac{1-p_1*\pi_b^-}{p_1*\pi_b^-}\right)^\varrho\right\}^{1/\varrho}}, \ldots, \frac{1}{1+\left\{\sum_{b=1}^t \delta_b\left(\frac{1-p_m*\pi_b^-}{p_m*\pi_b^-}\right)^\varrho\right\}^{1/\varrho}}$$
$$\leq \frac{1}{1+\left\{\sum_{b=1}^t \delta_b\left(\frac{1-p_1*\pi_b}{p_1*\pi_b}\right)^\varrho\right\}^{1/\varrho}}, \ldots, \frac{1}{1+\left\{\sum_{b=1}^t \delta_b\left(\frac{1-p_m*\pi_b}{p_m*\pi_b}\right)^\varrho\right\}^{1/\varrho}}$$
$$\leq \frac{1}{1+\left\{\sum_{b=1}^t \delta_b\left(\frac{1-p_1*\pi_b^+}{p_1*\pi_b^+}\right)^\varrho\right\}^{1/\varrho}}, \ldots, \frac{1}{1+\left\{\sum_{b=1}^t \delta_b\left(\frac{1-p_m*\pi_b^+}{p_m*\pi_b^+}\right)^\varrho\right\}^{1/\varrho}}.$$

Hence

$$V^- \leq mFDWG_\delta(\Omega_1, \Omega_2, \ldots, \Omega_t) \leq \Omega^+.$$

□

Theorem 4.6 (Monotonicity property) *Let* $\Omega_b = (p_1 * \pi_b, \ldots, p_m * \pi_b)$ *and* $\Omega_{\sigma(b)} = (p_{\sigma(1)} * \pi_{\sigma(b)}, \ldots, p_{\sigma(m)} * \pi_{\sigma(b)})$ $(b = 1, 2, \ldots, t)$ *be two sets of* t *mFSs such that* $\pi_b \leq \pi_{\sigma(b)}$ *for all b. Then*

$$\begin{aligned} &mFDWG_\delta(\Omega_1, \Omega_2, \ldots, \Omega_t) \leq \\ &mFDWG_\delta(\Omega_{\sigma(1)}, \Omega_{\sigma(2)}, \ldots, \Omega_{\sigma(t)}) \end{aligned} \tag{27}$$

where $\Omega_{\sigma(b)}$ *is any permutation of* Ω_b $(b = 1, 2, \ldots, t)$.

Proof The theorem can be proved obviously. □

Next, proceed with the mFDOWG operator.

Definition 4.7 Let $\Omega_b = (p_1 * \pi_b, \ldots, p_m * \pi_b)$ $(b = 1, 2, \ldots, t)$ be t mFSs. Then an mFSs Dombi ordered weighted geometric (mFDOWG) operator is a function $mFDOWG : \Theta^t \rightarrow \Theta$ defined as follows:

$$mFDOWG_\theta(\Omega_1, \Omega_2, \ldots, \Omega_t) = \bigotimes_{b=1}^t \left(\Omega_{\sigma(b)}\right)^{\theta_b} \tag{28}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_t)^T$ is a set of weight vector on Ω_b such that $\theta_b > 0$ and $\sum_{b=1}^t \theta_b = 1$. Also, $\sigma(1), \sigma(2), \ldots, \sigma(t)$ is the permutation of $(1, 2, \ldots, t)$ for which $\Omega_{\sigma(t-1)} \geq \Omega_{\sigma(t)}$ for all $b = 1, 2, \ldots, t$.

Theorem 4.8 *Let* $\Omega_a = (p_1 * \pi_a, \ldots, p_m * \pi_a)$ $(b = 1, 2, \ldots, t)$ *be* t *mFSs. Then collection of values of all mFSs under mFDOWG operator is also an mFS, which follows as*

$$mFDOWG_\theta(\Omega_1, \Omega_2, \ldots, \Omega_t) = \bigotimes_{b=1}^t \left(\Omega_{\sigma(b)}\right)^{\theta_b}$$
$$= \left(\frac{1}{1+\left\{\sum\limits_{b=1}^t \theta_b\left(\frac{1-p_1*\pi_{\sigma(b)}}{p_1*\pi_{\sigma(b)}}\right)^\varrho\right\}^{1/\varrho}}, \ldots, \frac{1}{1+\left\{\sum\limits_{b=1}^t \theta_b\left(\frac{1-p_m*\pi_{\sigma(b)}}{p_m*\pi_{\sigma(b)}}\right)^\varrho\right\}^{1/\varrho}} \right) \tag{29}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_t)$ *is the set of weight vector of* Ω_b $(b = 1, 2, \ldots, t)$ *such that* $\theta_b > 0$ *and* $\sum_{b=1}^t \theta_b = 1$. *Also,* $\sigma(1), \sigma(2), \ldots, \sigma(t)$ *is the permutation of* $(1, 2, \ldots, t)$ *for which* $\Omega_{\sigma(t-1)} \geq \Omega_{\sigma(t)}$ *for all* $b = 1, 2, \ldots, t$.

