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Inner bounds via simultaneous decoding in quantum network information theory

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Abstract. We prove new inner bounds for several multiterminal channels with classical inputs and quantum outputs. Our inner bounds are all proved in the one-shot setting and are natural analogues of the best classical inner bounds for the respective channels. For some of these channels, similar quantum inner bounds were unknown even in the asymptotic independent and identically distributed setting. We prove our inner bounds by appealing to a new classical–quantum joint typicality lemma established in a companion paper. This lemma allows us to lift to the quantum setting many inner bound proofs for classical multiterminal channels that use intersections and unions of typical sets.

Keywords. Quantum simultaneous decoder; one-shot inner bounds; broadcast channel; interference channel; network information theory.

1. Introduction

An important technical tool used in proving inner bounds in classical network information theory is the so-called conditional joint typicality lemma [1, 2]. What is equally important but often not emphasised are the implicit *union* and *intersection* arguments used in the inner bound proofs. For quantum channels, proving a joint typicality lemma that can withstand union and intersection arguments was a big bottleneck. As a result of this bottleneck, many inner bounds in classical network information theory were hitherto not known to be extendable to the quantum setting.

Most inner bounds in information theory were first proved in the traditional setting of many independent and identically distributed (iid) uses of a classical communication channel. Recently, attention has shifted to proving inner bounds in the *one-shot* setting where the classical or quantum channel can be used only once. This is the most general setting. The aim is to prove good one-shot inner bounds that ideally yield the best known inner bounds when restricted to the asymptotic iid and asymptotic non-iid (information spectrum) settings. In the one-shot setting, the importance of union and intersection arguments increases and they often need to be made explicit. This is because the technique of time sharing often used in the asymptotic iid setting does not apply in the one-shot setting. In other words, the one-shot setting forces us to look for so-called *simultaneous decoders* for multiterminal channels. The inner bound analyses for simultaneous decoders generally use union and intersection arguments.

Fawzi *et al* [3] and Sen [4] did construct a simultaneous decoder for the two-sender multiple access channel with classical inputs and quantum output (cq-MAC) but their constructions, which were given in the asymptotic iid setting, are not known to work in the one-shot setting. Qi, Wang and Wilde [5] constructed a one-shot simultaneous decoder for the cq-MAC with an arbitrary number of senders, but their achievable rates restricted to the asymptotic iid setting are inferior to the optimal rates obtained by Winter [6] using successive cancellation. Thus, for more than two senders a simultaneous decoder for the cq-MAC achieving optimal rates was hitherto unknown even in the asymptotic iid setting. A simultaneous decoder for the MAC with three senders is used as a crucial ingredient in the proof of the Han–Kobayashi inner bound for the interference channel [7], even in the asymptotic iid classical setting. Thus the lack of a simultaneous decoder for the asymptotic iid quantum setting is a bottleneck, which was sidestepped by Sen [4] by constructing a simultaneous decoder for a restricted type of three-sender cq-MAC, which sufficed to prove the Han–Kobayashi inner bound in the asymptotic iid setting for sending classical information over a quantum interference channel (q-IC). Hirche, Morgan and Wilde [8] also proved the Han–Kobayashi inner bound for sending classical information over a q-IC in the asymptotic iid setting. They did so using successive cancellation and polar coding. However, both Sen's and Hirche *et al*'s techniques are tied to the asymptotic iid setting and do not give any non-trivial inner bound for the interference channel in the one-shot setting. Additionally, those techniques do not seem to give any non-trivial inner bound for

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the entanglement-assisted interference channel even in the asymptotic iid quantum setting.

Very recently, in a companion paper, Sen [1] proved a one-shot quantum joint typicality lemma that possesses strong union and intersection properties. Using this lemma, he also constructed a one-shot simultaneous decoder for the cq-MAC with an arbitrary number of senders. In this paper[1], we use the quantum joint typicality lemma from [1] to obtain for the first time non-trivial one-shot inner bounds for sending classical information over several multiterminal quantum channels. The channels that we consider are the broadcast channel and interference channel, both without and with entanglement assistance. For both channels our one-shot quantum inner bounds are the natural analogues of the best known classical asymptotic iid inner bounds, and reduce to them in the iid limit.

1.1 *Organisation of the paper*

In the next section we state some preliminary facts, which will be useful throughout the paper. In section 3 we state two simple versions of Sen's quantum joint typicality lemma [1], which suffice for the applications in this paper. In section 4, we prove a one-shot Marton inner bound with common message [10] for sending classical information through unassisted as well as entanglement-assisted quantum broadcast channel (q-BC). section 5 proves the achievability of the Han–Kobayashi [7] and Chong–Motani–Garg–El Gamal [11] inner bounds for one-shot use of a cq-interference channel. Finally, we make some concluding remarks and list some open problems in section 6.

