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Synergetic study of inventory management problem in uncertain environment based on memory and learning effects

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Abstract. Due to the involvement of human intelligence in the inventory planning procedures, memory and learning from repeated tasks in the planning horizon are two important facts that make great impressions on the decision taken in reality. However, the concepts of learning and memory related to the inventory theory are rarely illustrated in literature and till date we have not noticed any work where the effects of memory and system learning have been explored simultaneously. Making an attempt to close the gap, the present paper extends an economic order quantity (EOQ) model into a memory and learning sensitive set-up. The primary structure of the model is established on the assumption that the demand of the EOQ model is constant in inventory run period and the same is a decreasing function of time in the shortage period. Introducing fractional calculus as a replacement of integer one, the notion of memory is included in the proposed theory. Finally, using Zadeh's extension principle, the fuzzzification of the fractional deterministic model is executed and ultimately the sense of learning based decision making is incorporated letting the demand to be a triangular dense fuzzy number. Here, considering different underlying scenarios, four different models have been illustrated and solved numerically. The α-cut method of defuzzification is used for the numerical simulation of two fuzzy models. It is worth mentioning that the joint impact of learning and memory creates positive results on the cost reduction objective of the proposed lot-sizing problem.

Keywords. Differential equation; Caputo's derivative; Riemann–Liouville integral; triangular dense fuzzy set; memory; learning experience.

1. Introduction

The word 'inventory' literally stands for the stock of items of a production or marketing system in the form of raw materials, semi-finished or finished products. Appropriate amount of stock helps any business organization towards its ultimate gain through smooth flow of trading. On the other hand, uncontrolled and inappropriate inventory may cause of discontinuity of the business flow (in the case of poor level of inventory) or leads the system to face losses (in the case of over stock of items). So, in the era of this high competition-market, the control over the inventory is a vital issue. In this context, after the introduction of classical economic order quantity model by Haris [1], lot-sizing (economic order) modelling approach had gained considerable attentions of the researchers and decision makers in the field of operation research. It is to be noted that the basic introductory models in the field of inventory management were constructed based on the assumption of constant demand with no shortage. On the presumption of constant demand, several other models [2–5] have also been developed allowing shortages of partial and fully backlogged types. However, in reality, the demand of the produced item depends on several other factors which are directly related with the inventory system. The price, stock and deterioration of items are such important facts which have much impression on the demand of the production and retailing business. Thus, the subsequent studies of order quantity models incorporated these components (price dependent [6–8], stock dependent with deterioration [9, 10], stock and price dependent with deterioration [11, 12]) in the list of the assumptions.

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In any real world physical or man-made scenario, uncertainty is an unavoidable issue that has been risen in the decision making process. Fuzzy theory introduced by Zadeh [13] and fuzzy decision making by Bellman and Zadeh [14] are smart and important tools to capture and quantify the real world uncertainty. In the way of searching the optimal lot-size in an inventory problem ensuring minimum possible cost (or maximum possible profit), the decision maker experiences ambiguity with the values of the involved parameters. In particular, the demand may not be predicted accurately. This leads the study of decision making of an inventory control problem towards the uncertain environments.

Human involvement in dynamical system makes it more complex to analyze in realty. Interaction among producers, retailers and customers may contribute much to the deviation of results predicted by the mathematical model that neglects the presence of human intelligence. Some studies [15–17] included the human factors in the inventory modelling. Memory from past and learning through repeated exercises on the same task are two important facts associated with the human involvement in the production, retailing and decision making sectors of management problem. Surprisingly, the investigations on these two effects are very rare in the existing literature. Moreover, among those rare studies, none shows interests on the experimentation of memory and learning effects simultaneously in uncertain scenario.

This present paper takes the challenge to measure the memory and learning effect on an EOQ model in an uncertain decision making situation. Here, the demand of the EOQ model is constant in inventory run period and the same is a decreasing function of time in the shortage period. Allowing the demand to be TDFS and describing the mathematical model through fractional differential equation under Caputo derivative and Riemann–Liouville integration, the idea of learning and memory respectively are cultivated in this paper. The graphical view of the ideas implemented in this paper is represented in figure 1.

The remaining part of this article is presented as follows: A brief literature review focusing on objective of the current study is scripted in section 2. Then, after a brief presentation on the general overview on the fractional calculus and the dense fuzzy number in section 3, notations and assumptions to formulate and describe the model are presented in section 4. Section 5 deals with the mathematical modeling of the EOQ problem in crisp environment as well as its corresponding fuzzy extension. In section 6, numerical analysis is carried out and sensitivity analysis in both tabular and graphical terms is presented. Major observations from the current study are spotted in section 7. Finally, concluding remark over the current article and

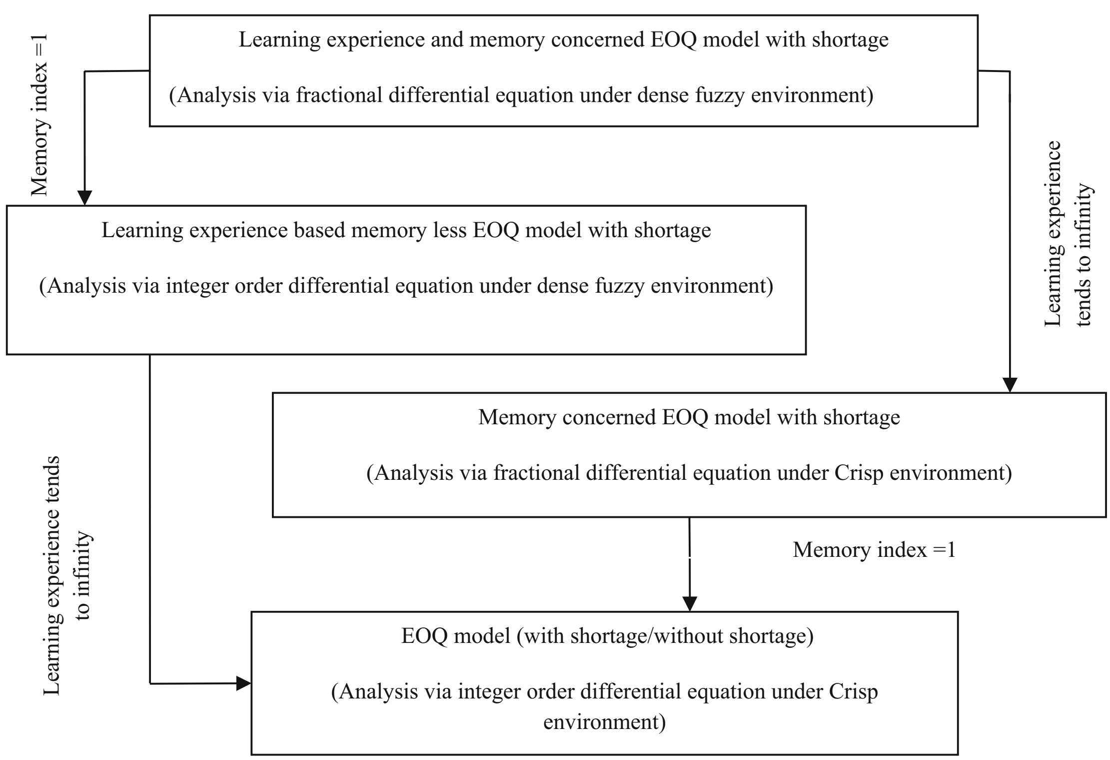

Figure 1. Models with and without memory and learning effect.

some directions towards future research are suggested in section 8.

2. Literature review

The theory in this paper is developed following an approach of accumulation of several ideas from different potential streams of mathematical research. So, this present section is designed with four components as sub sections in which the first three sub sections describe the literature reviews related to three different domains with closed attachment to the current work while the last sub section represents the research gaps and a sense of amalgamation of those ideas in this paper.

2.1 *Memory based derivative and fractional calculus*

Differential equations play an important role to mathematically formulate many real-life problems and natural phenomena in various fields of modern science, engineering and technology. However, most of the time knowledge and information are often imminent with the confusion associated with the physical phenomena. In many cases, it is observed that the system expressed through a set of differential equations fail to evaluate system behaviour due to ambiguity and uncertainties involved with the parameters that are used in the modelling of the system. Therefore, researchers introduced fuzzy differential equations to capture and handle different uncertainties associated with physical problems, which is generally done by considering the coefficients and/or initial values of the differential equations as different fuzzy numbers. On the other hand, in many scenarios, system's memory plays very crucial role on the progression of the system. In case of inventory management system, the past experiences of the managers or decision makers termed as 'memory' contribute on the decision taken at the present time. For an EOQ model with cost minimization objective, the memory on the demand pattern of a past planning horizon may help the dealer/retailer to develop a smarter approach of dealing with customers. Moreover, experiences of the customer through familiarity with same or similar kind of product, customer care service etc. may influence the demand of the customer. These complicated phenomena cannot be described well by the traditional integer order differential equations. In these cases, fractional order differential equations (FDEs) are proved to be better alternative of integer order differential equation. Having the ability to analyze the physical and real world more accurately, the fractional calculus has gained much interest for modeling of the dynamical problems in the various fields of science and engineering as well as economics and management [18–23]. The concepts of memory on the economic production quantity models with deterioration were implemented by Rahaman *et al* [24, 25]. On the other hand, Agarwal *et al* [26] was the first to propose the FDE with uncertainty. He considered Riemann–Liouville differentiability to develop the FDE. Hoa *et al* [27] described a fuzzy fractional initial valued problem under Caputo differentiability. The interval counterpart with generalized Hukuhara fractional differentiability was established by Lupulescu *et al* [28]. Furthermore, the generalized (fuzzy) fractional Laplace transform regarding the Riemaan-Liouville differentiability is introduced and improved to solve initial value problems in the recent studies [29–31]. Recently, Rahaman *et al* [32] have accounted the memory effect on an EPQ model in fuzzy environment letting the system to be described by fuzzy fractional differential equation under Riemann–Liouville derivative. In the above mentioned studies of fractional calculus in uncertainty, the notion of fuzzy derivative (Hukuhara, generalized Hukuhara) plays a very important role. However, Ahmad *et al* [33] developed fuzzy fractional differential equation using Zadeh's extension principle interpreting the fact that the fuzzy fractional differential equation needs not require the concept of the fuzzy derivative, which directed towards an alternative approach in solving the fuzzy fractional differential equations.

