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Layered Marangoni convection with the Navier slip condition

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Abstract. A new exact solution to the problem of Marangoni layered convection is obtained. This solution describes a layered steady-state flow of a viscous incompressible fluid at varying gradients of temperature and pressure. The velocity components depend only on the transverse coordinate; the temperature and pressure fields are three-dimensional. The Marangoni effect is observed on the upper free surface of the fluid layer. On the lower solid surface of the fluid layer, three different cases of defining boundary conditions are considered: the no-slip condition, the perfect slip condition and the Navier slip condition. The obtained exact solution is determined by the interaction of three flows: a flow caused by pressure drop (the Poiseuille flow), a flow caused by heating/cooling and the effect of the gravity force (the thermogravitational flow), and a flow caused by heating/cooling and the fluid surface tension effect (the thermocapillary flow). The obtained exact solutions in the case of each of the three types of boundary conditions specified on the lower surface are analyzed in detail. It has been proved that, when certain ratios of the boundary value problem parameters are fulfilled, the velocity components may acquire stagnation points, this being indicative of the presence of counterflow areas in the fluid layer under consideration. In particular, the presence of up to two stagnation points in each of the two longitudinal velocity components may cause a stratification of the velocity field in more than two regions. The obtained exact solution of the Marangoni layered convection problem can describe flows in thin films through the variation of the geometric anisotropy factor.

Keywords. Exact solution; Marangoni flow; viscous flow; vortex; slipping condition.

1. Introduction

Currently, the problem of determining characteristics and finding exact solutions for describing convective processes in fluids and media close to fluids continues to evoke constant interest among continuum mechanics researchers. In particular, subject matters concerning the existence of exact solutions to boundary value problems associated with the most well-known hydrodynamic equations, that is, the Navier–Stokes equations, are studied. Adding the heat equation and the incompressibility equation to the Navier–Stokes equations and taking into account the temperature dependence of fluid density in the model, we obtain a system of heat convection equations in the Oberbeck–Boussinesq approximation [1–4].

Convective processes caused by linear temperature distribution at the boundary were first described by Ostroumov [5] for regions with cylindrical symmetry. A solution describing advection caused by thermocapillary forces on the free surface was first obtained by Birikh [6]. Later, a wider class of exact solutions for describing convective processes in an incompressible fluid was considered [7, 8]. In this solution family, hydrodynamic fields are described by functions that are linear in terms of horizontal coordinates. This class of solutions was first proposed by Lin [9] for problems of magnetohydrodynamics. The Lin class was rediscovered for convection problems in [7, 10] and modified to describe thermal diffusion in [11]. A review of exact solutions to the Navier–Stokes equations and their classification for the case of a linear dependence of the velocity components on two spatial variables can be found in [12].

A distributed horizontal temperature gradient was considered earlier in the boundary value problems on the free surfaces of a fluid layer, but a similar pressure gradient was not taken into account [13]. The exact solution of the plane boundary value problem of the convective fluid flow in a two-layer system that takes into account the longitudinal

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pressure gradients at the boundary was investigated in [14]. In this paper, we construct a solution taking into account longitudinal pressure variation to describe the convective fluid flow in a three-dimensional case.

The choice of boundary conditions is a separate issue when problems for describing the flow of a fluid layer contacting a fixed solid surface are considered. By the mid-1900s, for solving such problems, it had become a practice to use the no-slip conditions at fluid–solid interfaces [15]. The no-slip condition is known to make the fluid velocity field at a fixed boundary vanish. However, experimental data [16, 17] often show the violation of the no-slip condition on a solid wall and, as a result, the slip condition is implemented. It should be noted that Navier used the slip condition to model fluid flow processes [18]. In recent decades, boundary conditions based on the slipping hypothesis have been gaining acceptance. According to this hypothesis, the tangential velocity component is non-zero and related to the stress tensor.

There are currently a great number of mathematical papers studying problems of hydrodynamics of Newtonian fluids, which take into account the slip conditions on solid boundaries, e.g. [16, 19–22]. Rajagopal [23] notes that it is of particular importance to take into account the effect of fluid slippage along the boundary.

In this paper we consider the Benard–Marangoni convection on the free surface of a viscous incompressible fluid layer with non-zero temperature and pressure gradients, specifying different types of boundary conditions (no-slip and slip) on a solid surface. Earlier the authors attempted to take into account the combination of the afore-described boundary conditions, but for unidirectional flows [24]. Therefore, the solution presented in this paper can be regarded as a generalization of the already available results to the case of non-dimensional flows. The combination "no-slip condition + Marangoni condition" was studied, for example, in the articles [25–36]. However, the authors are not aware of a single study in which a comprehensive investigation of all three variants of the Navier condition in combination with the thermocapillary effect is carried out. The obtained exact solution of the convection problem can describe flows in thin films [37–40] through the variation of the geometric anisotropy factor.