Example 4.9 Let $\Omega_1 = (0.3, 0.6, 0.8, 0.5)$, $\Omega_2 = (0.2, 0.7, 0.6, 0.4)$,
$\Omega_3 = (0.7, 0.5, 0.4, 0.2)$ be 4FSs with associated weight vector $\delta = (0.3, 0.5, 0.2)$ for 4FSs. As per the score function given in definition 2.2, we have
$\Phi(\Omega_1) = \frac{0.3+0.6+0.8+0.5}{4} = 0.5500$,
$\Phi(\Omega_2) = \frac{0.2+0.7+0.6+0.4}{4} = 0.4750$, and
$\Phi(\Omega_3) = \frac{0.7+0.5+0.4+0.2}{4} = 0.4500$.
Here $\Phi(\Omega_1) > \Phi(\Omega_2) > \Phi(\Omega_3)$, which imply that $\Omega_{\sigma(1)} = \Omega_1 = (0.3, 0.6, 0.8, 0.5)$,
$\Omega_{\sigma(2)} = \Omega_2 = (0.2, 0.7, 0.6, 0.4)$,
and $\Omega_{\sigma(3)} = \Omega_1 = (0.7, 0.5, 0.4, 0.2)$.
Then, by theorem 3.8 for $\varrho = 3$

$$mFDOWG_\theta(\Omega_1, \Omega_2, \Omega_3) = \bigotimes_{b=1}^3 \left(\Omega_{\sigma(b)}\right)^{\delta_b}$$
$$= \left(\frac{1}{1+\left\{\sum_{b=1}^3 \theta_b\left(\frac{1-p_1*\pi_{\sigma(b)}}{p_1*\pi_{\sigma(b)}}\right)^\varrho\right\}^{1/\varrho}}, \ldots, \frac{1}{1+\left\{\sum_{b=1}^3 \theta_b\left(\frac{1-p_m*\pi_{\sigma(b)}}{p_m*\pi_{\sigma(b)}}\right)^\varrho\right\}^{1/\varrho}} \right)$$
$$= \left(\begin{array}{c} \frac{1}{1+\left\{0.3\left(\frac{1-0.3}{0.3}\right)^3+0.5\left(\frac{1-0.2}{0.2}\right)^3+0.2\left(\frac{1-0.7}{0.7}\right)^3\right\}^{1/3}}, \\ \frac{1}{1+\left\{0.3\left(\frac{1-0.6}{0.6}\right)^3+0.5\left(\frac{1-0.7}{0.7}\right)^3+0.2\left(\frac{1-0.5}{0.5}\right)^3\right\}^{1/3}}, \\ \frac{1}{1+\left\{0.3\left(\frac{1-0.8}{0.8}\right)^3+0.5\left(\frac{1-0.6}{0.6}\right)^3+0.2\left(\frac{1-0.4}{0.4}\right)^3\right\}^{1/3}}, \\ \frac{1}{1+\left\{0.3\left(\frac{1-0.5}{0.5}\right)^3+0.5\left(\frac{1-0.4}{0.4}\right)^3+0.2\left(\frac{1-0.2}{0.2}\right)^3\right\}^{1/3}} \end{array} \right)$$
$$= \left\langle 0.2327, 0.5918, 0.5157, 0.2895 \right\rangle.$$

The following properties of mFDOWG can be proved easily.

Theorem 4.10 (Idempotency property) *Let* $\Omega_b = (p_1 * \pi_b, \ldots, p_m * \pi_b)$ $(b = 1, 2, \ldots, t)$ *be* t *mFSs. If all these* t *mFSs are equal, i.e.* $\Omega_b = \Omega$ *for all* $(b = 1, 2, \ldots, t)$, *then*

$$mFDWG_{\theta}(\Omega_1,\Omega_2,\ldots,\Omega_t)=\Omega. \quad (30)$$

Proof The theorem can be proved obviously. □

Theorem 4.11 (Boundedness property) *Let* $\Omega_b=(p_1*\pi_b,\ldots,p_m*\pi_b)$ $(b=1,2,\ldots,t)$ *be t mFSs. If* $\Omega^-=\min\{\Omega_1,\Omega_2,\ldots,\Omega_t\}$ *and* $\Omega^+=\max\{\Omega_1,\Omega_2,\ldots,\Omega_t\}$ *then*

$$\Omega^-\le mFDWG_{\theta}(\Omega_1,\Omega_2,\ldots,\Omega_t)\le\Omega^+. \quad (31)$$

Proof The theorem can be proved obviously. □

Theorem 4.12 (Monotonicity property) Let $\Omega_b=(p_1*\pi_b,\ldots,p_m*\pi_b)$ and $\Omega_{\sigma(b)}=(p_{\sigma(1)}*\pi_{\sigma(b)},\ldots,p_{\sigma(m)}*\pi_{\sigma(b)})$ $(b=1,2,\ldots,t)$ be two sets of t mFSs such that $\Omega_b\le\Omega_{\sigma(b)}$ for all b. Then

$$\begin{aligned}&mFDWG_{\theta}(\Omega_1,\Omega_2,\ldots,\Omega_t)\le\\&mFDWG_{\delta}(\Omega_{\sigma(1)},\Omega_{\sigma(2)},\ldots,\Omega_{\sigma(t)})\end{aligned} \quad (32)$$

where $\Omega_{\sigma(b)}$ is any permutation of Ω_b $(b=1,2,\ldots,t)$.

Proof The theorem can be proved obviously. □

Theorem 4.13 (Commutative property) *Let* $\Omega_b=(p_1*\pi_b,\ldots,p_m*\pi_b)$ *and* $\Omega_{\sigma(b)}=(p_{\sigma(1)}*\pi_{\sigma(b)},\ldots,p_{\sigma(m)}*\pi_{\sigma(b)})$ $(b=1,2,\ldots,t)$ *be two sets of t mFSs; then*

$$\begin{aligned}&mFDWG_{\theta}(\Omega_1,\Omega_2,\ldots,\Omega_t)\\&=mFDWG_{\theta}(\Omega_{\sigma(1)},\Omega_{\sigma(2)},\ldots,\Omega_{\sigma(t)})\end{aligned} \quad (33)$$

where $\Omega_{\sigma(b)}$ *is any arbitrary permutation of* Ω_b *for all* $(b=1,2,\ldots,t)$.