2.2 *Fuzzy inventory models*

Park [34] was the first to describe EOQ model under the consideration of fuzzy environment portraying the real circumstance regarding the ambiguity and impreciseness about the decision parameters, variables and several other components. Researchers like Vujosevic *et al* [35] and Hojati [36] carried out some interesting initial findings of inventory models in the fuzzy environment. Chen *et al* [37] was the first to build a fuzzy economic order quantity model with back order. Till now, several novel approaches contributed on the enrichment of literature related to fuzzy inventory model in terms of both inventory modelling and optimization techniques. In this context, Shekarian *et al* [38] have made a worthy review of fuzzy inventory models in details.

2.3 *Learning in fuzziness*

Learning through repeated tasks is an advanced component of human intelligence. Experiences gained by habituations result to the improvement of one's performances. In the managerial point of view, decisions must be matured through experience based learning. In this regard, Wright [39] was the first to identify the learning phenomena in industrial set-up. Subsequently, several more studies [40–43] regarding the learning effect on various inventory parameters enriched the literature. Glock *et al* [44] made an experiment on uncertainty learning for economic order quantity model with fuzzy demand function and noticed that more reliable and updated information can be accommodate to improve the decisions. The learning effect on an inventory system with imperfect production is measured in fuzzy random environment by Mahato [45]. Kazemi *et al* [46] contributed significantly on the reduction of deviation values of the uncertain demand through learning in a planning horizon. Kazemi *et al* [47] have evaluated the impact of human learning on fuzzy lot-sizing model. Shekarian *et al* [48] presented the results of an inspection on learning of an EOQ model with imperfect quality of products under different holding cost and fully fuzzy phenomena. Soni *et al* [49] also accounted the reduction of impreciseness regarding the fuzzy demand allowing learning capabilities in the modeling. Khatua *et al* [50] introduced a fuzzy optimal control model of process–product and innovation incorporating the concept of system learning. In the literature of fuzzy decision making, De and Beg [51] added a new concept of integrating the impact of learning experience by introducing the triangular dense fuzzy set (TDFS). In this approach, the impreciseness described by sequential representation can be diluted by allowing the system for executing repeated task. Later, De and Beg [52] hinted for the philosophical perspective of TDFS. Furthermore, De [53] extended the concept of triangular dense fuzzy set into the triangular dense fuzzy lock set (TDFLS). Recently, some studies [54–56] on the inventory control problems have been carried out utilizing the concepts of TDFS and TDFLS.

2.4 *Amalgamation of three different research directions*

After a brief survey on the literature related to the theory of fuzzy inventory modeling, concept of memory (in terms of fractional calculus) and experience based learning in imprecise environment, the following gaps and scopes to improve the ideas are identified:

i) Recently few studies [24, 25] incorporated the idea of memory in modeling of inventory control problem and these studies were only done in deterministic environment. Thus, this kind of study in imprecise environment is very much desirable.

ii) Some investigations [45–50, 54–56] cultivated the notion of learning only associated with inventory in fuzzy environments. But, these investigations neglected the memory effect on the inventory model.

iii) One recent investigation [32] on the measure of memory effect on the production phase of an EPQ model in uncertain environment has been done. However, this finding did not focus on the method of improvement on the decision incorporating system learning experience.

On the basis of above mentioned short-comings, we make an attempt to amalgamate the notions of memory, learning and impreciseness in a single inventory model. Here, we have developed an EOQ model where the demand is imprecise in nature. Also, it is assumed that there is an impact of memory on the proposed model. The fractional differential equation under Caputo differentiability in the environment of fuzzy uncertainty is utilized to represent the memory effect on the inventory model following the approaches described by Ahmad *et al* [33]. Also, the impreciseness and its dilution (due to availability of better experience) have been captured through a triangular dense fuzzy number as defined by De and Beg [51].

3. Mathematical preliminaries

This section is organized with three different sub sections. The sub-sections 3.1 contains some basic definitions and theories related to fractional calculus; brief description of dense fuzzy set has been presented in sub-sections 3.2 and fundamentals of fuzzy fractional differential equation including Zadeh's extension principle scheme have been presented in sub-sections 3.3.

3.1 *General overview on fractional calculus*

Definition 3.1.1 (*Riemann–Liouville integral*) [20] The left sided Riemann–Liouville integral of $y(x)$ of order $p-\beta_d$ (where $\beta_d > 0$ be a real number and $[\beta_d] = p$, the least integer greater than β_d) with the origin at $x = 0$ is given by

$$^{RL}I_x^{p-\beta_d} y(x) = \frac{1}{\Gamma(p-\beta_d)} \int_0^x (x-u)^{p-\beta_d-1} y(u)du$$

Definition 3.1.2 (*Riemann–Liouville derivative*) [20] The left-sided Riemann–Liouville derivative of $y(x)$ of order β_d ($\beta_d > 0$ be a real number and $[\beta_d] = p$) with the origin at $x = 0$ is given by

$$^{RL}D_x^{\beta_d} y(x) = D^{p\,RL} I_x^{p-\beta_d} y(x)$$

$$= \frac{1}{\Gamma(p-\beta_d)} \frac{d^p}{dx^p} \int_0^x (x-u)^{p-\beta_d-1} y(u)du$$

In particular, for $0 < \beta_d < 1$, $p = 1$ and then

$$^{RL}D_x^{\beta_d} y(x) = \frac{1}{\Gamma(1-\beta_d)} \frac{d}{dx} \int_0^x (x-u)^{-\beta_d} y(u)du$$

Definition 3.1.3 (*Caputo derivative*) [20] Let, $[\beta_d] = p$ and be real number. Then, the left-sided Caputo derivative of $y(x)$ of order β_d with the origin at $t = 0$ is given by

$$ {}^{C}D_x^{\beta_d} y(x) = {}^{RL}I_x^{p-\beta_d} y^p(x) $$

$$ = \frac{1}{\Gamma(p-\beta_d)} \int_0^x (x-u)^{p-\beta_d-1} y^p(u) du. $$

where $y^p(x)$ is the p-th order integer derivative of $y(x)$. In particular, for $0 < \beta_d < 1$,

$$ {}^{C}D_x^{\beta_d} y(x) = \frac{1}{\Gamma(1-\beta_d)} \int_0^x (x-u)^{-\beta_d} y'(u) du. $$

Proposition 3.1.1 (Laplace transform of Caputo derivative) [20] *The L.T. of* ${}^{C}D_x^{\beta_d} y(x)$ *is given by*

$$ \mathcal{L}\left({}^{C}D_x^{\beta_d} y(x); s\right) = s^{\beta_d} Y(s) - \sum_{k=0}^{p-1} s^{\beta_d - 1 - k} y^k(0), $$

where $Y(s)$ *reprents the Laplace transform of* $y(x)$.

3.2 *Brief description of dense fuzzy set*

The learning experience based study to reduce the ambiguity affecting the decision of an inventory management problem is incorporated in terms of triangular dense fuzzy set (TDFS).

Definition 3.2.1 [51] Let $\tilde{A}$ be fuzzy number whose components are elements of the Cartesian product $\mathbb{R} \times \mathbb{N}$, (where $\mathbb{R}$ and $\mathbb{N}$ represent the set of real numbers and natural number respectively) with membership grade satisfying the functional relation $\mu : \mathbb{R} \times \mathbb{N} \to [0,1]$. If $\mu(x,n) \to 1$, as $n \to \infty$, for some $x \in \mathbb{R}$ and $n \in \mathbb{N}$, then $\tilde{A}$ is called dense fuzzy set.

In particular if $\tilde{A}$ is triangular in the above definition, it is called triangular dense fuzzy set (TDFS). Furthermore, if $\mu(x,n)$ attain the highest membership degree 1. for some $n \in \mathbb{N}$, then it is called the normalized triangular dense fuzzy set (NTDFS).

Example 3.2.1 Let us take a NTDFS as follows $\tilde{A} = \left\langle a\left(1 - \frac{\rho}{1+n}\right), a, a\left(1 + \frac{\sigma}{1+n}\right) \right\rangle$, where $\rho, \sigma \in (0,1)$, a be a real number and n be a natural number. Then, its membership function is given by

$$ \mu(x,n) = \begin{cases} 0, & for\, x \left\langle a\left(1 - \frac{\rho}{1+n}\right) or x \right\rangle a\left(1 + \frac{\sigma}{1+n}\right) \\ \dfrac{x - a\left(1 - \frac{\rho}{1+n}\right)}{\frac{a\rho}{1+n}}, & for\, a\left(1 - \frac{\rho}{1+n}\right) \le x \le a \\ \dfrac{a\left(1 + \frac{\sigma}{1+n}\right) - x}{\frac{a\sigma}{1+n}}, & for\, a \le x \le a\left(1 + \frac{\sigma}{1+n}\right) \end{cases} $$

3.2.1 *Defuzzification of TDFLS based on α-cut method [51]*

The α-cut of a TDFS $\tilde{A} = \left\langle a\left(1 - \frac{\rho}{1+n}\right), a, a\left(1 + \frac{\sigma}{1+n}\right) \right\rangle$ is given by $[A_L, A_R] = \left[a + (\alpha - 1)\frac{a\rho}{n+1}, a + (1-\alpha)\frac{a\sigma}{n+1}\right]$ and then the index value of $\tilde{A}$ is given by

$$ I(\tilde{A}) = \frac{1}{2N} \sum_{n=1}^{N} \int_0^1 \{A_L + A_R\} d\alpha $$

$$ = \frac{1}{2N} \sum_{n=1}^{N} \int_0^1 \{\left(a + (\alpha-1)\frac{a\rho}{n+1}\right) + \left(a + (1-\alpha)\frac{a\sigma}{n+1}\right)\} d\alpha $$

$$ = a - \frac{a(\rho - \sigma)}{4N} \sum_{n=1}^{N} \frac{1}{n+1} $$

Since $\frac{1}{N}\sum_{n=1}^{N} \frac{1}{n+1} \to 0$ when $N \to \infty$. then, $I(\tilde{A}) \to a$. This approach does not give only a defuzzified value of a fuzzy number, but also carries the insight theme of the paper. That is, repetition of the same task can create a more clear perception of the decision on the parameters which initially appears to be hesitant to him/her. More learning leads the system towards a crisp decision with no deviation at all.