2. Problem statement

To solve the problem of the convective flow of a viscous incompressible fluid, we write a system of equations consisting of the Navier–Stokes equation in the Boussinesq approximation, the heat equation and the incompressibility equation in the coordinate form:

$$
\begin{aligned}
&\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z} \\
&\quad = -\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right);\\
&\frac{\partial V_y}{\partial t} + V_x\frac{\partial V_y}{\partial x} + V_y\frac{\partial V_y}{\partial y} + V_z\frac{\partial V_y}{\partial z} \\
&\quad = -\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right);\\
&\frac{\partial V_z}{\partial t} + V_x\frac{\partial V_z}{\partial x} + V_y\frac{\partial V_z}{\partial y} + V_z\frac{\partial V_z}{\partial z} \\
&\quad = -\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right) + g\beta T;
\end{aligned}
\tag{1}
$$

$$
\frac{\partial T}{\partial t} + V_x\frac{\partial T}{\partial x} + V_y\frac{\partial T}{\partial y} + V_z\frac{\partial T}{\partial z} = \chi\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right);
\tag{2}
$$

$$
\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0.
\tag{3}
$$

Here V_x, V_y and V_z are the velocities parallel to the corresponding coordinate axes of the rectangular Cartesian coordinate system $Oxyz$; $P = P(x, y, z)$ is the deviation of pressure from hydrostatic, taken relative to the constant average density of the fluid ρ; T is the deviation from the average temperature; ν and χ are the coefficients of kinematic viscosity and of thermal diffusivity of the fluid, respectively; g is the acceleration of gravity and β is the temperature coefficient of volume expansion for the fluid.

In the system of equations (1) the Boussinesq hypothesis is adopted, which makes it possible to relate the specific mass force of gravity to the temperature field by means of the coefficient of volume expansion, and this relationship is represented by a linear dependence. In addition, taking into account the law of conservation of mass of a moving volume of fluid in the form (3) limits the range of applicability of model (1)–(3) since the concept of incompressibility of a fluid is used only for non-fast flowing fluids (the average flow velocity is about 1 m/s). However, taking into account the fact that, for example, the open ocean currents with a speed of 5.5 km/h (which is approximately equal to 1.53 m/s) and more are considered strong, the incompressible fluid approximation turns out to be applicable to describe the vast majority of ocean currents.

Let us make the following assumptions regarding the motion of a viscous fluid described by the afore-mentioned system of equations (1)–(3). We will further consider only stable flows. This assumption will make it possible to consider all the functions appearing in systems (1)–(3) to be independent of time, and, therefore, will make it possible to neglect the partial derivative with respect to time, leaving only the terms of the convective derivative in the Navier–Stokes equations (1) and the heat equation (2). We also assume that the fluid flow occurs inside an infinite horizontal layer (see figure 1) of constant thickness h and is sheared; moreover the fluid layers are directed parallel to the horizontal plane Oxy. This assumption allows us to consider the vertical velocity V_z identically equal to zero, which in turn also leads to a simplification of the form of system (1)–(3).

The exact solution of the system (1)–(3) is sought in the form [41–43]

$$\begin{aligned} &V_x = U(z); \quad V_y = V(z); \quad V_z = 0; \\ &P = P_0(z) + P_1(z)x + P_2(z)y; \\ &T = T_0(z) + T_1(z)x + T_2(z)y. \end{aligned} \tag{4}$$

Note that solutions of the form (4) describe three-dimensional (in coordinates) distributions of the temperature and pressure fields, which make it possible to take into account the inhomogeneity of the distribution of these physical characteristics within each sub-layer. We also note that the dependence of the velocities U, V (as well as the coefficients of linear forms describing the pressure P and the temperature T) on the vertical coordinate z can have an arbitrary character, i.e. be polynomial, exponential, trigonometric or a general function.

In addition the fact that real processes in a fluid are anisotropic gives grounds to use the thin layer approximation, i.e. consider the vertical size h (see figure 1) to be much less than the characteristic horizontal size of the layer. This assumption allows us to neglect the deformation of the upper free surface of the considered fluid layer.

We substitute the class of exact solutions (4) into the nonlinear system (1)–(3). The resulting expressions projected onto the axes Ox, Oy and Oz are written in the following form:

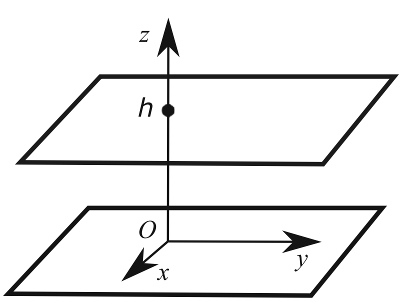

Figure 1. Fluid flow scheme.

$$\begin{aligned} &\nu\frac{\partial^2 U}{\partial z^2} = P_1; \quad \nu\frac{\partial^2 V}{\partial z^2} = P_2; \\ &\frac{\partial P_0}{\partial z} + \frac{\partial P_1}{\partial z}x + \frac{\partial P_2}{\partial z}y = g\beta(T_0 + T_1 x + T_2 y); \\ &UT_1 + VT_2 = \chi\left(\frac{\partial^2 T_0}{\partial z^2} + \frac{\partial^2 T_1}{\partial z^2}x + \frac{\partial^2 T_2}{\partial z^2}y\right). \end{aligned} \tag{5}$$

Let us present similar terms in relations (5) and write the resulting expressions as the following linear forms:

$$A_k + B_k x + C_k y = 0.$$

Let us further use the method of undefined coefficients, and equate the expressions B, C in front of the independent variables x, y, as well as the homogeneous term A to zero. As a result of this operation we obtain the following system, consisting of eight nonlinear equations, for the determination of the eight unknown functions (the equations in the system are written in the order of integration):

$$\begin{aligned} &T_1'' = 0; \quad P_1' = g\beta T_1; \\ &T_2'' = 0; \quad P_2' = g\beta T_2; \\ &\nu U'' = P_1; \quad \nu V'' = P_2; \\ &\chi T_0'' = UT_1 + VT_2; \quad P_0' = g\beta T_0. \end{aligned} \tag{6}$$

The resulting system (6) is a 13th-order system; therefore, to determine the integration constants, it is necessary to set 13 boundary conditions.