Proof The theorem can be proved obviously. □

In definitions 4.1 and 4.7, mFDWG operator took weights of mFv; again mFDOWG operator's weights imply the ordered position of mFv instead of the weights of mFv themselves. Hence we define another operator, namely mF Dombi hybrid geometric (mFDHWG) operator, with the qualitative use of both the mFDWG and mFDOWG operators.

Definition 4.14 Let $\Omega_b=(p_1*\pi_b,\ldots,p_m*\pi_b)$ $(b=1,2,\ldots,t)$ be t mFSs. Then an mFDHWG operator is a function $mFDHWG:\Theta^t\to\Theta$ defined as follows:

$$mFDHWG_{\delta,\theta}(\Omega_1,\Omega_2,\ldots,\Omega_t)=\bigotimes_{b=1}^{t}\left(\dot{\Omega}_{\sigma(b)}\right)^{\theta_b}. \quad (34)$$

Also, $\sigma(1),\sigma(2),\ldots,\sigma(t)$ is the permutation of $(1,2,\ldots,t)$ for which $\Omega_{\sigma(t-1)}\ge\Omega_{\sigma(t)}$ for all $b=1,2,\ldots,t$ for mFS π_b and $\theta=(\theta_1,\theta_2,\ldots,\theta_t)^T$ is the associated weighted vector of the mFSs $(\Omega_1,\Omega_2,\ldots,\Omega_t)$ such that $\theta_b>0$ and $\sum_{b=1}^{t}\theta_b=1$. $\dot{\Omega}_b$ are biggest mFNs where $\dot{\Omega}_b=(t\delta)\Omega_t$, $(b=1,2,\ldots,t)$ for which $\delta=(\delta_1,\delta_2,\ldots,\delta_t)^T$ is the weight vector such that $\delta_b>0$ and $\sum_{b=1}^{t}\delta_b=1$.

Theorem 4.15 *Let* $\Omega_b=(p_1*\pi_b,\ldots,p_m*\pi_b)$ $(b=1,2,\ldots,t)$ *be t mFSs. Then collection of values of mFNs* π_b *using mFDHWG operator is also an mFS. Further, we get*

$$mFDHWG_{\delta,\theta}(\Omega_1,\Omega_2,\ldots,\Omega_t)=\bigotimes_{b=1}^{t}\left(\dot{\Omega}_{\sigma(b)}\right)^{\theta_b}$$
$$=\left(\begin{array}{c}\dfrac{1}{1+\left\{\sum\limits_{b=1}^{t}\theta_b\left(\frac{1-p_1*\dot{\pi}_{\sigma(b)}}{p_1*\dot{\pi}_{\sigma(b)}}\right)^{\varrho}\right\}^{1/\varrho}},\\ \ldots,1-\dfrac{1}{1+\left\{\sum\limits_{b=1}^{t}\theta_b\left(\frac{1-p_m*\dot{\pi}_{\sigma(b)}}{p_m*\dot{\pi}_{\sigma(b)}}\right)^{\varrho}\right\}^{1/\varrho}}\end{array}\right). \quad (35)$$

Proof This theorem can be proved obviously. □

Example 4.16 Let $\Omega_1=(0.4,0.6,0.7)$, $\Omega_2=(0.7,0.5,0.8)$, $\Omega_3=(0.3,0.4,0.2)$ be 3FSs with associated weight vector and $\delta=(0.4,0.2,0.4)^T$ be weight vector of these 3FSs, and associated weighted vector be $\theta=(0.3,0.2,0.5)^T$. Then, by theorem 4.15, aggregated mFSs for $(\varrho=3)$ are

$$\begin{aligned}\dot{\Omega}_1=&\Big\langle\Big(\frac{1}{1+\left\{3\times0.3\times\left(\frac{1-0.4}{0.4}\right)^3\right\}^{1/3}},\\&\frac{1}{1+\left\{3\times0.3\times\left(\frac{1-0.6}{0.6}\right)^3\right\}^{1/3}},\\&\frac{1}{1+\left\{3\times0.3\times\left(\frac{1-0.7}{0.7}\right)^3\right\}^{1/3}}\Big)\Big\rangle\\=&\big\langle0.2477,0.7895,0.9338\big\rangle,\end{aligned}$$

$$\begin{aligned}\dot{\Omega}_2=&\Big\langle\Big(\frac{1}{1+\left\{3\times0.2\times\left(\frac{1-0.7}{0.7}\right)^3\right\}^{1/3}},\\&\frac{1}{1+\left\{3\times0.2\times\left(\frac{1-0.5}{0.5}\right)^3\right\}^{1/3}},\\&\frac{1}{1+\left\{3\times0.2\times\left(\frac{1-0.8}{0.8}\right)^3\right\}^{1/3}}\Big)\Big\rangle\\=&\big\langle0.9549,0.6250,0.9907\big\rangle,\end{aligned}$$

$$\begin{aligned}\dot{\Omega}_3=&\Big\langle\Big(\frac{1}{1+\left\{3\times0.5\times\left(\frac{1-0.3}{0.3}\right)^3\right\}^{1/3}},\\&\frac{1}{1+\left\{3\times0.5\times\left(\frac{1-0.4}{0.4}\right)^3\right\}^{1/3}},\\&\frac{1}{1+\left\{3\times0.5\times\left(\frac{1-0.2}{0.2}\right)^3\right\}^{1/3}}\Big)\Big\rangle\\=&\big\langle0.0499,0.1649,0.0103\big\rangle.\end{aligned}$$

Scores of Ω_t (t=1,2,3) are calculated as follows:

$\Phi(\dot{\Omega}_1)=\frac{0.2477+0.7895+0.9338}{3}=0.6570,$

$\Phi(\dot{\Omega}_2) = \frac{0.9549+0.6250+0.9907}{3} = 0.8569,$
$\Phi(\dot{\Omega}_3) = \frac{0.0499+0.1649+0.0103}{3} = 0.0750.$
Since $\Phi(\dot{\Omega}_2) > \Phi(\dot{\Omega}_1) > \Phi(\dot{\Omega}_3)$,
$\dot{\Omega}_{\sigma(1)} = \dot{\Omega}_2 = \left\langle 0.2477, 0.7895, 0.9338 \right\rangle,$
$\dot{\Omega}_{\sigma(2)} = \dot{\Omega}_1 = \left\langle 0.9549, 0.6250, 0.9907 \right\rangle,$
$\dot{\Omega}_{\sigma(3)} = \dot{\Omega}_3 = \left\langle 0.0499, 0.1649, 0.0103 \right\rangle.$ Therefore, aggregated value of mFSs ($\varrho = 3$) by the definition of mFDHWG operator is

$$mFDHWG_{\delta,\theta}(\Omega_1,\Omega_2,\Omega_3) = \bigotimes_{t=1}^{3}(\dot{\Omega}_{\sigma(t)})^{\theta_t}$$
$$= \begin{pmatrix} \frac{1}{1+\left\{\sum\limits_{b=1}^{t}\theta_b\left(\frac{1-p_1*\dot{\pi}_{\sigma(b)}}{p_1*\dot{\pi}_{\sigma(b)}}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, \quad \frac{1}{1+\left\{\sum\limits_{b=1}^{t}\theta_b\left(\frac{1-p_m*\dot{\pi}_{\sigma(b)}}{p_m*\dot{\pi}_{\sigma(b)}}\right)^{\varrho}\right\}^{1/\varrho}} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{1+\left\{0.4\left(\frac{1-0.2477}{0.2477}\right)^3+0.2\left(\frac{1-0.9549}{0.9549}\right)^3+0.4\left(\frac{1-0.0499}{0.0499}\right)^3\right\}^{1/3}}, \\ \frac{1}{1+\left\{0.4\left(\frac{1-0.7895}{0.7895}\right)^3+0.2\left(\frac{0.4575}{1-0.4575}\right)^3+0.4\left(\frac{1-0.6250}{0.6250}\right)^3\right\}^{1/3}}, \\ \frac{1}{1+\left\{0.4\left(\frac{1-0.9338}{0.9338}\right)^3+0.2\left(\frac{1-0.9907}{0.9907}\right)^3+0.4\left(\frac{1-0.0103}{0.0103}\right)^3\right\}^{1/3}} \end{pmatrix}$$
$$= \left\langle 0.0665, 0.5704, 0.0139 \right\rangle. \tag{36}$$

5. Model for MADM using mF data

In this area, we develop an MADM procedure using mF Dombi aggregation operators in which attribute weights are real numbers and values are in mFSs. Let $Q = \{Q_1, Q_2, \ldots, Q_s\}$ be a set of alternatives and $g = \{g_1, g_2, \ldots, g_t\}$ be a set of attributes. Let $\delta = (\delta_1, \delta_2, \ldots, \delta_t)$ be a set of weight vector of the attribute g_b $(b = 1, 2, \ldots, t)$ assigned by decision makers (DMs) such that $\delta_t > 0$ and $\sum_{b=1}^{t}\delta_t = 1$. Suppose that $D = (p_1 * \pi_{st}, \ldots, p_m * \pi_{st})_{a\times b}$ is the mF decision matrix, where $p_r * \pi_{st}$ $(r = 1, 2, \ldots, m)$ is the membership degree for the alternatives Q_b that fulfills the attribute g_t and $p_r * Q$.

We propose to solve the MADM problem using mF data, and mFDWA and mFDWG operators in the accompanying algorithm.

Step 1. We employ the decision data stated in matrix D, and the operator mfDWA:

$$\Upsilon_a = mFDWA_\delta(\Omega_{a1},\Omega_{a2},\ldots,\Omega_{ab}) = \bigoplus_{b=1}^{t}\left(\delta_b\Omega_b\right)$$
$$= \begin{pmatrix} 1-\frac{1}{1+\left\{\sum\limits_{b=1}^{t}\delta_b\left(\frac{p_1*\pi_b}{1-p_1*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, 1-\frac{1}{1+\left\{\sum\limits_{b=1}^{t}\delta_b\left(\frac{p_m*\pi_b}{1-p_m*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}} \end{pmatrix} \tag{37}$$

or

$$\Upsilon_a = mFDWG_\delta(\Omega_{a1},\Omega_{a2},\ldots,\Omega_{ab}) = \bigotimes_{b=1}^{t}\left(\Omega_b\right)^{\delta_b}$$
$$= \begin{pmatrix} \frac{1}{1+\left\{\sum\limits_{b=1}^{t}\delta_b\left(\frac{1-p_1*\pi_b}{p_1*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}}, \\ \ldots, \frac{1}{1+\left\{\sum\limits_{b=1}^{t}\delta_b\left(\frac{1-p_m*\pi_b}{p_m*\pi_b}\right)^{\varrho}\right\}^{1/\varrho}} \end{pmatrix} \tag{38}$$

to accumulate Υ_a $(a = 1, 2, \ldots, s)$ of the alternatives Q_s.