3.3 *Initial valued fuzzy fractional differential equation*

3.3.1 *Zadeh's extension principle*

Definition 3.3.1 (*Extension principle*) Suppose, $f : A \to B$ be crisp function. Let, $\tilde{F}$ be a fuzzy set in A. Then, extension principle provides the guaranty to define the fuzzy set $\tilde{G} = f(\tilde{F})$ in B For, f to be a strictly monotone function, formation of $\tilde{G} = f(\tilde{F})$ is quite easy. The extension of f thegiven b

$$ f(\tilde{F})(b) = \begin{cases} \tilde{F}(f^{-1}(b)), & when\, b \in Range(f) \\ 0, & when\, b \notin Range(f) \end{cases} $$

However, when f is a non monotone function, the above extension is not so easy. In this situation, more accurately the extension is given as the following:

$$ f(\tilde{F})(b) = \begin{cases} \sup_{a \in f^{-1}(b)} \tilde{F}(a), & when\, b \in Range(f) \\ 0, & when\, b \notin Range(f) \end{cases}, $$ where

$f^{-1}(b) = \{a \in A : f(a) = b\}$.

According to the Zadeh's extension principle [13], every crisp function can be generalized (extended) to take fuzzy set as arguments.

Theorem 3.3.1 [57] *Let, $\tilde{A}$ be the space of all convex and compact set defined on A If $\psi : R^n \to R^n$ be a crisp continuous function, then the fuzzy extension $\tilde{\psi} : \widetilde{R^n} \to \widetilde{R^n}$ is well defined and the α-cut of $\tilde{\psi}$ is given by*

$$\left[\tilde{\psi}(\tilde{A})\right]_\alpha = \tilde{\psi}\left([\tilde{A}]_\alpha\right), where, \tilde{\psi}(\tilde{A}) = \left\{\tilde{\psi}(a) : a \in \tilde{A}\right\}.$$

3.3.2 *Fuzzy differential equation* Consider a crisp differential equation

$$\begin{cases} y'(x) = \psi(x, y(x)) \\ y(x_0) = y_0 \end{cases} \quad (3.3.1)$$

Where $\psi : [x_0, x_1] \times A \to R^n$ is a continuous crisp function. Suppose $\tilde{\psi} : [x_0, x_1] \times \tilde{A} \to \widetilde{R^n}$ be the function obtained by using extension principle on ψ, where $\tilde{A} \in \widetilde{R^n}$.

Then, by theorem 3.3.1, $\left[\tilde{\psi}(x, \tilde{y}(x))\right]_\alpha = \tilde{\psi}\left(x, [\tilde{y}(x)]_\alpha\right)$.

Now we are interested to solve the fuzzy differential equation given by

$$\begin{cases} \tilde{y}'(x) = \tilde{\psi}(x, \tilde{y}(x)) \\ \tilde{y}(x_0) = \tilde{y}_0 \end{cases} \quad (3.3.2)$$

According to Chalco-Cano and Roman-Flores [58] and Roman-Flores *et al* [57]**,** the solution of the fuzzy differential equation is as following:

First solve the equation (3.3.1) and suppose $y(x)$ be the solution of (3.3.1). Then applying extension principle on $y(x)$, we obtained the fuzzy extension $\tilde{y}(x)$ as a solution of (3.3.2).

3.3.3 *Fuzzy fractional differential equation* A parallel fractional version corresponding to the approach discussed in sub section 3.3.2 was given by Ahmad *et al* [33]. Here, we present the said approach for solving the fuzzy fractional differential equation briefly which seems to be a nice application of extension principle in the theory of fractional calculus.

Consider the fuzzy fractional differential equation

$$\begin{cases} {}^C D_x^{\beta_d} \tilde{y}(x) = \tilde{\psi}(x, \tilde{y}(x)) \\ \tilde{y}(x_0) = \tilde{y}_0 \end{cases} \quad (3.3.3)$$

Where $\beta_d \in (0, 1]$ and $x > 0$ and $\tilde{\psi} : [x_0, x_1] \times \tilde{A} \to \widetilde{R^n}$ be the fion obtained by using extension principle on $: [x_0, x_1] \times A \to R^n$, a continuous crisp function.

The corresponding fractional differential equation is

$$\begin{cases} {}^C D_x^{\beta_d} y(x) = \psi(x, y(x)) \\ y(x_0) = y_0 \end{cases} \quad (3.3.4)$$

Solving (3.3.4) using the Laplace and inverse Laplace transformation, we get the solution of (3.3.4) is given by

$$y(x) = \phi(x) \quad (3.3.5)$$

Using the Zadeh's extension principle to fuzzify the solution given by (3.3.5), we obtain the solution of the fuzzy fractional differential equation as $\tilde{y}(x) = \tilde{\phi}(x)$.

4. Assumptions and notations for defining the problem

Fractional calculus (in particular under Riemaan–Liouville and Caputo's approach) has the finest interpretation that it can capture the memory of dynamical system. In this article, we have mathematically presented an inventory management system having memory in terms of fractional derivative. The current planning of inventory strategies must be motivated by the previous memory on the demand pattern which makes an EOQ model to be memory sensitive. Also, here the EOQ model with fully backlogged shortage is assumed to be uniform in the inventory run period and is a decreasing function of time during the shortage period. Very often in the real market, during the time of shortage of the product, the demand of that item bounds to be decreased slowly which makes the assumption a very natural fact. Also, here the demand pattern is assumed to be uncertain at the very initial phase of the inventory cycle and the ambiguity regarding the demand is assumed to be removed gradually as time progresses due to the learning experience of the decision maker. This important fact is presented in the problem by considering the demand to be triangular dense fuzzy number. Here, two important issues, say the memory effect and learning experience, are incorporated in the modeling of the inventory system.

Assumptions

Apart from the memory and learning experience (to be represented by fractional calculus and TDFN respectively), the following basic assumptions have been considered to develop the proposed model:

i) Demand of the product is uniform on the inventory running phase and gradually decreasing in the time of shortage. The demand can be written as a function of time as the following

$$D(t) = \begin{cases} a, & 0 \le t \le t_1 \\ \dfrac{a}{\sqrt{t}}, & t_1 < t \le T \end{cases}$$

ii) Shortage is allowed and considered to be fully backlogged.
iii) Replenishment rate is infinite but lot size is finite.
iv) The time horizon is infinite.
v) Lead time is zero.
vi) The system is memory concerned.

Notations

To describe our proposed problem we use the following notations with certain units and description (displayed in Table 1):

5. Formulation and description of the mathematical model

5.1 *Transformation of integer order dynamical model to that of fractional order*

A memory affected dynamical system can be described by the following differential equation:

$$\frac{dy}{dt} = \int_0^t k(t,s) f(s, y(s)) ds \quad (5.1.1)$$

Where $k(t,s)$ is the kernel function. In the classical differential equation (to describe a memory less model), $k(t,s) = \delta(t,s)$, the usual dta function and (5.1.1) is taken the form

$$\frac{dy}{dt} = f(t, y(t)) \quad (5.1.2)$$

When the kernel is taken to be $k(t,s) = \frac{(t-s)^{\beta_d - 2}}{\Gamma(\beta_d - 1)}$ (where $0 < \beta_d \leq 1$) in (5.1.1) then,

$$\frac{dy}{dt} = \int_0^t \frac{(t-s)^{\beta_d - 2}}{\Gamma(\beta_d - 1)} f(s, y(s)) ds$$

$$\text{Or,} \quad \frac{dy}{dt} = {}^{RL}I_t^{\beta_d - 1} f(t, y(t)) \quad (5.1.3)$$

The right hand side is the Riemann–Liouville integral of f of order $\beta_d - 1$.

Therefore,

$${}^{C}D_t^{\beta_d} y(t) = {}^{RL}I_t^{1-\beta_d}\left(\frac{dy}{dt}\right) = {}^{RL}I_t^{1-\beta_d}\left({}^{RL}I_t^{\beta_d - 1} f(t, y(t))\right)$$

$$= f(t, y(t)$$

That is

$${}^{C}D_t^{\beta_d} y(t) = f(t, y(t) \quad (5.1.4)$$

The equation (5.1.4) represents the equivalent fractional differential equation under Caputo's derivative of the equation (5.1.1).

5.2 *Mathematical modelling of fractional (arbitrary) order EOQ*

The inventory cycle starts with the initial inventory level S_1 and gradually decreases due to meeting up the uniform demand a during the time interval $0 \leq t \leq t_1$ At $t = t_1$, the inventory level becomes zero and then shortage occurs. During the period $t_1 < t \leq T$, the shortage level increases due to the demand $\frac{a}{\sqrt{t}}$ and reaches to the highest level of shortage $Q - S_1$ (where Q is the lot size) at $t = T$, where the shortage phase as well as the wentory cle is stopped. The phenomenon is described by the figure 2.

The memory effected EOQ model with shortage is governed by the differential equations

$${}^{C}D_t^{\beta_d} \mathrm{I}(t) = -a, \text{for}, 0 \leq t \leq t_1 \quad (5.2.1)$$

$$\text{And } {}^{C}D_t^{\beta_d} \mathrm{J}(t) = \frac{a}{\sqrt{t}}, \text{for} \quad (5.2.2)$$

Table 1. Notations used to describe the proposed model.

Notations	Units	Descriptions
h_c	\$/unit	Holding cost per unit time
o_c	\$/unit	Ordering cost per cycle
s_c	\$/unit	Shortage cost per unit time
D	Unit	Demand rate per cycle
T	Month	Total time cycle
t_1	Month	Inventory run time
Q	Unit	Lot size of the inventory
S_1	Unit	Inventory level at the starting of the cycle
β_d	Unit	Differential memory index
β_i	Unit	Integral memory index
Decision variables		
T	Month	Total time cycle
t_1	Month	Production time
Q	Unit	Highest level of inventory
Objective function		
TAP_{β_d,β_i}	\$	Total average profit

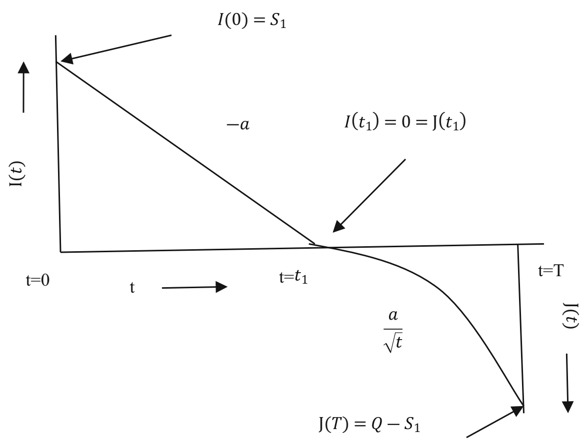

Figure 2. Diagram of the proposed EOQ model.