3. Boundary value problem

We assume that the lower boundary of the infinite horizontal layer of the fluid given by the equation $z = 0$ is absolutely solid and fixed. The temperature at the lower boundary is equal to the zero reference value. Besides, the Navier slip condition is set on the lower boundary.

In view of the structure (4) of the selected generalized class of solutions, the Navier slip conditions are written in the following form:

$$\begin{aligned} &T_0(0) = T_1(0) = T_2(0) = 0; \\ &\alpha\frac{\partial U}{\partial z}\Big|_{z=0} = U(0); \quad \alpha\frac{\partial V}{\partial z}\Big|_{z=0} = V(0). \end{aligned} \tag{7}$$

Here, α is a dimensional slip factor (slip length).

The values of the temperature and pressure, as well as the horizontal temperature and pressure gradients, are determined on the upper (free) surface $z = h$. Besides, tangential stresses induced by the thermocapillary effect are specified on the upper boundary of the fluid layer. According to the structure of the class of solutions (4), these conditions take the following form:

$$T_0(h) = 0; \quad T_1(h) = A; \quad T_2(h) = B;$$
$$P_0(h) = S_0; \quad P_1(h) = S_1; \quad P_2(h) = S_2; \tag{8}$$
$$\eta \frac{\partial U}{\partial z}|_{z=h} = -\sigma T_1(h); \quad \eta \frac{\partial V}{\partial z}|_{z=h} = -\sigma T_2(h).$$

Here, σ and η are the temperature coefficients of surface tension and dynamic viscosity, respectively. Note that specifying a non-uniform pressure distribution at the upper boundary of the layer under consideration allows one to construct generalizations of the classical unidirectional Poiseuille flow [44] caused by the pressure drop when passing from one region of fluid flow to another. From the point of view of the physics of the process such a boundary condition arises, for example, when a fluid flows out through a slot with a slowly varying gap width or at the interface between a fluid and a gas.

The solution of the equation system (6), satisfying the boundary conditions (7), (8), is polynomial, i.e.

$$\begin{aligned}
T_1 &= \frac{Az}{h}; \quad P_1 = \frac{Ag\beta}{2h}\left(z^2 - h^2\right) + S_1;\\
T_2 &= \frac{Bz}{h}; \quad P_2 = \frac{Bg\beta}{2h}\left(z^2 - h^2\right) + S_2;\\
U &= \frac{S_1}{2\nu}\left[z^2 - 2h(z+\alpha)\right] + \frac{Ag\beta}{24\nu h}\left[z^4 - 6h^2z^2 + 8h^3(z+\alpha)\right]\\
&\quad - \frac{A\sigma}{\eta}(z+\alpha);\\
V &= \frac{S_2}{2\nu}\left[z^2 - 2h(z+\alpha)\right] + \frac{Bg\beta}{24\nu h}\left[z^4 - 6h^2z^2 + 8h^3(z+\alpha)\right]\\
&\quad - \frac{B\sigma}{\eta}(z+\alpha);\\
T_0 &= \frac{AS_1 + BS_2}{120h\nu\chi}\left[3z^5 - 10hz^4 - 20hz^3\alpha + h^3z(7h + 20\alpha)\right]\\
&\quad + \frac{(A^2+B^2)g\beta}{5040h^2\nu\chi}\left[5z^7 - 63h^2z^5 + 140h^3z^4\right.\\
&\quad \left. + 280h^3z^3\alpha - 2h^5z(41h + 140\alpha)\right] +\\
&\quad \frac{(A^2+B^2)\sigma}{12h\chi\eta}(h-z)z\left[h^2 + (h+z)(z+2\alpha)\right];\\
P_0 &= S_0 - \frac{(AS_1 + BS_2)g\beta}{240h\nu\chi}(z-h)^2\left[4h^4 - z^4 + \right.\\
&\quad \left. 2hz^2(z+5\alpha) + 2h^3(4z+5\alpha) + 5h^2z(z+4\alpha)\right]\\
&\quad - \frac{(A^2+B^2)g\beta\sigma}{120h\eta\chi}(z-h)^2\left[3h^3 + 2hz(2z+5\alpha) + \right.\\
&\quad \left. + z^2(2z+5\alpha) + h^2(6z+5\alpha)\right]\\
&\quad + \frac{(A^2+B^2)g^2\beta^2}{40320h^2\nu\chi}(z-h)^2\left[183h^6 - 69h^2z^4 + 10hz^5 + \right.\\
&\quad 5z^6 + h^4z(221z + 1120\alpha) + h^5(366z + 560\alpha)\\
&\quad \left. + 4h^3z^2(19z + 140\alpha)\right].
\end{aligned} \tag{9}$$

In what follows, we focus on the investigation of the velocity field (4) of the convective fluid flow. It follows from the analysis of the formulas (9) that each velocity component is determined by the interaction of three flows: a flow caused by pressure drop (the Poiseuille flow), a flow caused by heating/cooling and the effect of the gravity force (the thermogravitational flow) and a flow caused by heating/cooling and the fluid surface tension effect (the thermocapillary flow). At a certain value of α, each of these flows separately can admit the appearance of a stagnation point. The overlap of these flows significantly complicates the topology of the velocity field.