Step 2. Compute the score function $\Phi(\Upsilon_a)$ $(a = 1, 2, \ldots, s)$ based on overall mF information Υ_a $(a = 1, 2, \ldots, s)$ in order to rank all the alternatives Q_a $(a = 1, 2, \ldots, s)$ to select the best choice Q_a. If there is no variation between score functions $\Phi(\Upsilon_a)$ and $\Phi(\Upsilon_b)$ then we proceed to compute accuracy degrees of $\Psi(\Upsilon_a)$ and $\Psi(\Upsilon_b)$ based on overall mF information of Υ_a and Υ_b, and rank the alternatives Q_a depending on the accuracy degrees of $\Psi(\Upsilon_a)$ and $\Psi(\Upsilon_b)$.

Step 3. Rank all the alternatives Q_a $(a = 1, 2, \ldots, s)$ in order to choose the leading one(s) in a manner conforming with $\Phi(\Upsilon_a)$ $(a = 1, 2, \ldots, s)$.

Step 4. Stop.

6. Numerical example

For this part, we utilize the proposed approach to solve an MADM problem.

6.1 *Selection of suitable location for a thermal power station*

Thermal power stations are used to generate electricity from chemical energy into mechanical energy. A company selects a suitable location for this purpose. Assume that the company has four possible locations Q_1, Q_2, Q_3, Q_4, and Q_5 is treated as an alternative for likely thermal power stations. An expert team of engineers of the company selects the best choice. The expert team of engineers selects the best location under the following main criteria.

g_1 : Infrastructures
g_2 : Environmental conditions
g_3 : Social impact
g_4 : Governmental policies.

Each of the criteria is subdivided into three sub-criteria to form a 3-polar fuzzy set. The Infrastructure depends on the availability of coal, availability of water, and availability of transportation facilities. The Environmental conditions depend on ambient temperature, humidity, and air velocities. Social impact depends on education facilities, hospital facilities, and health care facilities. The Governmental policies are subdivided into licensing policies, institutional finance, and Government subsidies. They have no dominant power over each other; DMs are required to exempt five possible locations Q_a $(a = 1, 2, \ldots, 4, 5)$ under the mentioning attributes whose weights (0.4, 0.3, 0.1, 0.2) are addressed by DMs, the decision matrix $D = (\Upsilon_{ab})_{5\times 4}$, which is provided in table 1, where Υ_{ab} are in the form of 3FNs.

To chose suitable location Q_a $(a = 1, 2, \ldots, s)$, apply mFDWA and mFDWG operators to model MADM, which can be computed as follows:

Step 1. Let $\varrho = 1$; using the mFDWA operator, overall accumulated values Υ_a of Q_a $(a = 1, 2, \ldots, 5)$ are $\Upsilon_1 = (0.5556, 0.6471, 0.5906)$, $\Upsilon_2 = (0.7610, 0.6000, 0.7500)$, $\Upsilon_3 = (0.5200, 0.5714, 0.6104)$, $\Upsilon_4 = (0.6000, 0.5950, 0.5556)$, $\Upsilon_5 = (0.6806, 0.8886, 0.4831)$.

Step 2. Score values of $\Phi(\Upsilon_a)$ $(a = 1, 2, \ldots, 5)$ are as follows: $\Phi(\Upsilon_1) = 0.5978$, $\Phi(\Upsilon_2) = 0.7037$, $\Phi(\Upsilon_3) = 0.5673$, $\Phi(\Upsilon_4) = 0.5835$, $\Phi(\Upsilon_5) = 0.6202$.

Step 3. Rank results of Q_a $(a = 1, 2, \ldots, 5)$ in accordance with the score values of $\Phi(\Upsilon_a)$ $(a = 1, 2, \ldots, 5)$ of the overall mFSs as $Q_2 \succ Q_5 \succ Q_1 \succ Q_4 \succ Q_3$.

Step 4. Q_2 is suggested as the most favourable location.

If the *mFDWG* operator is used instead, in a similar manner, the problem is solved as follows:

Step 1. Let $\varrho = 1$; using the mFDWG operator to compute the overall accumulated values of Q_a $(a = 1, 2, \ldots, 5)$ $\Upsilon_1 = (0.5455, 0.5813, 0.4688)$, $\Upsilon_2 = (0.6409, 0.4870, 0.6729)$, $\Upsilon_3 = (0.5000, 0.4970, 0.4423)$, $\Upsilon_4 = (0.5000, 0.4988, 0.4773)$, $\Upsilon_5 = (0.5490, 0.6774, 0.4444)$.

Step 2. Compute score values $\Phi(\Upsilon_a)$ $(a = 1, 2, \ldots, 5)$ of Υ_a $(a = 1, 2, \ldots, 5)$ as $\Phi(\Upsilon_1) = 0.5319$, $\Phi(\Upsilon_2) = 0.6003$, $\Phi(\Upsilon_3) = 0.4798$, $\Phi(\Upsilon_4) = 0.4920$, $\Phi(\Upsilon_5) = 0.5569$.

Step 3. Rank all Q_a $(a = 1, 2, \ldots, 5)$ in the value of score functions $\Phi(\Upsilon_a)$ $(a = 1, 2, \ldots, 5)$ of the overall mFSs as $Q_2 \succ Q_5 \succ Q_1 \succ Q_4 \succ Q_3$.

Step 4. Select and return Q_2 as the suitable location.

This computation shows from ranking order of the alternatives that Q_2 is the most desirable location when mFDWA and mFDWG operators are used.

Sensitivity of the parameter $\varrho \in [1, 10]$ on the ranking order of the alternatives Q using, respectively, mFDWA and mFDWG operators is given in tables 2 and 3.