With the following conditions,

$$\begin{cases} I(0) = S_1 \\ I(t_1) = 0 \\ J(t_1) = 0 \\ J(T) = Q - S_1 \end{cases} \tag{5.2.3}$$

Taking the Laplace transformation of (5.2.1) about the origin at $t = 0$,

$$s^{\beta_d}\tilde{I}(s) - s^{\beta_d-0-1}I(0) = -\frac{a}{s}. \tag{5.2.4}$$

Where $\tilde{I}(s)$ is the Laplace transformation of $I(t)$.
Or,

$$s^{\beta_d}\tilde{I}(s) = s^{\beta_d-1}S_1 - \frac{a}{s}$$

Or,

$$\tilde{I}(s) = \frac{S_1}{s} - \frac{a}{s^{\beta_d+1}}$$

Or,

$$I(t) = L^{-1}\{\tilde{I}(s)\} = S_1 - \frac{at^{\beta_d}}{\Gamma(\beta_d+1)} \tag{5.2.5}$$

Again, using the ending condition $I(t_1) = 0$, (5.2.5) gives

$$S_1 = \frac{at_1^{\beta_d}}{\Gamma(\beta_d+1)} \tag{5.2.6}$$

And therefore, (5.2.5) turns to

$$q(t) = a\frac{t_1^{\beta_d} - t^{\beta_d}}{\Gamma(\beta_d+1)}, 0 \le t \le t_1 \tag{5.2.7}$$

Again, taking Laplace transformation of (5.2.2) considering $t = t_1$ to be the origin,

$$s^{\beta_d}\tilde{J}(s) - s^{\beta_d-0-1}J(t_1) = \frac{a\sqrt{\pi}}{\sqrt{s}}$$

Or,

$$\tilde{J}(s) = \frac{a\sqrt{\pi}}{s^{\beta_d+\frac{1}{2}}}$$

Therefore,

$$J(t) = \frac{a\sqrt{\pi}(t-t_1)^{\beta_d-\frac{1}{2}}}{\Gamma\left(\beta_d+\frac{1}{2}\right)} \tag{5.2.8}$$

Using $J(T) = Q - S_1$, from (5.2.8),

$$Q - S_1 = \frac{a\sqrt{\pi}(T-t_1)^{\beta_d-\frac{1}{2}}}{\Gamma\left(\beta_d+\frac{1}{2}\right)} \tag{5.2.9}$$

Again, from (5.2.6) and (5.2.9),

$$Q = \frac{at_1^{\beta_d}}{\Gamma(\beta_d+1)} + \frac{a\sqrt{\pi}(T-t_1)^{\beta_d-\frac{1}{2}}}{\Gamma\left(\beta_d+\frac{1}{2}\right)} \tag{5.2.10}$$

So, total holding cost,

$$HC = c_h{}^{RL}I_t^{\beta_i}I(t_1)$$

$$= \frac{c_h a}{\Gamma(\beta_i)}\int_0^{t_1}(t_1-s)^{\beta_i-1}\frac{t_1^{\beta_d} - s^{\beta_d}}{\Gamma(\beta_d+1)}ds$$

$$= \frac{c_h a}{\Gamma(\beta_i)\Gamma(\beta_d+1)}\left[\frac{t_1^{\beta_d+\beta_i}}{\beta_i} - t_1^{\beta_d+\beta_i}\int_0^1 s^{\beta_d}(1-s)^{\beta_i-1}ds\right]$$

$$= \frac{c_h a t_1^{\beta_d+\beta_i}}{\Gamma(\beta_i)\Gamma(\beta_d+1)}\left[\frac{1}{\beta_i} - \mathrm{B}(\beta_d+1,\beta_i)\right]$$

$$= c_h a t_1^{\beta_d+\beta_i}\left[\frac{1}{\Gamma(\beta_i+1)\Gamma(\beta_d+1)} - \frac{1}{\Gamma(\beta_d+\beta_i+1)}\right] \tag{5.2.11}$$

Total shortage cost,

$$SC = c_s{}^{RL}I_t^{\beta_i}J(T)$$

$$= \frac{c_s}{\Gamma(\beta_i)}\int_0^T(T-s)^{\beta_i-1}\frac{a\sqrt{\pi}(s-t_1)^{\beta_d-\frac{1}{2}}}{\Gamma\left(\beta_d+\frac{1}{2}\right)}ds$$

$$= \frac{c_s a\sqrt{\pi}}{\Gamma(\beta_i)\Gamma\left(\beta_d+\frac{1}{2}\right)}\int_0^T(T-s)^{\beta_i-1}(s-t_1)^{\beta_d-\frac{1}{2}}ds$$

$$= \frac{c_s a\sqrt{\pi}(T-t_1)^{\beta_d+\beta_i-\frac{1}{2}}}{\Gamma(\beta_i)\Gamma\left(\beta_d+\frac{1}{2}\right)}\int_0^1 (1-u)^{\beta_i-1}u^{\beta_d-\frac{1}{2}}du$$

$$= \frac{c_s a\sqrt{\pi}(T-t_1)^{\beta_d+\beta_i-\frac{1}{2}}\,\mathrm{B}\left(\beta_d+\frac{1}{2},\beta_i\right)}{\Gamma(\beta_i)\Gamma\left(\beta_d+\frac{1}{2}\right)}$$

$$= \frac{c_s a\sqrt{\pi}(T-t_1)^{\beta_d+\beta_i-\frac{1}{2}}}{\Gamma\left(\beta_d+\beta_i+\frac{1}{2}\right)} \tag{5.2.12}$$

Total ordering cost

$$OC = c_0 \tag{5.2.13}$$

So, total average cost

$$TAC = \frac{1}{T}(OC+HC+SC)$$

$$= af+g \tag{5.2.14}$$

Where,

$$\begin{cases} f = \dfrac{c_h t_1^{\beta_d+\beta_i}}{T}\left[\dfrac{1}{\Gamma(\beta_i+1)\Gamma(\beta_d+1)} - \dfrac{1}{\Gamma(\beta_d+\beta_i+1)}\right] + \dfrac{c_s\sqrt{\pi}(T-t_1)^{\beta_d+\beta_i-\frac{1}{2}}}{T\Gamma\left(\beta_d+\beta_i+\frac{1}{2}\right)} \\ g = \dfrac{c_0}{T} \end{cases} \tag{5.2.15}$$

So the optimization problem is

$$\begin{cases} Min TAC = af+g \\ 0<t_1<T \\ \text{Subject to the constraints given by}(5.1.15), (5.1.10) \text{ and } (5.1.6) \end{cases}$$

5.3 *Fuzzy model*

Assuming the demand to be uncertain at the initial phase of the inventory cycle and the pattern to be clearer to the decision maker (DM) as time passes, we actually consider the demand to be a dense fuzzy number. Let us consider a to be a triangular dense fuzzy number given by

$\tilde{a} = \left\langle a_0\left(1-\frac{\rho}{n+1}\right), a_0, a_0\left(1+\frac{\sigma}{n+1}\right)\right\rangle$ and its membership function is given by

$$\mu(\tilde{a}) = \begin{cases} \dfrac{a - a_0\left(1-\frac{\rho}{n+1}\right)}{\frac{a_0\rho}{n+1}}, & \text{when } a_0\left(1-\frac{\rho}{n+1}\right)<a<a_0 \\ \dfrac{a_0\left(1+\frac{\sigma}{n+1}\right)-a}{\frac{a_0\sigma}{n+1}}, & \text{when } a_0<a<a_0\left(1+\frac{\sigma}{n+1}\right) \\ 0, & \text{otherwise} \end{cases} \tag{5.3.1}$$

Then, α-cut of $\tilde{a}$ is $\left[a_0+(\alpha-1)\frac{a_0\rho}{n+1}, a_0+(1-\alpha)\frac{a_0\sigma}{n+1}\right]$ and corresponding index value is

$$I(\tilde{a}) = \frac{1}{2N}\sum_{n=1}^{N}\int_0^1\left\{\left(a_0+(\alpha-1)\frac{a_0\rho}{n+1}\right) + \left(a_0+(1-\alpha)\frac{a_0\sigma}{n+1}\right)\right\}d\alpha$$

$$= \frac{1}{2N}\sum_{n=1}^{N}\left[2a_0 - \frac{a_0(\rho-\sigma)}{2(n+1)}\right] = a_0 - \frac{a_0(\rho-\sigma)}{4N}\sum_{n=1}^{N}\frac{1}{n+1} \tag{5.3.2}$$

Now (5.2.13) gives $a = \frac{TAC-g}{f}$ and then $\tilde{a} \cong \widetilde{\frac{TAC-g}{f}}$.

Then, using (5.3.1), the membership function of $\widetilde{TAC}$ is given by

$$\mu(\widetilde{TAC}) = \begin{cases} \dfrac{\frac{TAC-g}{f} - a_0\left(1-\frac{\rho}{n+1}\right)}{\frac{a_0\rho}{n+1}}, & \text{when } a_0\left(1-\frac{\rho}{n+1}\right)f+g<C<a_0f+g \\ \dfrac{a_0\left(1+\frac{\sigma}{n+1}\right) - \frac{TAC-g}{f}}{\frac{a_0\sigma}{n+1}}, & \text{when } a_0f+g<TAC<a_0\left(1+\frac{\sigma}{n+1}\right)f+g \\ 0, & \text{otherwise} \end{cases} \tag{5.3.3}$$

So, the α-cut of $\widetilde{TAC}$ is $\left[a_0f+g+(\alpha-1)\frac{a_0\rho f}{n+1}, a_0f+g+(1-\alpha)\frac{a_0\sigma f}{n+1}\right]$ and the index value of the objective function is

$$I(\widetilde{TAC}) = \frac{1}{2N}\sum_{n=1}^{N}\int_0^1\left\{\left(a_0f+g+(\alpha-1)\frac{a_0\rho f}{n+1}\right) + \left(a_0f+g+(1-\alpha)\frac{a_0\sigma f}{n+1}\right)\right\}d\alpha$$

$$= a_0f+g-\frac{a_0(\rho-\sigma)f}{4N}\sum_{n=1}^{N}\frac{1}{n+1} \tag{5.3.4}$$

5.4 *Particular cases of the fuzzy arbitrary order model*

Case 1 *when* $N\to\infty$.