The velocity components U and V are proportional (velocity $U = \gamma V$) when the following double equality holds:

$$\frac{S_1}{S_2} = \frac{A}{B} \neq 0.$$

The flow becomes unidirectional, and the velocity direction is determined by the angle $\gamma = arctg(U/V)$ measured from the axis Ox. In this case, the dimension of the velocity analysis problem can be reduced. Also note that, according to (9), with a simultaneous substitution of S_2 for S_1 and B for A, an expression for the velocity component V can be obtained from the expression for U. Therefore, the results of studying the velocity component U can be easily extended to the case of considering the velocity component V. In view of this reasoning, we will focus on studying the behavior of the velocity $V_x = U$.

Note that, in the isothermal case $(A = B = 0)$, the velocity component U is determined only by the Poiseuille flow

$$U = \frac{S_1}{2\nu}\left[z^2 - 2h(z+\alpha)\right].$$

Moreover, the Poiseuille flow does not presume the occurrence of stagnation points.

Let us further study the effect of slip length α and layer thickness h on the velocity profile in the non-isothermal case. For this purpose we consider the limiting case $S_1 = 0$ for which, by virtue of (9), the velocity component U is described by the expression

$$U = A\left[\frac{g\beta}{24\nu h}\left[z^4 - 6h^2z^2 + 8h^3(z+\alpha)\right] - \frac{\sigma}{\eta}(z+\alpha)\right]. \tag{10}$$

The form of the velocity component (10) clearly illustrates that the absence of a Poiseuille-type flow leads to the fact that, as distinct from the solution (9), the temperature gradient A has no effect on the position of the critical point in the fluid layer for the velocity component. Note that, even when $\sigma = 0$, the velocity expression (10) is able to describe the appearance of stagnation points. Figure 2 shows characteristic profiles of the velocity U for various

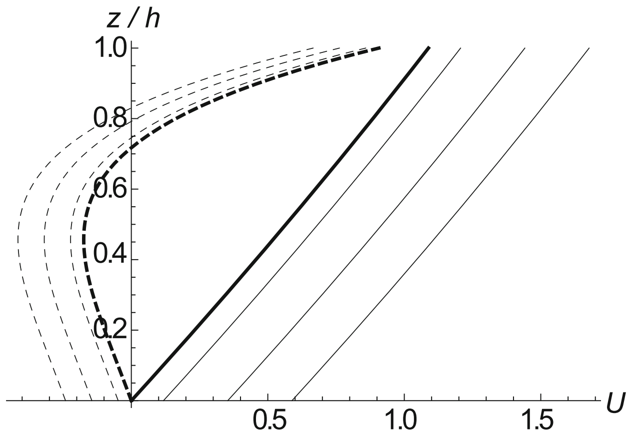

Figure 2. Profiles of the velocity U in a particular case $S_1 = 0$ m/s^2 with different values of h: $h = 10$ m (solid lines) and $h = 0.1$ m (dashed lines) (four slip length variants are taken in the calculations: $\alpha_0 = 0$ m (thick curves), $\alpha_1 = 0.1$ m, $\alpha_2 = 0.3$ m and $\alpha_3 = 0.5$ m).

values of h (the average values of the parameters for water at a temperature of 20°C are considered in the calculations).

The number of stagnation points for a fixed value of α is different for different values of h. This is due to the change in the shape of the velocity component profile. There exists such a value of layer thickness above which the linear term $A\sigma(z+\alpha)/\eta$ dominates in the expression (10). Otherwise, the velocity component profile is determined by the second term. For the selected fluid (water at 20°C) the value of the critical height is comparable to unity in the order of magnitude, but this value may differ for other fluids and for other temperature conditions.

4. Velocity field analysis for the no-slip condition at the lower boundary

In the limiting case $\alpha = 0$, the Navier condition in the boundary conditions (7) degenerates into the classical no-slip condition $U(0) = 0$. As a result, the velocity U takes the following form:

$$U = z\left[\frac{S_1}{2\nu}(z-2h) + \frac{Ag\beta}{24h\nu}\left(z^3 - 6h^2 z + 8h^3\right) - \frac{A\sigma}{\eta}\right]. \tag{11}$$

As mentioned earlier, each velocity component is determined by the interaction of three flows. The case of the absence of the Poiseuille flow (the longitudinal pressure gradient at the upper boundary is zero) was discussed in detail in [45]. The possibility of the existence of counterflows induced in a fluid layer by the interaction of the thermocapillary and thermogravitational flows was demonstrated.

Next, we investigate the behavior of the fluid flow in the horizontal layer under study additionally taking into account the Poiseuille flow. The analysis of the structure of expression (11) allows us to conclude that the contribution of the Poiseuille flow described by the monotonic polynomial $(z-2h)$ is that, in particular, the number of stratification points in the velocity component U increases by one. Thus, the total number of stagnation points in the velocity component defined by equation (11) can reach two (see figure 3).

The layer thickness h affects the contribution of the nonlinear term to the final velocity (11) in the same way as it does in expression (10). Figure 4 shows the profiles of the velocity components U for different values of fluid layer thickness h.

It is obvious that the profiles are nonlinear and that there is no symmetry in their position relative to the axis Oz. In addition, if the profile shape at a specific layer thickness h is known, it is easy to evaluate the pressure gradient at which the velocity component U has a stagnation point in the fluid layer under study.