6.2 *Analysis on the effect of parameter ϱ on decision making results*

Here, the working parameter's operational behavior ϱ on MADM based on mFDWA and mFDWG operators is addressed in tables 2 and 3. Table 2 shows that when ϱ is changed for the mFDWA operator, the corresponding favorable alternative remains the same. Thus, when $\varrho = 1$, $Q_2 \succ Q_5 \succ Q_1 \succ Q_4 \succ Q_3$. When $\varrho = 2$, $Q_2 \succ Q_5 \succ Q_1 \succ Q_3 \succ Q_4$. When $\varrho = 3$, $Q_2 \succ Q_3 \succ Q_4 \sim Q_5 \succ Q_1$. For $\varrho = 4$, the ranking order is $Q_2 \succ Q_5 \succ Q_3 \succ Q_4 \succ Q_1$. For $5 \leq \varrho \leq 10$, $Q_2 \succ Q_4 \succ Q_3 \succ Q_1 \succ Q_5$. Therefore, we have seen that in all the cases, the favorable location is Q_2. In table 3, when $\varrho = 1$ is changed for the mFDWG operator, the ranking order is $Q_2 \succ Q_5 \succ Q_1 \succ Q_4 \succ Q_3$. Also, for $2 \leq \varrho \leq 10$ the ranking order is $Q_2 \succ Q_5 \succ Q_1 \succ Q_3 \succ Q_4$. Thus, Q_2 is again the best alternative for a suitable location. In both cases, the best choice is Q_2 though order sequences are different.

The proposed MADM problems based on mFDWA and mFDWG operators that show different values of working parameters could change corresponding ranking orders of the alternatives for the mFDWG operator, which is more responsive, for various discounts of parameters could not change raking forms corresponding to the mFDWG operator, which is less effective by ϱ for the so called MADM model.

6.3 *Comparative results*

The proposed study is compared to the present existing problems introduced by Waseem *et al* [27] in *m*-polar fuzzy environment. In this study, they have imposed *m*-polar fuzzy Hamacher weighted average (mFHWA) operator and mFHWG operator. In this comparison study, the evaluation matrix in the form of 3-polar fuzzy numbers is given in table 1. We apply mFHWA and mFHWG operators to aggregate 3-polar fuzzy information, given in table 4. Score values computed using definition 2.2 are provided in table 5, and their corresponding comparison ranking order of the alternatives is shown in table 6.

The proposed model's effectiveness is compared to that of the present method [27], which uses *m*-polar fuzzy Hamacher aggregation operators, and their justification with the existing operator is given in tables 7 and 8. It is noticed that the current models can describe fuzzy information without any difficulty. Still, it does not comfortably

Table 1. Three-polar fuzzy decision matrix.

	Q_1	Q_2	Q_3	Q_4	Q_5
g_1	(0.5,0.6,0.5)	(0.7,0.4,0.6)	(0.5,0.4,0.4)	(0.5,0.7,0.4)	(0.7,0.6,0.4)
g_2	(0.6,0.7,0.4)	(0.5,0.7,0.8)	(0.6,0.7,0.4)	(0.4,0.5,0.7)	(0.6,0.9,0.5)
g_3	(0.5,0.8,0.3)	(0.6,0.8,0.9)	(0.5,0.4,0.9)	(0.5,0.6,0.6)	(0.4,0.7,0.4)
g_4	(0.6,0.4,0.8)	(0.9,0.4,0.6)	(0.4,0.6,0.5)	(0.8,0.3,0.4)	(0.4,0.6,0.5)

Table 2. Effect of parameters on ranking orders using the mFDWA operator.

ϱ	$\Phi(\Upsilon_1)$	$\Phi(\Upsilon_2)$	$\Phi(\Upsilon_3)$	$\Phi(\Upsilon_4)$	$\Phi(\Upsilon_5)$	Ranking order
1	0.5978	0.7037	0.5673	0.5835	0.6202	$Q_2 \succ Q_5 \succ Q_1 \succ Q_4 \succ Q_3$
2	0.6308	0.7518	0.6281	0.6261	0.6431	$Q_2 \succ Q_5 \succ Q_1 \succ Q_3 \succ Q_4$
3	0.6539	0.7825	0.6566	0.6550	0.6550	$Q_2 \succ Q_3 \succ Q_4 \sim Q_5 \succ Q_1$
4	0.6694	0.8016	0.6734	0.6731	0.6748	$Q_2 \succ Q_5 \succ Q_3 \succ Q_4 \succ Q_1$
5	0.6802	0.8141	0.6841	0.6848	0.6705	$Q_2 \succ Q_4 \succ Q_3 \succ Q_1 \succ Q_5$
6	0.6882	0.8228	0.6915	0.6929	0.6746	$Q_2 \succ Q_4 \succ Q_3 \succ Q_1 \succ Q_5$
7	0.6943	0.8292	0.6971	0.6987	0.6779	$Q_2 \succ Q_4 \succ Q_3 \succ Q_1 \succ Q_5$
8	0.6990	0.8341	0.7015	0.7031	0.6804	$Q_2 \succ Q_4 \succ Q_3 \succ Q_1 \succ Q_5$
9	0.7028	0.8379	0.7049	0.7065	0.6824	$Q_2 \succ Q_4 \succ Q_3 \succ Q_1 \succ Q_5$
10	0.7059	0.8409	0.7077	0.7092	0.6841	$Q_2 \succ Q_4 \succ Q_3 \succ Q_1 \succ Q_5$

Table 3. The effect of parameter on ranking order using mFDWG operator.