Then,

$$I(\widetilde{TAC}) \to a_0f+g \tag{5.4.1}$$

This is the case of crisp arbitrary order model. Here, long run of the inventory process makes the decision maker learned through experience and the ambiguity regarding the demand pattern is removed. Also, the model is described by the fractional order calculus and hence it is memory concerned.

Case 2 *when* $\beta_d = \beta_i = 1$
Then,

$$I(\widetilde{TAC}) \to a_0 f' + g' - \frac{a_0(\rho-\sigma)f'}{4N}\sum_{n=1}^{N}\frac{1}{n+1} \quad (5.4.2)$$

Where

$$\begin{cases} f' = \dfrac{c_h t_1^2}{2T} + \dfrac{4s_c(T-t_1)^{\frac{3}{2}}}{3T} \\ g' = \dfrac{c_0}{T} \end{cases} \quad (5.4.3)$$

Also, then the optimal lot size and the initial inventory level are given by

$$Q' = at_1 + 2a(T-t_1)^{\frac{1}{2}} \quad (5.4.4)$$

And

$$S_1' = at_1 \quad (5.4.5)$$

This is the case of memory less fuzzy dense model.
Case 3 *when* $\beta_d = \beta_i = 1$ *and* $N \to \infty$
Then, $I(\widetilde{TAC}) \to a_0 f' + g'$.
This is the case of the memory less crisp model.

5.5 *Solution Methodology*

The solution procedure in this current study can be presented as the following:

Algorithm

Step 1 Input the values of the parameters corresponding to the integer order crisp problem, say problem 1.

Step 2 Solve the problem 1 for TAC^*, T^*, t_1^*, Q^* and store the optimal solution as

$$Sol_{int,crisp} = \begin{cases} TAC^*_{int,crisp} = TAC^* \\ T^*_{int,crisp} = T^* \\ t^*_{1,\ \ int,crisp} = t_1^* \\ Q^*_{int,crisp} = Q^* \end{cases}$$

Step 3 Initialize $Sol_{best} = Sol_{int,crisp}$.

Step 4 Apply the dense fuzzy rule on problem 1 considering demand to be Triangular dense fuzzy set and call it as model 2.

Step 5 Input the values of the additional parameters to describe the fuzzy model.

Step 6 Solve the problem 2 for TAC^*, T^*, t_1^*, Q^* and store the optimal solution as

$$Sol_{int,fuzzy} = \begin{cases} TAC^*_{int,fuzzy} = TAC^* \\ T^*_{int,fuzzy} = T^* \\ t^*_{1,\ \ int,fuzzy} = t_1^* \\ Q^*_{int,fuzzy} = Q^* \end{cases}$$

Step 7 Check whether $TAC^*_{int,fuzzy} < TAC^*_{best}$. If yes go to step 8 else go to step 9.

Step 8 Store $Sol_{best} = Sol_{int,fuzzy}$.

Step 9 Store $Sol_{best} = Sol_{best}$.

Step 10 Consider the fractional calculus to describe the model and call it the model 3.

Step 11 Input the values of the additional parameters (differential and integral memory index) to describe the fractional model.

Step 12 Solve the problem 3 for TAC^*, T^*, t_1^*, Q^* and store the optimal solution as

$$Sol_{frac,crisp} = \begin{cases} TAC^*_{frac,crisp} = TAC^* \\ T^*_{frac,crisp} = T^* \\ t^*_{1\ \ frac,crsip} = t_1^* \\ Q^*_{frac,crisp} = Q^* \end{cases}$$

Step 13 Check whether $TAC^*_{frac,crisp} < TAC^*_{best}$. If yes go to step 14 else go to step 15.

Step 14 Store $Sol_{best} = Sol_{frac,Crisp}$.

Step 15 Store $Sol_{best} = Sol_{best}$.

Step 16 Simultaneously, fractional calculus and dense fuzzy demd is to be considered to describe the model 1 and call it the model 4.

Step 17 Input the values of the additional parameters to describe fuzzy fractional model.

Step 18 Solve the problem 4 for TAC^*, T^*, t_1^*, Q^* and store the optimal solution as

$$Sol_{frac,fuzzy} = \begin{cases} TAC^*_{frac,fuzzy} = TAC^* \\ T^*_{frac,fuzzy} = T^* \\ t^*_{1\ \ frac,fuzzy} = t_1^* \\ Q^*_{frac,fuzzy} = Q^* \end{cases}$$

Step 19 Check whether $TAC^*_{frac,fuzzy} < TAC^*_{best}$. If yes go to step 20 else go to step 21.

Step 20 Store $Sol_{best} = Sol_{frac,fuzzy}$. Go to step 22.

Step 21 Store $Sol_{best} = Sol_{best}$. Go to step 22.

Step 22 Select the global best solution among four problems as $Sol_{gbest} = Sol_{best}$.

Step 23 End.

Flow chart

The pictorial diagram (flow chart) of the above described algorithm is presented by figure 3.

6. Numerical illustration

6.1 *Numerical simulation*

For numerical illustration, we consider the following numerical values of the parameters involved in the proposed model:

(i) Crisp integer order model: Let, $c_0 = 5000$, $c_h = 3.5$, $s_c = 1.5$ and $a = 800$.
(ii) Fuzzy integer order model: Let, $c_0 = 5000$, $c_h = 3.5, s_c = 1.5$, $a_0 = 800$, $\rho = 0.3$, $\sigma = 0.2$. and $N = 10$.

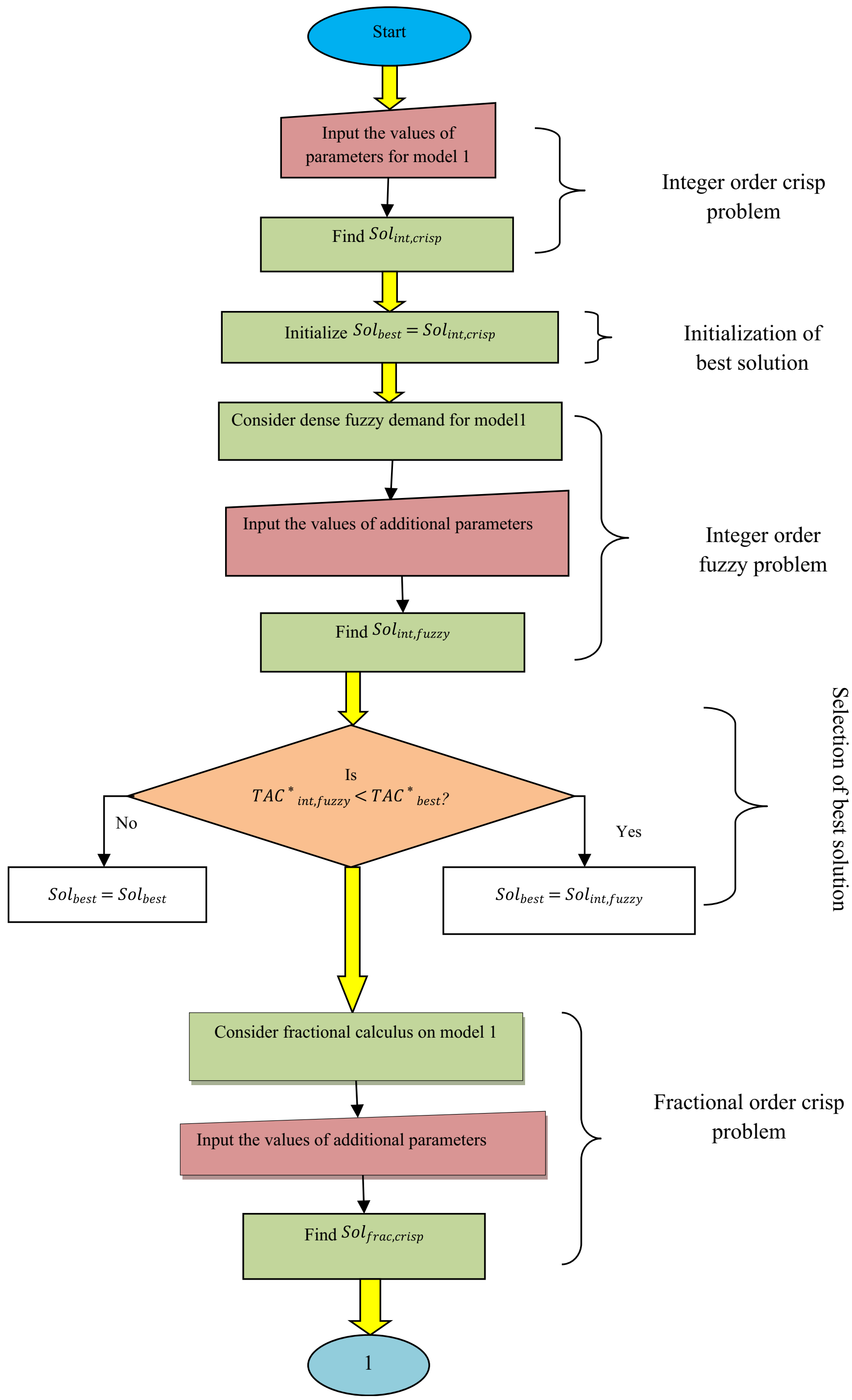

Figure 3. Flow chart for solution methodology.

(iii) Crisp fractional order model: Let, $c_0 = 5000$, $c_h = a00.\beta_d^* = \beta_i^* = 0.9$.

(iv) Fzy fractional order model: Let, $c_0 = 5000$, $c_h = 3.5$,$s_c = 1.5$, $a_0 = 800$, $\rho = 0.3$, $\sigma = 0.2$, $N = 10$ and $\beta_d^* = \beta_i^* = 0.9$.

The optimum value of the objective function and the decision variables are presented in the table 2.