2. Preliminaries

All Hilbert spaces in this paper are finite dimensional. The symbol $\oplus$ always denotes the orthogonal direct sum of Hilbert spaces. For a subspace X of a Hilbert space $\mathcal{H}$, let $\Pi_X^{\mathcal{H}}$ denote the orthogonal projection in $\mathcal{H}$ onto X. When clear from the context, we may use Π_X instead of $\Pi_X^{\mathcal{H}}$ for brevity of notation.

A quantum state or a density matrix in a Hilbert space $\mathcal{H}$ refers to a Hermitian, positive semidefinite linear operator on $\mathcal{H}$ with trace equal to one. A POVM element Π in $\mathcal{H}$ refers to a Hermitian positive semidefinite linear operator on $\mathcal{H}$ with eigenvalues between 0 and 1. Stated in terms of inequalities on Hermitian operators, $\mathbb{0} \leq \Pi \leq \mathbb{1}$ where $\mathbb{0}$, $3\mathbb{1}$ denote the zero and identity operators on $\mathcal{H}$.

Let $\|v\|_2$ denote the ℓ_2-norm of a vector $v \in \mathcal{H}$. For an operator A on $\mathcal{H}$ we use $\|A\|_1$ to denote the Schatten ℓ_1-norm, also known as trace norm, of A, which is nothing but the sum of singular values of A. We use $\|A\|_\infty$ to denote the Schatten ℓ_∞-norm, also known as operator norm, of A, which is nothing but the largest singular value of A. For operators A, B on $\mathcal{H}$, we have the inequality

$$|\operatorname{Tr}[AB]| \leq \|AB\|_1 \leq \min\{\|A\|_1\|B\|_\infty, \|A\|_\infty\|B\|_1\}.$$

Let $\mathcal{X}$ be a finite set. A *classical–quantum* (hereafter called cq for short) state on $\mathcal{X}\mathcal{H}$ refers to a quantum state of the form $\rho^{\mathcal{X}\mathcal{H}} = \sum_{x \in \mathcal{X}} p_x |x\rangle\langle x|^{\mathcal{X}} \otimes \rho_x^{\mathcal{H}}$ where x ranges over computational basis vectors of $\mathcal{X}$ viewed as a Hilbert space, $\{p_x\}_{x \in \mathcal{X}}$ is a probability distribution on $\mathcal{X}$ and the operators ρ_x for all $x \in \mathcal{X}$ are quantum states in $\mathcal{H}$. We will also use the terminology that ρ is classical on $\mathcal{X}$ and quantum on $\mathcal{H}$. In this paper, superscripts in the notation for a quantum state will denote the Hilbert space in which it lies. A similar convention will be used for classical probability distributions.

In this paper all quantum operations will be trace-non-increasing completely positive superoperators, generalising unitary evolution, POVM measurement and tracing out subsystems. For brevity, we will use the term *superoperator* to denote such operations. An expression like $\mathfrak{C}^{A_1A_2 \to B_1B_2B_3}$ will denote a superoperator taking operators on $A_1 \otimes A_2$ to operators on $B_1 \otimes B_2 \otimes B_3$. We use $\mathbb{I}^A$ to denote the identity superoperator on A. When there is a need for very precise notation, we will use expressions like $(\mathfrak{C}^{A_1A_2 \to B}(\rho^{A_1A_2}))^B$ to denote the (possibly subnormalised) quantum state in B obtained by applying the superoperator $\mathfrak{C}$ to quantum state $\rho^{A_1A_2}$.

For a positive integer c, we will use $[c]$ to denote the set $\{1, 2, \ldots, c\}$. If $c = 0$, we define $[c] := \{\}$. We shall study systems that are classical on $\mathcal{X}^{\otimes[c]}$ and quantum on $\mathcal{H}$. If $\mathbf{x}$ is a computational basis vector of $\mathcal{X}^{\otimes[c]}$, for a subset $S \subseteq [c]$, $\mathbf{x}_S$ will denote its restriction to the system $\mathcal{X}^{\otimes S}$. Thus, $\mathbf{x} \equiv \mathbf{x}_{[c]}$. We also use $\mathbf{x}_S$ to denote computational basis vectors of $\mathcal{X}^{\otimes S}$ without reference to the systems in $[c] \setminus S$. The notation $(\cdot)^{\otimes S}$ denotes a tensor product only for the coordinates in S. We will use the notation $(S_1, \ldots, S_l) \subseteq [c]$ to denote a collection of subsets $S_1, \ldots, S_l$, $l > 0$ of $[c]$. Note that order does not matter in describing a collection of subsets of $[c]$.