ϱ	$\Phi(\Upsilon_1)$	$\Phi(\Upsilon_2)$	$\Phi(\Upsilon_3)$	$\Phi(\Upsilon_4)$	$\Phi(\Upsilon_5)$	Ranking order
1	0.5319	0.6003	0.4798	0.4920	0.5569	$Q_2 \succ Q_5 \succ Q_1 \succ Q_4 \succ Q_3$
2	0.5087	0.5717	0.4630	0.4604	0.5353	$Q_2 \succ Q_5 \succ Q_1 \succ Q_3 \succ Q_4$
3	0.4904	0.5542	0.4517	0.4389	0.5204	$Q_2 \succ Q_5 \succ Q_1 \succ Q_3 \succ Q_4$
4	0.4760	0.5429	0.4437	0.4246	0.5102	$Q_2 \succ Q_5 \succ Q_1 \succ Q_3 \succ Q_4$
5	0.4648	0.5352	0.4376	0.4147	0.5029	$Q_2 \succ Q_5 \succ Q_1 \succ Q_3 \succ Q_4$
6	0.4561	0.5298	0.4328	0.4078	0.4277	$Q_2 \succ Q_5 \succ Q_1 \succ Q_3 \succ Q_4$
7	0.4522	0.5257	0.4289	0.4021	0.4936	$Q_2 \succ Q_5 \succ Q_1 \succ Q_3 \succ Q_4$
8	0.4435	0.5226	0.4258	0.3979	0.4905	$Q_2 \succ Q_5 \succ Q_1 \succ Q_3 \succ Q_4$
9	0.4389	0.5201	0.4232	0.3946	0.4879	$Q_2 \succ Q_5 \succ Q_1 \succ Q_3 \succ Q_4$
10	0.4352	0.5181	0.4211	0.3919	0.4859	$Q_2 \succ Q_5 \succ Q_1 \succ Q_3 \succ Q_4$

Table 4. Aggregated values using mFHWA (mFHWG) operators on 3FNs.

Alternative	mFHWA	mFHWG
Q_1	(0.5520, 0.6243, 0.5353)	(0.5486, 0.6020, 0.4958)
Q_2	(0.7041, 0.5541, 0.7127)	(0.6627, 0.5154, 0.6859)
Q_3	(0.5134, 0.5446, 0.5032)	(0.5067, 0.5191, 0.4593)
Q_4	(0.5522, 0.5669, 0.5253)	(0.5199, 0.5344, 0.4986)
Q_5	(0.5928, 0.7372, 0.4514)	(0.5709, 0.6980, 0.4480)

Table 5. Score of the alternatives using mFHWA and mFHWG operators.

Alternative	mFHWA	mFHWG
Q_1	0.5705	0.5488
Q_2	0.6569	0.6213
Q_3	0.5204	0.4950
Q_4	0.5481	0.5176
Q_5	0.5938	0.5721

make the aggregation process of data flexible by a parameter of the proposed model. However our proposed method quickly describes fuzzy information and information aggregation process is made more flexible by a parameter. Therefore, our proposed MADM method for mFDWA and mFDWG operators investigated improved resilience in real utilization. Thus, the advanced aggregation operators implement a new flexible measure for DMs to control *m*-polar fuzzy MADM problems.

7. Conclusion

In this article we develop some new aggregation operators, namely mFDWA operator, mFDOWA operator, mFDHWA operator, mFDWG operator, mFDOWG operator, and mFDHWG operator. Idempotency, monotonicity, boundedness, and commutativity properties related to the mentioned operators are discussed here. We have utilized these operators to solve MADW problems with mFSs information. Finally, the feasibility and effectiveness of the proposed approach have been demonstrated by considering a numerical example. Some comparative studies have been done in this direction to show its efficiency. Also, the parameter's sensitivity will give various choices to the DMs to select their desirable alternatives. Therefore, our proposed model has the advantage of depicting more fuzzy information and can derive stable computing results. In further study the proposed model can be developed for other operators environment such as Bonferroni mean operator, Heronian mean operator, Interactive operator, Einstein operator, etc. in *m*-polar hesitant fuzzy TOPSIS-

Table 6. Ranking order of the alternatives.

Operators	Ranking orders
mFHWA	$Q_2 \succ Q_5 \succ Q_1 \succ Q_4 \succ Q_3$
mFHWG	$Q_2 \succ Q_5 \succ Q_1 \succ Q_4 \succ Q_3$

Table 7. Comparison of methods.

Operators	Ranking orders
[27] mFHWA	$Q_2 \succ Q_5 \succ Q_1 \succ Q_4 \succ Q_3$
[27] mFHWG	$Q_2 \succ Q_5 \succ Q_1 \succ Q_4 \succ Q_3$
Proposed mFDWA	$Q_2 \succ Q_5 \succ Q_1 \succ Q_4 \succ Q_3$
Proposed mFDWG	$Q_2 \succ Q_5 \succ Q_1 \succ Q_4 \succ Q_3$

Table 8. Characteristic comparisons with some of the existing methods.

Methods	Whether describing fuzzy information easier	Whether information aggregation more flexible by a parameter
Waseem *et al* [27]	√	×
Proposed method	√	√

based group decision making [28], and also can be applied in economic model, business and management areas, intelligent diagnosis, three-way decision making, and other environments with uncertainties.