The optimum (minimum) values of the objective function in different scenarios are presented graphically in figure 2. From table 2 and figure 4, we observe an interesting fact that the minimum total average cost decreases as we go through the model from up to down in table 2 and left to right in figure 4. Here, cost can be minimized more considering the triangular dense fuzzy demand instead of crisp

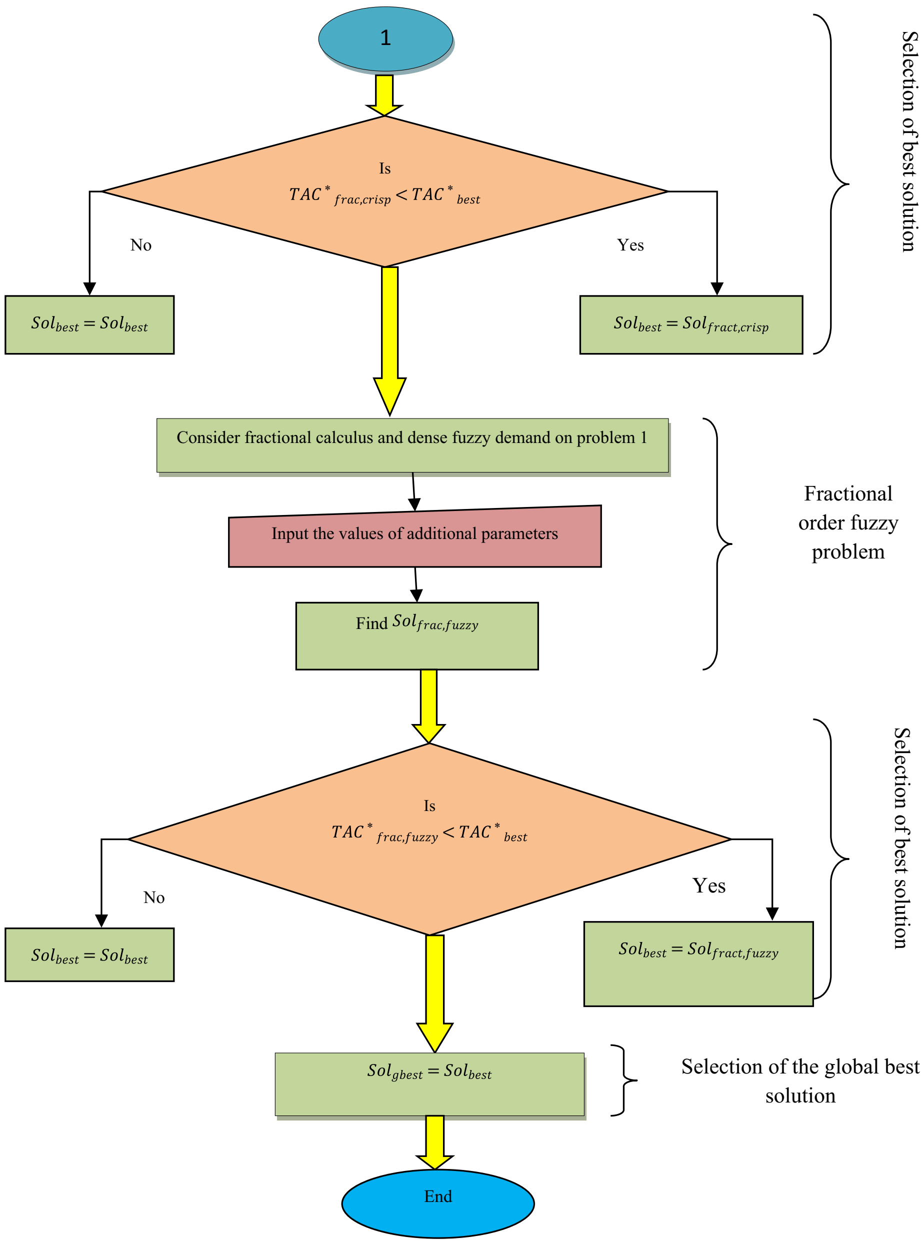

Figure 3. continued

Table 2. Optimal solution of the models in different scenarios.

Model	t_1^*	T^*	Q^*	S_1^*	β_d^*	β_i^*	TAC^*
Crisp integer order model	1.284	3.529	3424.72	1027.42	1	1	3595.96
Fuzzy integer order model	1.287	3.541	3431.56	1029.56	1	1	3584.94
Crisp fractional order model	1.524	4.923	5125.79	1215.18	0.9	0.9	3420.90
Fuzzy fractional order model	1.527	4.943	5135.69	1217.24	0.9	0.9	3408.79

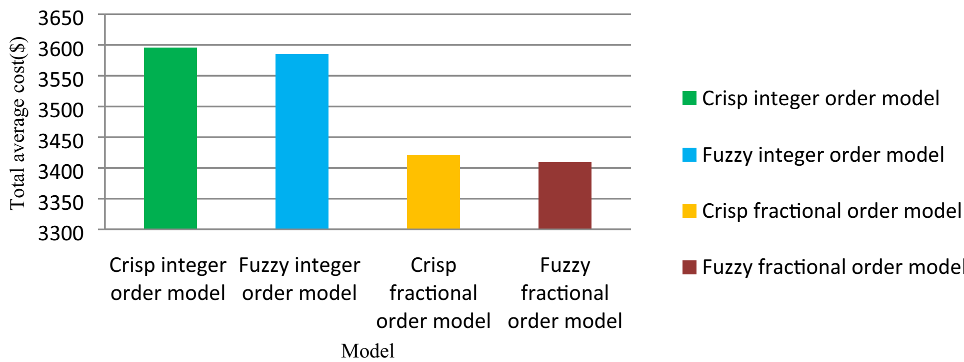

Figure 4. Total average cost (TAC) in different models.

Table 3. Sensitivity with respect to the memory (both differential and integral) index.

$\beta_d^* = \beta_i^*$	t_1^*	T^*	Q^*	S_1^*	TAC^*
1	1.287	3.541	3431.56	1029.56	3584.94
0.9	1.527	4.943	5135.69	1217.24	3408.79
0.8	1.775	13.743	6349.91		
0.7	0.11464×10^{14}	0.1×10^{31}	0.119821×10^{13}	0.1195893×10^{13}	2.20
0.6	0.00	0.1678455×10^{30}	1870277	0.00	0.00
0.5	0.00	0.6871656×10^{19}	2126.94	0.00	0.00
0.4	0.00	0.11464×10^{14}	79.61	0.00	0.00
0.3	0.00	0.7750909×10^{12}	7.66	0.00	0.00
0.2	0.266	0.2780237×10^{12}	669.28	668.68	0.00
0.1	0.473	0.260448×10^{12}		780.26	780.22

Table 4. Sensitivity with respect to the differential memory index.

β_d^*	β_i^*	t_1^*	T^*	Q^*	S_1^*	TAC^*
1	1	1.287	3.541	3431.56	1029.56	3584.94
0.9	1	1.397	4.048	5025.59	1117.81	3522.74
0.8	1	1.551	4.937	5657.02	1240.88	3399.67
0.7	1	1.760	6.822	6807.90	1407.92	3187.93
0.6	1	1.996	13.184	9624.28	1596.75	2831.97
0.5	1	2.565	0.1096731×10^{11}	0.2513405×10^{9}	2052.06	2116.20
0.4	1	0.000	0.1932066×10^{28}	0.1054921×10^{18}	0.00	4.11
0.3	1	0.000	0.385291×10^{25}	0.471082×10^{16}	0.00	0.023
0.2	1	0.000	0.6220013×10^{22}	0.1907953×10^{15}	0.00	0.00
0.1	1	0.000	0.5569871^{1}	0.5664106	0.00	0.02

for a memory less model. However, considering a very short memory on the crisp model, we get better result in the view of cost minimization. The third model is better not only over the crisp integer order model but also the dense fuzzy integer model. So, the presence of small memory affects the model more positively in terms of reducing the cost. Finally, memory effected learning based model i.e., the fourth model shows the best result among the four models we considered. The fourth model is better than the third model where memory is considered but the learning experience is absent. This motivates us to make the conclusion that dealing the EOQ model with fractional differential equation under dense fuzzy environment is a more acceptable and smart approach.

Table 5. Sensitivity with respect to the integral memory index.

β_d^*	β_i^*	t_1^*	T^*	Q^*	S_1^*	TAC^*
1	1	1.287	3.541	3431.56	1029.56	3584.94
1	0.9	1.397	4.048	4664.64	1124.01	3522.74
1	0.8	1.551	4.937	4637.25	1220.33	3399.67
1	0.7	1.760	6.822	4511.89	1307.80	3187.93
1	0.6	1.996	13.184	4201.78	1355.42	2831.97
1	0.5	2.200	0.2174636×10^{14}	3404.22	1277.28	2116.20
1	0.4	0.000	0.1×10^{31}	1.99	0.00	2.20
1	0.3	0.000	0.1×10^{31}	0.00	0.00	0.00
1	0.2	0.000	0.1×10^{31}	0.00	0.00	0.00
1	0.1	0.000	0.240654×10^{24}	0.00	0.00	0.00

Table 6. Sensitivity with respect to the parameters used to describe dense fuzzy demand.

Parameter	Change (%)	t_1^*	T^*	Q^*	S_1^*	TAC^*
$N = 10$	+ 30	1.530	4.940	5134.27	1216.94	3410.48
	+ 20	1.530	4.941	5134.70	1217.03	3410.00
	+ 10	1.530	4.942	5135.16	1217.12	3409.39
	− 10	1.527	4.944	5136.29	1217.36	3408.00
	− 20	1.527	4.945	5137.00	1217.51	37.14
	− 30	1.527	4.947	5137.82	1217.68	3406.13
$\rho = 0.3$	+ 30	1.529	4.961	5144.65	1219.10	3397.79
	+ 20	1.528	4.955	5141.66	1218.48	3401.44
	+ 10	1.527	4.949	5138.67	1217.86	3405.09
	− 10	1.526	4.937	5132.71	1216.61	3412.39
	− 20	1.525	4.931	5129.74	1216.00	3416.04
	− 30	1.524	4.925	5126.77	1215.39	3419.69
$\sigma = 0.2$	+ 30	1.525	4.931	5129.74	1216.00	3416.04
	+ 20	1.525	4.935	5131.72	1216.41	3413.61
	+ 10	1.526	4.939	5133.70		3411.18
	− 10	1.527	4.947	5137.67	1217.65	3406.31
	− 20	1.528	4.951	5139.66	1218.06	3403.88
	− 30	1.528	4.955	5141.66	1218.48	3401.44

6.2 *Sensitivity analysis*

The sub section 6.1 establishes the effectiveness of the fourth model over the others. So, in this subsection we have performed the sensitivity analysis for the fuzzy fractional order model. Here, we analyse the sensitivity of the optimum solution (with respect to cost minimization) through different angles.