We will need Winter's gentle measurement lemma [12].

Fact 1 ([13]) *Let* Λ *be a POVM element and* ρ *be a quantum state such that* $\operatorname{Tr}[\Lambda\rho] \geq 1 - \epsilon$. *Then,* $\left\|\rho - \Lambda^{1/2}\rho\Lambda^{1/2}\right\|_1 \leq 2\sqrt{\epsilon}$.

We recall the definition of the *hypothesis testing relative entropy* given by Wang and Renner [14]. Very similar quantities were defined and used in earlier works [15, 16].

Definition 1 Let α, β be two quantum states in the same Hilbert space. Let $0 \leq \epsilon < 1$. Then the *hypothesis testing relative entropy* of α with respect to β is defined by

[1]Journal version of [9].

$$D_H^{\epsilon}(\alpha\|\beta) := \max_{\Pi:\, \mathrm{Tr}\ [\Pi\alpha] \ge 1-\epsilon} -\log \mathrm{Tr}\ [\Pi\beta],$$

where the maximisation is over all POVM elements Π acting on the Hilbert space.

The definition quantifies the minimum probability of 'accepting' β by a POVM element Π that 'accepts' α with probability of at least $1-\epsilon$. From the definition, it is easy to see that if $\epsilon < \epsilon'$, $D_H^{\epsilon}(\alpha\|\beta) < D_H^{\epsilon'}(\alpha\|\beta)$. We now define the *hypothesis testing mutual information* of a bipartite quantum state ρ^{AB}.

Definition 2 Let $0 \le \epsilon < 1$. Let ρ^{AB} be a quantum state in a bipartite system AB. The *hypothesis testing mutual information* is defined as $I_H^{\epsilon}(A:B)_{\rho} := D_H^{\epsilon}(\rho^{AB}\|\rho^A \otimes \rho^B)$.

For a cq-state, we can define the *hypothesis testing conditional mutual information*.

Definition 3 Let $0 \le \epsilon < 1$. Let ρ^{ABC} be a state that is classical on A and quantum on BC. It can be expressed as $\rho^{ABC} = \sum_a p(a)|a\rangle\langle a|^A \otimes \rho_a^{BC}$. Consider a state σ^{ABC} that is classical on A and quantum on BC defined as $\sigma^{ABC} = \sum_a p(a)|a\rangle\langle a|^A \otimes \rho_a^B \otimes \rho_a^C$. The *hypothesis testing conditional mutual information* is defined as $I_H^{\epsilon}(B:C|A)_{\rho} := D_H^{\epsilon}(\rho^{ABC}\|\sigma^{ABC})$.

Let $0 \le \epsilon \le 1$. Let P, Q be probability distributions on the same sample space $\mathcal{X}$. For non-negative vectors v_1, v_2 supported on $\mathcal{X}$, we use the notation $v_1 \le v_2$ to denote $v_1(x) \le v_2(x)$ for all sample points $x \in \mathcal{X}$. We now define the smooth max relative entropy of P with respect to Q. The definition here is obtained by taking the classical version of the quantity defined by Datta [17], coupled with the observation that there exists a minimising P' in the definition satisfying $P' \le P$. This condition will be useful when we prove a one-shot mutual covering lemma in Fact 2.

Definition 4 The ϵ-smooth max relative entropy of P with respect to Q is defined as

$$D_{\infty}^{\epsilon}(P\|Q) := \min_{0 \le P' \le P:\|P-P'\|_1 \le \epsilon} \max_{x \in \mathcal{X}} \log \frac{P'(x)}{Q(x)},$$

where $\frac{0}{0} := 1$. Note that $D_{\infty}^{\epsilon}(P\|Q)$ can be $+\infty$ if the support of P is not contained in the support of Q and ϵ is small.

For completeness, we recall Datta's definition of ϵ-smooth max relative entropy of quantum state ρ with respect to quantum state σ:

$$D_{\infty}^{\epsilon}(\rho\|\sigma) := \min_{\rho':\|\rho-\rho'\|_1 \le \epsilon} \min_{\lambda \in \mathbb{R}: \rho \le 2^{\lambda}\sigma} \lambda.$$

Note that $D_{\infty}^{\epsilon}(\rho\|\sigma)$ can be $+\infty$ if the support of ρ is not contained in the support of σ and ϵ is small.

For a joint probability distribution P on the sample space $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$, we define the smooth max conditional mutual information as follows.