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References

[1] Zadeh L A 1965 Fuzzy sets. *Information and Control* 8: 338–353
[2] Atanassov K T 1999 On intuitionistic fuzzy sets theory. *Studies in Fuzziness and Soft Computing* 283. Berlin–Heidelberg: Springer-Verlag
[3] Garg H 2018 Generalised Pythagorean fuzzy geometric interactive aggregation operators using Einstein operations and their application to decision making. *Journal of Experimental and Theoretical Artificial Intelligence* 30: 763–794
[4] Jana C, Pal M, Karaaslan F and Wang J Q 2020 Trapezoidal neutrosophic aggregation operators and its application in multiple attribute decision making process. *Scientia Iranica E* 27(3): 1655–1673
[5] Jana C and Pal M 2019 Assessment of enterprise performance based on picture fuzzy Hamacher aggregation operators. *Symmetry* 11(1): 75, https://doi.org/10.3390/sym11010075
[6] Zhao H, Xu Z S, Ni M F and Liu S S 2010 Generalized aggregation operators for intuitionistic fuzzy sets. *International Journal of Intelligent Systems* 25(1): 1–30
[7] Zhao X F and Wei G W 2013 Some intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute decision making. *Knowledge-Based Systems* 37: 472–479
[8] Xu Z S 2007 Intuitionistic fuzzy aggregation operators. *IEEE Transaction on Fuzzy Systems* 15(6): 1179–1187
[9] Xu Z S and Yager R R 2006 Some geometric aggregation operators based on intuitionistic fuzzy sets. *International Journal of General Systems* 35(4): 417–433
[10] Yager R R 1998 On ordered weighted averaging aggregation operators in multicriteria decision making. *IEEE Transactions on Systems, Man, and Cybernetics* 18(1): 183–190
[11] Yager R R and Kacprzyk J 1997 *The ordered weighted averaging operators: theory and applications*. Boston, M.A.: Kluwer
[12] Zhang W R 1994 Bipolar fuzzy sets and relations: a computational framework for cognitive modelling and multiagent decision analysis. In: *Proceedings of the IEEE Conference*, pp. 305–309
[13] Zhang W R 1998 Bipolar fuzzy sets. In: *Proceedings of FUZZYIEEE*, pp. 835–840
[14] Jana C, Pal M and Wang J Q 2019a Bipolar fuzzy Dombi aggregation operators and its application in multiple attribute decision making process. *Journal of Ambient Intelligence and Humanized Computing* 10: 3533–3549
[15] Jana C, Pal M and Wang J Q 2020 Bipolar fuzzy Dombi prioritized aggregation operators in multiple attribute decision making. *Soft Computing* 24: 3631–3646
[16] Wei G W, Alsaedi F E, Tasawar H and Alsaedi A 2018 Bipolar fuzzy Hamacher aggregation operators in multiple attribute decision making. *International Journal of Fuzzy Systems* 20(1): 1–12
[17] Gao H, Wei G W and Huang Y H 2018 Dual hesitant bipolar fuzzy Hamacher prioritized aggregation operators in multiple attribute decision making. *IEEE Access* 6(1): 11508–11522
[18] Xu X R and Wei G W 2017 Dual hesitant bipolar fuzzy aggregation operators in multiple attribute decision making. *International Journal of Knowledge-Based and Intelligent Engineering Systems* 21: 155–164
[19] Jana C, Pal M and Wang J Q 2019 A robust aggregation operator for multi-criteria decision-making method with bipolar fuzzy soft environment. *Iranian Journal of Fuzzy Systems* 16(6): 1–16
[20] Chen J, Li S, Ma S and Wang X 2014 *m*-polar fuzzy sets: an extension of bipolar fuzzy sets. *Scientific World Journal* 2014: 416530
[21] Al-Masarwah A and Ahmad A G 2019 *m*-Polar fuzzy ideals of *BCK/BCI*-algebras. *Journal of King Saud University-Science* 31(4): 1220–1226

[22] Akram M, Farooq A and Shum K P 2016 On *m*-polar fuzzy Lie subalgebras. *Italian Journal of Pure and Applied Mathematics* 36: 445–454

[23] Ghorai G and Pal M 2016 Some properties of *m*-polar fuzzy graphs. *Pacific Science Review A: Natural Science and Engineering* 18(1): 38–46

[24] Khameneh A Z and Kilicman A 2018 *m*-polar fuzzy soft weighted aggregation operators and their applications in group decision-making. *Symmetry* 10: 636

[25] Akram M and Waseem N 2019 Novel approach in decision-making with *m*-polar fuzzy ELECTRE-I. *International Journal of Fuzzy Systems* 21: 1117–1129

[26] Akram M 2019 *m*-polar fuzzy graphs. In: *Studies in Fuzziness and Soft Computing*. Cham, Switzerland: Springer, p. 371

[27] Waseem N, Akram M and Alcantud J C R 2019 Multi-attribute decision-making based on *m*-polar fuzzy Hamacher aggregation operators. *Symmetry* 11: 1498

[28] Akram M, Adeel A and Alcantud J C R 2019 Multi-criteria group decision-making using an *m*-polar hesitant fuzzy TOPSIS approach. *Symmetry* 11(6): 795

[29] Akram M, Shumaiza and Alcantud J C R 2020 An *m*-polar fuzzy PROMETHEE approach for AHP-assisted group decision-making. *Mathematical and Computational Applications* 25(2): 26

[30] Dombi J 1982 A general class of fuzzy operators, the DeMorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators. *Fuzzy Sets and Systems* 8: 149–163

[31] Liu P D, Liu J L and Chen S M 2017 Some intuitionistic fuzzy Dombi Bonferroni mean operators and their application to multi-attribute group decision making. *Journal of the Operational Research Society*, https://doi.org/10.1057/s41274-017-0190-y

[32] Jana C, Senapati T, Pal M and Yager R R 2019 Picture fuzzy Dombi aggregation operators: application to MADM process. *Applied Soft Computing* 74(1): 99–109

[33] Jana C, Senapati T and Pal M 2019 Pythagorean fuzzy Dombi aggregation operators and its applications in multiple attribute decision-making. *International Journal of Intelligent Systems* 34(9): 2019–2038

[34] Jana C, Muhiuddin G and Pal M 2019 Some Dombi aggregation of *q*-rung orthopair fuzzy numbers in multiple-attribute decision making. *International Journal of Intelligent Systems* 34(12): 3220–3240