5. Boundary value problem with perfect slip conditions at the lower boundary

The boundary conditions for the velocity components take the following form in the case of perfect slip at the lower boundary:

$$\frac{\partial U}{\partial z}\Big|_{z=0} = 0;\ \frac{\partial V}{\partial z}\Big|_{z=0} = 0;$$
$$\eta\frac{\partial U}{\partial z}\Big|_{z=h} = -\sigma T_1(h);\ \eta\frac{\partial V}{\partial z}\Big|_{z=h} = -\sigma T_2(h).$$

All these boundary conditions are Neumann-type boundary

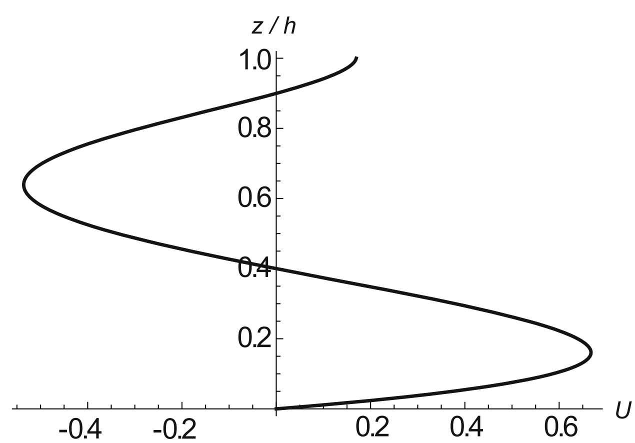

Figure 3. Profile of the velocity component U in the limiting case of two stagnation points.

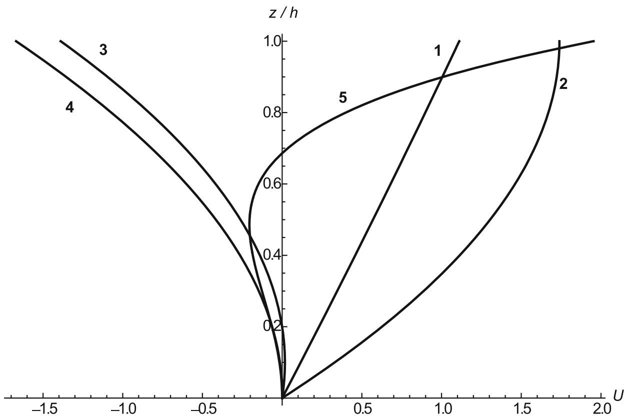

Figure 4. Profiles of the velocity component U for fixed $S_1 = -0.7$ m/s^2 and different values of h: (curve 1—$h = 10$ m, curve 2—$h = 1$ m, curve 3—$h = 0.1$ m, curve 4—$h = 0.01$ m, curve 5—$h = 0.0001$ m).

conditions; therefore, the question of their consistency arises. It can be proved that, if the condition

$$S_1 = \frac{g\beta\eta h^2 - 3\nu\sigma}{3h\eta}A \tag{12}$$

is met, the velocity component U can be determined with an accuracy up to a constant term as

$$U = \frac{1}{24h\nu}\left[Ag\beta z^4 + \left(12hS_1 - 6Ag\beta h^2\right)z^2\right] + C. \tag{13}$$

The compatibility condition (12) imposed on the boundary conditions means that, if the equality (12) holds, the stated boundary value problem has a solution, but a non-unique one, since the velocity component U depends on the constant term C. It is necessary to add another boundary condition for the velocity component to be determined uniquely. This condition should be either Dirichlet-type or Robin-type, since only these boundary conditions include not only the derivative but also the function itself (the velocity component in this case).

As an additional condition, we consider the fluid flow rate through the layer thickness:

$$Q = \int_0^h U dz. \tag{14}$$

If $Q = 0$, the boundary value problem solution asymptotically describes the flow in a closed layer. Such flows include, for example, the distribution of a viscous adhesive (with the properties of a non-Newtonian fluid) in a sandwich composite used for a tighter adhesive bond between the layers of the composite. Note that the flow rate value is related to the average fluid flow velocity. The substitution of solution (13) into condition (14) allows us to define the constant C as

$$C = \frac{-20h^3S_1 + 9Ag\beta h^4 + 120Q\nu}{120h\nu}.$$

In this case, the velocity component U has the following form:

$$U = \frac{1}{120h\nu}\left[5Ag\beta z^4 + \left(60hS_1 - 30Ah\beta h^2\right)z^2 + \left(-20h^3S_1 + 9Ag\beta h^4 + 120Q\nu\right)\right]. \tag{15}$$

Let us now analyze the behavior of velocity component (15) with an arbitrary fluid flow rate Q through the layer thickness. Let us begin with the particular case when the temperature gradient along the longitudinal coordinate at the upper boundary of the fluid layer is equal to zero ($A = 0$). In this case, the velocity component expression (15) can be written as

$$U = \frac{1}{6h\nu}\left[3hS_1z^2 - h^3S_1 + 6Q\nu\right].$$

Then, a single stagnation point can exist only when the following inequality is fulfilled:

$$\left(-h^3S_1 + 6Q\nu\right)\left(h^3S_1 + 3Q\nu\right) < 0.$$

The return flow is not observed if the longitudinal pressure gradient is zero.