The dispersion of the optimum solution with respect to the change of the memory index (both integral and differential) is shown in table 3. In table 3, as the memory index decreases (the memory of the dynamical model increases), the value of the total average cost also decreases. In other words, more strong memory results in a better one in the perspective of cost minimization. Others parameters (inventory run time, total time cycle, lot size and initial inventory level) are gradually increased with the improvement of memory up to certain memory index ($\beta_d^* = \beta_i^* = 1$). After that, the total time cycle grow faster than the inventory run time which makes the inventory run phase to be absent in the time cycle. Though, the cost minimizing pattern is followed as the increasing nature of memory, the situation occurred due to the presence of much memory is not an acceptable incident in reality.

Tables 4 and 5 present the sensitivity of the optimum values of objective function and the decision variables with respect to the differential and integral memory index respectively. The integral and differential memory indexes are set to 1 in tables 4 and table 5. The observation and analysis for these two tables are same as in table 3 except one thing. That is, in both tables 4 and 5, the acceptable results are obtained up to the memory index 0.6, whereas in the table 3, it was 0.8. Moreover, another

Table 7. Sensitivity with respect to the crisp parameters.

Crisp parameter	Change (%)	t_1^*	T^*	Q^*	S_1^*	TAC^*
$c_0 = 5000$	+ 30	1.681	6.095	5668.91	1327.26	31.29
	+ 20	1.632	5.717	5501.97	1292.96	3596.61
	+ 10	1.581	5.334	5324.77	1256.41	3506.09
	− 10	1.468	4.544	4932.60	1174.95	3303.29
	− 20	1.404	4.137	4712.80	1128.94	3188.03
	− 30	1.334	3.720	4472.55	1078.32	3060.66
$c_h = 3.5$	+ 30	1.155	5.044	5074.03	946.71	3544.13
	+ 20	1.260	5.013	5092.59	1023.56	3506.53
	+ 10	1.382	4.979	5113.06	1112.72	3462.03
	− 10	1.700	4.905	5160.75	1341.18	3344.02
	− 20	1.911	4.868	5188.58	1490.13	3264.18
	− 30	2.172	4.837	5219.73	1671.96	3163.92
$s_c = 1.5$	+ 30	1.786	3.952	5647.08	1402.03	3865.07
	+ 20	1.709	4.224	5507.45	1347.58	3731.34
	+ 10	1.623	4.550	5338.06	12.15	3579.75
	− 10	1.420	5.421	4897.10	1140.40	3216.80
	− 20	1.303	6.010	4619.09	1055.26	3002.40
	− 30	1.175	6.751	4298.40	961.48	2764.03

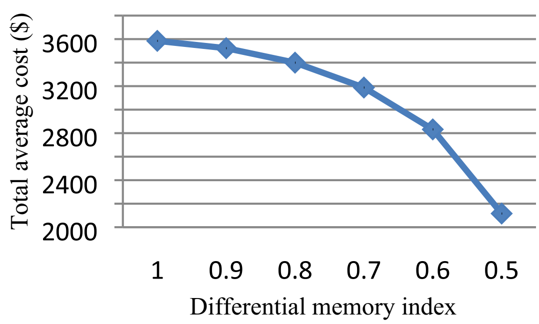

Figure 5. TAC with respect to differential memory index.

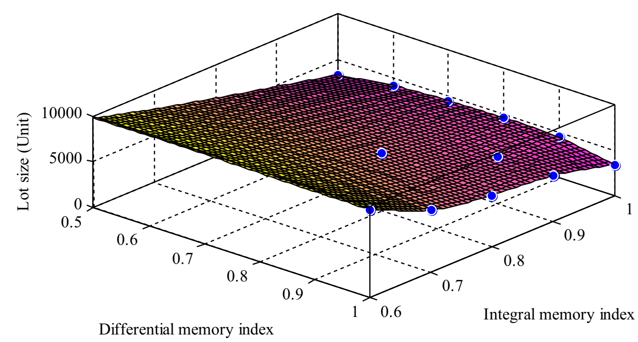

Figure 7. Lot-size with respect to memory index.

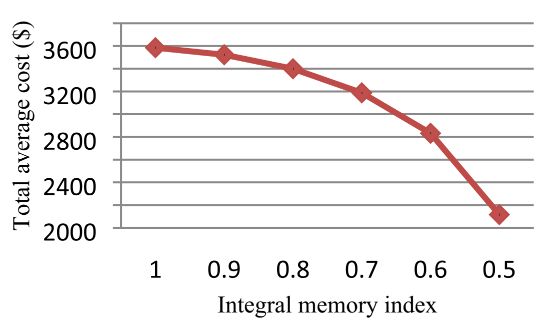

Figure 6. TAC with respect to differential memory index.

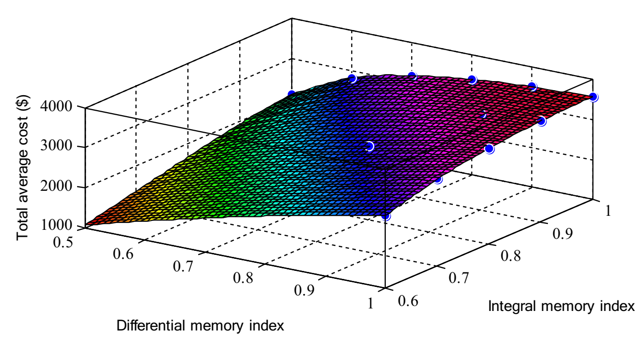

Figure 8. Total average cost with respect to memory index.

important observation is that the values of the total average cost presented in tables 4 and 5 are same. So, the total average cost is symmetric function with respect to the differential and integral memory index.

Table 6 presents the variation of the optimal values of the objective function and decision variables with respect to the parameter involved in defining the triangular dense fuzzy

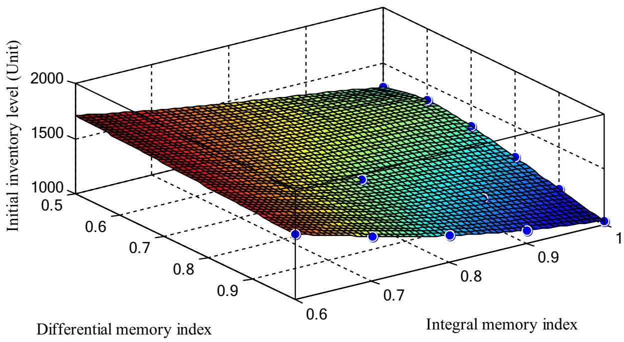

Figure 9. Initial inventory level with respect to memory index.

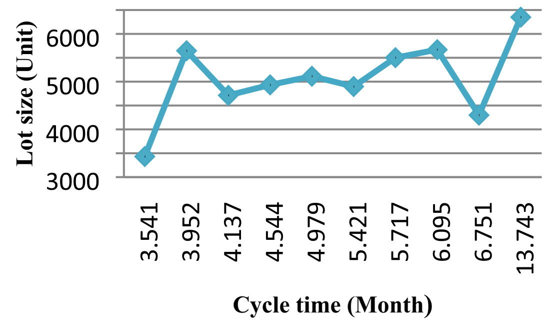

Figure 11. Lot size with respect to time cycle.

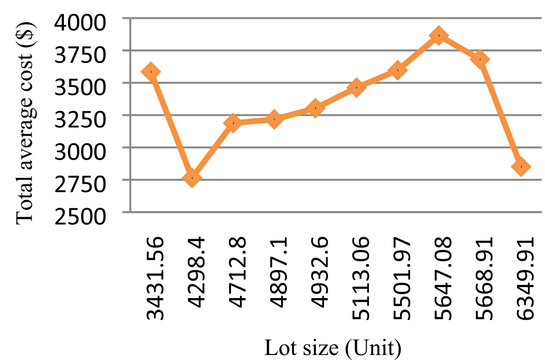

Figure 12. TAC with respect to lot size.

demand. The total average cost is regulated by changing the values of N ,ρ and σ. In table 6, the values of these three parameters are changed from $-$ 30% to + 30%. The change in the values of decision variables and the objective function is very small with respect to the change of these parameters. As the number of interaction (N) between supplier and customers increases the cost also increases. This observation restricts the model locally to do small interaction in the perspective of cost minimization. However, as the number of interaction progresses, the uncertainty about the demand is reduced which helps the decision maker to take a more appropriate decision. Thus, more interaction for learning experience makes a negative impact on the objective of cost minimization. But it helps the decision maker to take more accurate decision in a fuzzy circumstance.

Table 7 indicates the change of the optimal solution with respect to the crisp parameters, say c_0, c_h. and s_c. All the three parameters are changed from $-$ 30% to 30%. For the change of the values of c_0 in the mentioned scale, the total average cost is also changed to a wider region of \$3060.66 to \$3681.29. Similarly, changes in the other two non fuzzy parameters also result in much change in the solution.

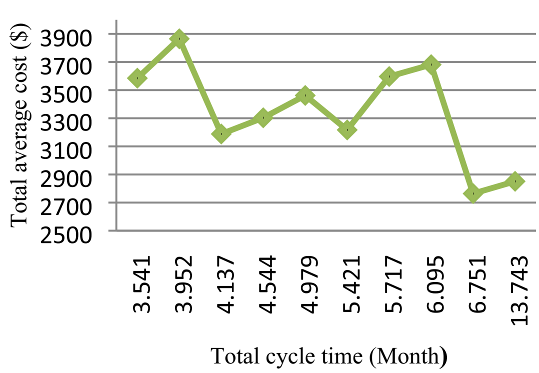

Figure 10. TAC with respect to time cycle.

6.3 *Graphical illustration of model output*

In this sub-section we graphically represent the output of the fractional order EOQ model with shortage under dense fuzzy environment.