If the temperature gradient is nonzero ($A \neq 0$), we can represent the velocity component U(15) as

$$U = \frac{Ag\beta}{24h\nu}\left[z^4 + \left(\frac{12h}{g\beta}\frac{S_1}{A} - 6h^2\right)z^2 + \left(-\frac{4h^3}{g\beta}\frac{S_1}{A} + \frac{9}{5}h^4 + \frac{24\nu}{g\beta}\frac{Q}{A}\right)\right]. \tag{16}$$

Next, we introduce the dimensionless vertical coordinate $Z = z/h \in [0, 1]$. The velocity component U (16) then has the following form:

$$\begin{aligned} U &= \frac{Ag\beta h^3}{24\nu}\left[Z^4 + 2\left(\frac{6}{g\beta h}\frac{S_1}{A} - 3\right)Z^2 + \left(-\frac{4}{g\beta h}\frac{S_1}{A} + \frac{9}{5} + \frac{24\nu}{g\beta h^4}\frac{Q}{A}\right)\right] \\ &= \frac{Ag\beta h^3}{24\nu}\left[\left(Z^2 + \frac{6}{g\beta h}\frac{S_1}{A} - 3\right)^2 - b\right], \end{aligned} \tag{17}$$

where

$$b = \frac{4\left(3Ag\beta(3Ag\beta h^4 - 10Q\nu) - 40Ag\beta h^3S_1 + 45h^2S_1^2\right)}{5A^2g^2\beta^2h^4}.$$

The study of the spectral properties of the polynomial (17) allows us to conclude that the number of zeros in this polynomial in the interval $(0, 1)$ does not exceed two. Consequently, the velocity component U can have two

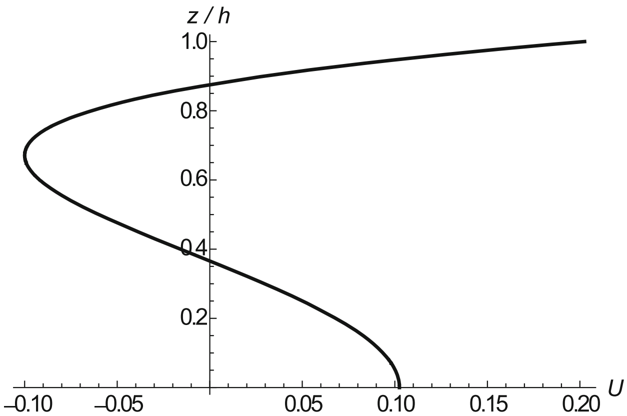

Figure 5. Profile of the velocity U with perfect slip at the lower boundary of the layer.

stagnation points inside the fluid layer under study (see figure 5).

Note that, due to the structure of velocity class (4), the vorticity components

$$\Omega_x = -\frac{dV}{dz} = -\frac{1}{h}\frac{dV}{dZ}; \quad \Omega_y = \frac{dU}{dz} = \frac{1}{h}\frac{dU}{dZ}$$

coincide with tangential stresses with an accuracy up to a constant factor:

$$\tau_{yz} = \eta\frac{dV}{dz} = \frac{\eta}{h}\frac{dV}{dZ}; \quad \tau_{xz} = \eta\frac{dU}{dz} = \frac{\eta}{h}\frac{dU}{dZ}.$$

The tangential stresses can change their sign through the layer thickness. Indeed, the stress

$$\tau_{xz} = \frac{\eta}{h}\frac{dU}{dZ} = \frac{Ag\beta h^2\eta}{6\nu}\left(Z^2 + \frac{6}{g\beta h}\frac{S_1}{A} - 3\right)Z$$

can vanish inside the layer at only one point

$$Z = \sqrt{3 - \frac{6}{g\beta h}\frac{S_1}{A}},$$

if $g\beta h/3 < S_1/A < g\beta h/2$. The tangential stress changes its type at this point (from tensile to compressive), and the vortex Ω_y changes its direction to the opposite. As this takes place, the velocity component U can have one or two stagnation points or none at all.

If the condition $g\beta h/3 < S_1/A < g\beta h/2$ is not fulfilled the vortex direction remains constant, and the stress τ_{xz} retains its type through the entire layer thickness. Herewith, the velocity component U can change its sign once inside the fluid layer.

6. Boundary value problem with the Navier slip condition at the lower boundary

Let us study the properties of the exact solution (9) for an arbitrary $\alpha \in (0, \infty)$. Here, we reduce the obtained solutions (9) to a dimensionless form to analyze the velocity field. We choose, respectively, h, l and $g\beta Al^3/\nu$ as the characteristic scales of the vertical length, the horizontal length and the velocity and introduce new dimensionless variables as follows: $X = x/l, Y = y/l, Z = z/h$. As a result, in view of solution (9), the velocities can be written in a dimensionless form as

$$U = \frac{\delta^2}{2}\frac{Ga}{Gr}\left[Z^2 - 2\left(Z + \frac{a}{\delta}\right)\right] + \frac{\delta^3}{24}\left[Z^4 - 6Z^2 + 8\left(Z + \frac{a}{\delta}\right)\right] - \frac{\delta Mg}{GrPr}\left(Z + \frac{a}{\delta}\right);$$
$$V = \frac{\delta^2\xi}{2}\frac{Ga}{Gr}\left[Z^2 - 2\left(Z + \frac{a}{\delta}\right)\right] + \frac{\delta^3\zeta}{24}\left[Z^4 - 6Z^2 + 8\left(Z + \frac{a}{\delta}\right)\right] - \frac{\delta\zeta Mg}{GrPr}\left(Z + \frac{a}{\delta}\right). \quad (18)$$

Here,

$$Ga = \frac{S_1 l^3}{\nu^2}, \; Gr = \frac{g\beta Al^4}{\nu^2}, \quad Mg = \frac{A\sigma l^2}{\eta\chi}, \; Pr = \frac{\nu}{\chi}$$

are the Galileo, Grashof, Marangoni and Prandtl dimensionless numbers, respectively; $\zeta = B/A, \xi = S_2/S_1$ are the ratios of the longitudinal components of the temperature and pressure gradients, respectively; $a = \alpha/l$ is a dimensionless parameter characterizing the slip length relative to the thickness of the fluid layer under study; $\delta = h/l$ is the ratio of the vertical to horizontal characteristic dimension.

Each of the three terms of the velocity component U describes the characteristic flows in the fluid layer. We denote these terms as follows:

$$U_1 = \frac{\delta^2}{2}\frac{Ga}{Gr}\left[Z^2 - 2\left(Z + \frac{a}{\delta}\right)\right];$$
$$U_2 = \frac{\delta^3}{24}\left[Z^4 - 6Z^2 + 8\left(Z + \frac{a}{\delta}\right)\right];$$
$$U_3 = \frac{\delta Mg}{GrPr}\left(Z + \frac{a}{\delta}\right).$$

The first term U_1 describes the Poiseuille flow caused by pressure drop, the second term U_2 describes the thermogravitational flow and the third term U_3 describes the thermocapillary flow. All the terms in the expressions (18) are monotonic functions. We study these flows separately and their interaction for the existence of stagnation points.

In the interval $Z \in (0, 1)$ the functions U_1, U_2 and U_3 cannot take the zero value. The polynomial (18) can have up to two zeros in the interval $(0, 1)$ depending on the values of the dimensionless similarity numbers Ga, Gr, Mg, Pr, i.e. the velocity component U can have two stagnation points in the layer $(0, 1)$. Similar conclusions are valid for the velocity component V.

As an example, figure 6 shows the profiles of the functions U (solid line) and V (dashed line) for $\delta = 0.0843433$, $a = 0.00843433$, $Ga = -1.64194$, $Gr = -49.889$, $Pr = 6.7$, $Mg = 0.18$, $\xi = 0.882$, $\zeta = 0.85$. Figures 7 and 8 show the stream lines and the hodograph of the velocity vector V, respectively, with the same values of the parameters. Figures 6–8 illustrate the counterflow areas in the fluid layer under study.

We give a physical interpretation of the obtained solution (18). Under certain conditions for the similarity numbers Mg, Pr, Ga and Gr, the velocity components U and V can vanish at the boundaries $Z = 0$ and $Z = 1$ of the fluid layer. Thus, the velocity component U is equal to zero on the lower solid surface when the following condition is met:

$$Mg = \frac{\delta Pr}{3}(\delta Gr - 3Ga). \tag{19}$$

Similarly, the velocity component V is zero on the lower surface when the following condition is met:

$$Mg = \frac{\delta Pr}{3\zeta}(\delta\zeta Gr - 3\xi Ga). \tag{20}$$

The tangential stresses defined by the relationships

$$\begin{aligned} \tau_{XZ} &= \frac{1}{\delta}\frac{dU}{dZ} = \frac{\delta Ga}{Gr}(Z-1) \\ &\quad + \frac{\delta^2}{6}\left(Z^3 - 3Z + 2\right) - \frac{Mg}{GrPr}; \\ \tau_{YZ} &= \frac{1}{\delta}\frac{dV}{dZ} = \frac{\delta\xi Ga}{Gr}(Z-1) \\ &\quad + \frac{\delta^2\zeta}{6}\left(Z^3 - 3Z + 2\right) - \frac{Mg\zeta}{GrPr} \end{aligned} \tag{21}$$

vanish due to the boundary condition (7) when the conditions (19) and (20) are fulfilled, respectively. Making a

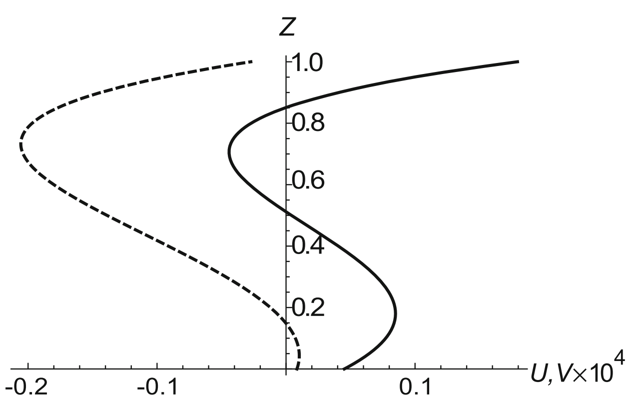

Figure 6. Profiles of the velocity components U (solid lines) and V profiles (dashed lines).

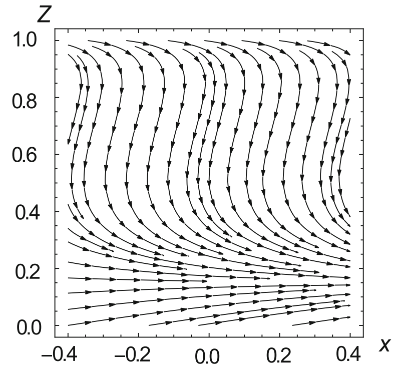

Figure 7. Stream lines.