The variation of the total average cost (TAC) with respect to the differential and integral memory index have been plotted in figures 5 and 6 respectively where the index values varyng in the range from 1 to 0.5.

The more index value indicates less memory and vice versa. In both cases of figures 5 and figure 6, variation of TAC is represented by a strictly decreasing curve with

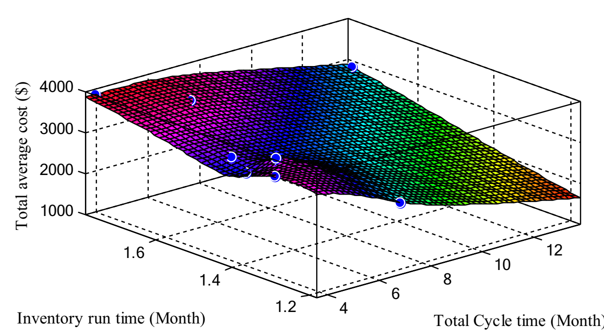

Figure 13. Inter-dependency among total average cost, inventory run time and total time cycle.

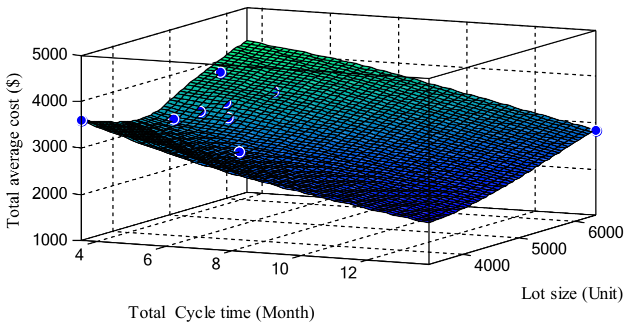

Figure 14. Inter-dependency among total average cost, lot-size and total time cycle.

respect to the strength of memory. This indicates that within the specified region of memory parameter, better memory helps to achieve better solution in the cost minimization perspective. Figures 7, 8 and 9 depict three-dimensional presentations of lot size, total average profit and initial inventory level with respect to the differential and integral memory index.

From the surface graph of figure 7, it is observed that lot-size increases as the value of the memory index decreases (i.e., when the memory strength increases). Also, the presence of differential memory (taking integral memory index as 1) gives the least lot-size whereas that of both the memory gives the best and the presence of integral memory (taking differential memory index as 1) gives the medium values of lot size. The lowest peak of lot size is 3431.56 units given for the differential and integral memory to be 1 (memory less case) and the highest peak of lot size is 9624.28 units obtained for the integral memory index to be 0.6 and differential memory index to be 1.

The surface graph of figure 8 indicates that as the memory index (both differential and integral) decreases (i.e., strength of memory increases), the values of the total average cost (*TAC*) decreases. It is also noted that the effort towards decreasing the cost, contribution of both memory indexes are equal. The highest peak of the *TAC* is \$358.944 obtained for the differential as well as the integral memory index to be 1 and the lowest peak of *TAC* is \$2116.2 obtained for the integral memory to be 1 and differential memory to be 0.5.

The surface graph of figure 9 indicates that as the memory index (both differential and integral) decreases (i.e., strength of memory increases), the values of the initial inventory level (S_1) increases with an exception of a sudden slope at the differential memory index 0.5 and integral memory index 1 which gives 1277.28 units as the initial inventory level. The highest pk of S_1 is 1596.75 units obtained for the integral memory index to be 0.6 and differential memory index to be 1 and the lowest peak of S_1 is 1029.56 unit obtained for the both the memory index to be 1.

Figure 10 represents a two-dimensional plot of total average cost (*TAC*) with respect to the total cycle time (T). The curve of *TAC* faces several ups and downs against the increasing nature of T. Highest peak of *TAC* is \$3865.07 for T =3.952 months. Also, the lowest value of *TAC* it afied region is \$2764.03 given at T =6.751 months.

The figure 11 presents the graphical plot of Lot size (Q) with respect to the total cycle time (T) which has also several ups and downs in the values of Q. However, the lowest value of Lot size 3431.56 units is given for the total cycle time 3.541 months which is eventually the least value of T in the considered region. Also, same thing happed for the highest peak of Q which is 6349.91 units and is set to the total cycle time 13.743 month. This value of T is also the largest one among the considered values of T.

In the figure 12, total average cost is plotted against the lot size. Initially, the curve is decreasing between the Lot size 3431.56 units and 4298.40 units and then gradually increases up to the value of the lot size 5647.08 units. From the values 5647.08 units to 6349.91 units, the curve for *TAC* again decreases. The lowest values of *TAC* are \$2764.03 for Q =4298.4 units and the highest peak of *TAC* is \$3865.07 for Q=5647.98 units.

The three-dimensional figure 13 portrays the relational inter-dependency of TAC, T and t_1. When, the value of T is around 7 and t_1 is around 1.2; the value of *TAC* will be minimum. Also, when the value of T is around 4 and t_1. is around 1.8, the value of *TAC* wilbmimum.

The relational inter-dependency of TAC, T and Q is shown in figure 14. When, the values of T is near 7 months and Q is near 4300 units, the value of *TAC* will be minimum. Also, When, the values of T is about 4 months and Q is about 5650 units, the value of *TAC* will be maximum.

7. Discussion on fundamental insights

In this research article, we have embedded and cultivated the idea of memory and learning pattern of inventory management system in an EOQ model through notion of fractional differential equation under Caputo derivative and Riemann–Liouville in an imprecise environment. Here we have noticed the following important findings.

(i) The implementation of the learning experience based study through the TDFS on integer order crisp model gives a better result compared to the integer order crisp model in the view point of cost minimization.
(ii) Considering the memory sensitivity on the crisp model proves its effectiveness over both the memory less fuzzy and crisp model.
(iii) The proposed EOQ model gives best result when we incorporate both the memory sensitivity and

Table 8. Values of gamma functions corresponding to the memory index (up to 5 places of decimals).

Differential memory index (β_d)	Integral memory index (β_i)	$\Gamma(\beta_d+1)$	$\Gamma(\beta_i+1)$	$\Gamma(\beta_d+\beta_i+1)$	$\Gamma(\beta_d+\beta_i+0.5)$	
1	1	1	1	2	1.32934	623
0.9	0.9	0.96177	0.96177	1.67649	1.16671	0.88726
0.8	0.8	0.93138	0.93138	1.42962	1.04649	0.89747
0.7	0.7	0.90864	0.90864	1.24217	0.96177	0.91817
0.6	0.6	0.89352	0.89352	1.10180	0.90864	0.95135
0.5	0.5	0.88623	0.88623	1	0.88623	1
0.4	0.4	0.88726	0.88726	0.93138	0.89747	1.06863
0.3	0.3	0.89747	0.89747	0.89352	0.95135	1.16423
0.2	0.2	0.91817	0.91817	0.88726	1.06863	1.29806
0.1	0.1	0.95135	0.95135	0.91817	1.29806	1.48919
1	0.9	1	0.96177	1.82736	1.24217	0.88623
1	0.8	1	0.93138	1.67649	1.16671	0.88623
1	0.7	1	0.90864	1.54469	1.10180	0.88623
1	0.6	1	0.89352	1.42962	1.04649	0.88623
1	0.5	1	0.88623	1.32934	1	0.88623
1	0.4	1	0.88726	1.24217	0.96177	0.88623
1	0.3	1	0.89747	1.16671	0.93138	0.88623
1	0.2	1	0.91817	1.10180	0.90863	0.88623
1	0.1	1	0.95135	1.04649	0.89352	0.88623
0.9	1	0.96177	1	1.82736	1.24217	0.88726
0.8	1	0.93138	1	1.67649	1.16671	0.89747
0.7	1	0.90864	1	1.54469	1.10180	0.91817
0.6	1	0.89352	1	1.42962	1.04649	0.95135
0.5	1	0.88623	1	1.32934	1	1
0.4	1	0.88726	1	1.24217	0.96177	1.06863
0.3	1	0.89747	1	1.16671	0.93138	1.16423
0.2	1	0.91817	1	1.10180	0.90863	1.29806
0.1	1	0.95135	1	1.04649	0.89352	1.48919

learning based experience through interaction. However, too much memory on the model creates a scenario which is not acceptable because it nullifies the inventory run phase.

(iv) More interactions between supplier and the customers help to turn the uncertain model to its deterministic form. So, good interactions between supplier and the customers are very much desirable for better and precise decision in the long run.

(v) However, through more interactions the total average cost may be increased. This indicates a tendency of failure to minimize the cost for too much interaction.

Therefore, the fundamental insights for a smart decision maker (DM) from the analysis of the pattern spotted in this paper are as follows:

a) DM must be careful to utilize memory selecting feasible measure of memory index.
b) DM also has to be concerned about the appropriate means and measure of interaction for minimizing the cost as well as to minimize ambiguity in decision making.

8. Conclusion

In this present paper, an EOQ model with shortage is described by the fractional differential equation, which incorporates the notion of memory involved in the decision making procedure. The demand is assumed to be uncertain and it is considered that the learning experience helps on the crystallization over the ambiguities. The uncertain phenomena and its convergence towards deterministic scenario are modelled by the dense fuzzy approach of experience based learning. We have improved the EOQ model into the fuzzy fractional dynamic model by using the Zadeh's extension principle. It is established in this study that the experience based learning and memory motivated decision impressively helps to achieve better minimum average cost, which is the most interesting theoretical finding of this paper. However, the authors accepted some limitations in this current experimentation. The whole study is developed on the hypothetical data instead of data from the practical fields of market and management sector. In future, we will engage ourselves to validate the logical finding established in the current paper by collecting data from real domain of business world. Besides, some more scopes for future study in this direction are as follows: This EOQ model can be replaced by more complicated and realistic model in the current phenomena. The approach of dealing the fuzzy fractional model can be experimented through other possible approaches like Adomian decomposition, fuzzy Laplace transformation, etc. Fractional order modelling in fuzzy phenomena may be examined for other dynamical model described by integer order calculus in crisp and uncertain situations.

Appendix

Let $x \in \mathbb{C}$, then the Gamma function is given by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \text{ for } Re(x) > 0.$$

In table 8 we represent the value of gamma fctions cresponding to the values of the integral and differential memory indices. Here, we used the online gamma function calculator to obtain the values of the gamma functions and consider the approximation up to 5 decimal digits.

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