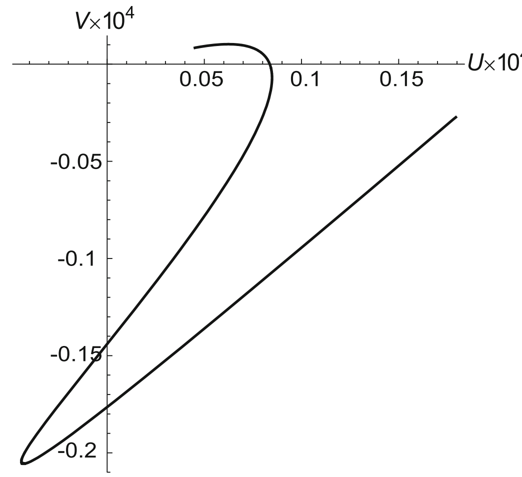

Figure 8. Hodograph of the velocity vector $\mathbf{V} = (U; V; 0)$.

reverse transition to dimensional variables in (21), we obtain the values of layer thickness h at which the tangential stresses τ_{XZ} and τ_{YZ} vanish at the lower boundary of the fluid layer. These layer thickness values can be determined from the following algebraic equations:

$$\begin{aligned} \frac{h^2}{3l^2} - \frac{S_1 h}{g\beta A l^2} - \frac{\nu\sigma}{g\beta l^2\eta} &= 0; \\ \frac{h^2\zeta}{3l^2} - \frac{S_1\xi h}{g\beta A l^2} - \frac{\nu\sigma\zeta}{g\beta l^2\eta} &= 0. \end{aligned} \tag{22}$$

Thus, there are no tangential stresses on the lower rigid surface at the following fluid layer thickness values:

$$\begin{aligned} h_{1,2} &= \frac{3S_1\eta \pm \sqrt{9S_1^2\eta^2 + 12A^2 g\beta\eta\nu\sigma}}{2Ag\beta\eta}; \\ h_{3,4} &= \frac{3S_1\xi\eta \pm \sqrt{9S_1^2\xi^2\eta^2 + 12A^2\zeta^2 g\beta\eta\nu\sigma}}{2A\zeta g\beta\eta}. \end{aligned} \tag{23}$$

For the values of layer thickness $h_i(i = \overline{1,4})$ determined by the expressions (23) to have a physical meaning, several conditions must be met: the non-negativity of the radicands in (23):

$$S_1^2 \geq -\frac{4}{3}\frac{A^2 g\beta\nu\sigma}{\eta};$$
$$S_1^2 \geq -\frac{4}{3}\frac{A^2\zeta^2 g\beta\nu\sigma}{\eta\xi^2}, \tag{24}$$

and such ratios of the pressure and temperature gradients specified at the upper boundary that the resulting layer thickness expressions are positive. The expressions in the right-hand sides of the inequalities (24) may assume both positive and negative values in view of the fact that the temperature coefficient of surface tension may be both positive and negative in the case of abnormal (non-Newtonian) fluids [1].

It can be concluded from the analysis that, if the fluid layer thickness h is equal to h_1 or h_2, the tangential stress τ_{XZ} is zero on the solid lower surface. If the fluid layer thickness h takes one of the values h_3 or h_4, the tangential stress τ_{YZ} is zero on the lower surface. Obviously, with a certain choice of parameters, the tangential stresses τ_{XZ} and τ_{YZ} may vanish simultaneously on the lower surface specified by the equation $Z = 0$. It is possible to observe this effect in fluids whose surface tension coefficient decreases with increasing temperature [1].

Similarly, the point $Z = 1$ is the zero of functions U and V if, respectively, the following conditions are met:

$$Mg = \frac{\delta Pr}{24(a+\delta)}[\delta(8a+3\delta)Gr - 12(\delta+2a)Ga];$$
$$Mg = \frac{\delta Pr}{24(a+\delta)\zeta}[\delta(8a+3\delta)\zeta Gr - 12(\delta+2a)\xi Ga]. \tag{25}$$

Note that, due to the boundary condition (8), the tangential stresses τ_{XZ} and τ_{YZ} can vanish at the upper boundary of the fluid layer if the thermocapillary effect can be neglected (the horizontal temperature gradients at the upper boundary prove to be zero).

7. Conclusion

In this paper, an exact solution has been obtained for the three-dimensional problem of the convective flow of a viscous incompressible fluid. The velocity components have been determined as functions of the transverse coordinate. The temperature and pressure in a fluid layer have been set by linear functions in two longitudinal coordinates. On the upper free surface of the fluid layer, the Benard–Marangoni convection with non-zero temperature and pressure gradients has been considered. Three types of boundary conditions have been considered on the lower solid surface of the fluid layer: the no-slip condition, the perfect slip condition and the Navier slip condition. It has been shown how temperature and pressure variation on the free surface and the value of the slip length α influence the occurrence of stagnation points and counterflow regions in the fluid layer flow. The resulting solution is applicable to the description of both large-scale flows and flows in thin films through the variation of the geometric anisotropy factor.

List of symbols

A, B	values of temperature gradients on upper boundary of fluid layer
g	acceleration of gravity
Ga, Gr	Galileo and Grashof dimensionless numbers
h	thickness of fluid layer
Ma	Marangoni dimensionless number
P	pressure, divided by average fluid density
P_0, P_1, P_2	components of pressure field
Pr	Prandtl dimensionless number
Q	fluid flow rate through the layer thickness
S_1, S_2	values of pressure gradients on upper boundary of fluid layer
T	temperature
T_0, T_1, T_2	components of temperature field
$\mathbf{V}$	velocity vector
V_x, V_y, V_z	projections of velocity vector on coordinate axis
U, u, V	components of velocity field
x, y, z	Cartesian coordinates
Z	dimensionless vertical coordinate
α	dimensional slip factor (slip length)
β	temperature coefficient of volume expansion
δ	ratio of the vertical to horizontal characteristic dimension
ζ	ratio of the longitudinal components of the temperature gradients
η	coefficient of dynamic viscosity
ν	coefficient of kinematic viscosity of the fluid
ξ	ratio of the longitudinal components of the pressure gradients
σ	coefficient of temperature surface tension
τ_{xz}, τ_{yz}	tangential stresses
χ	coefficient of thermal diffusivity of the fluid
Ω_x, Ω_y	vorticity components

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