Definition 5 The ϵ-smooth max mutual information between random variables X and Y conditioned on Z under the joint distribution P is defined as

$$I_{\infty}^{\epsilon}(X:Y|Z)_P := D_{\infty}^{\epsilon}(P^{XYZ}\|P^Z \times (P^X|Z) \times (P^Y|Z))$$

where the superscripts denote the sample spaces of the respective probability distributions, and $P^Z \times (P^X|Z) \times (P^Y|Z)$ denotes the probability distribution on $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ obtained by first taking a sample according to the marginal on $\mathcal{Z}$ followed by independently taking a pair of samples according to the marginals on $\mathcal{Y}$ and $\mathcal{Z}$ conditioned on the chosen sample from $\mathcal{Z}$.

We now define the so-called 'restricted smooth conditional max mutual information' for a quantum state ρ^{ZXY} that is classical on Z and quantum on X and Y. Our definition is the conditional version of a quantity defined in [18].

Definition 6 The ϵ-smooth restricted conditional max mutual information between X and Y conditioned on Z under the state ρ^{XYZ}, where X, Y are quantum and Z is classical, is defined as

$$I_{\infty}^{\epsilon,\delta}(X:Y|Z)_{\rho} := D_{\infty}^{\epsilon,\delta}(\rho^{XYZ}\|\rho^Z \otimes \rho^X|Z \otimes \rho^Y|Z),$$

where the cq-state $\rho^Z \otimes \rho^X|Z \otimes \rho^Y|Z$ is obtained by taking a sample z according to the marginal probability distribution ρ^Z followed by the tensor product of the marginal quantum states $\rho^X|z$ and $\rho^Y|z$ obtained by conditioning on z, and the smoothing in the definition of $D_{\infty}^{\epsilon,\delta}(\cdot\|\cdot)$ is done only over cq-states $(\rho')^{XYZ}$ ϵ-close to ρ^{XYZ} satisfying $(\rho')^{XZ} \le (1+\delta)\rho^{XZ}$ and $(\rho')^{YZ} \le (1+\delta)\rho^{YZ}$.

We next state a *one-shot mutual covering lemma*, which strengthens the one-shot mutual covering lemma of Radhakrishnan *et al* [19, Lemma 3]. Our mutual covering lemma is closely related to the *bipartite convex split lemma* of Anshu, Jain and Warsi [18] specialised to the classical setting. We state it in this form so that it may be useful for other problems in network information theory. For the broadcast channel, it allows us to give a clean one-shot proof of Marton's inner bound with the added advantage of decoding Alice's 'input random variables' exactly and not just 'up to the band' as in the traditional forms of Marton's inner bound.

Fact 2 (One-shot mutual covering lemma) *Let (U_0, U_1, U_2) be a triple of random variables in the sample space $\mathcal{U}_0 \times \mathcal{U}_1 \times \mathcal{U}_2$ with joint distribution function $P^{U_0U_1U_2}$. Let $0 < \epsilon < 1$. Define $I_{\infty} := I_{\infty}^{\epsilon}(U_1:U_2|U_0)_P$. Let r_1, r_2 be positive integers such that*

$$r_1 + r_2 \geq I_\infty + 2\log\frac{1}{\epsilon}.$$

We now define two probability distributions on the set $\mathcal{U}_0 \times (\mathcal{U}_1)^{2^{r_1}} \times (\mathcal{U}_2)^{2^{r_2}} \times [2^{r_1}] \times [2^{r_2}]$ as follows:

1. For the first distribution $(P_1)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}K_1K_2}$, define a new pair of random variables (K_1, K_2) taking uniformly random values in $[2^{r_1}] \times [2^{r_2}]$. Choose first a sample u_0 according to the marginal P^{U_0}. Choose independently a sample (k_1, k_2) from (K_1, K_2). Let

$$\vec{U}_1^{-k_1}|u_0 := (U_1(1) \times \cdots \times U_1(k_1 - 1) \times U_1(k_1 + 1) \times \cdots \times U_1(2^{r_1}))|u_0$$

be $(2^{r_1} - 1)$ independent copies of the random variable $U_1|(U_0 = u_0)$. Similarly, define

$$\vec{U}_2^{-k_2}|u_0 := (U_2(1) \times \cdots \times U_2(k_2 - 1) \times U_2(k_2 + 1) \times \cdots \times U_2(2^{r_2}))|u_0$$

to be $(2^{r_2} - 1)$ independent copies of the random variable $U_2|(U_0 = u_0)$. Let $(U_1(k_1), U_2(k_2))|u_0$ denote the distribution $P^{U_1U_2}|(U_0 = u_0)$ on the (k_1, k_2)th copy. This completes the definition of the probability distribution $(P_1)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}K_1K_2}$ denoted in brief by

$$(P_1)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}K_1K_2} := K_1K_2U_0((U_1(k_1), U_2(k_2))|U_0)(\vec{U}_1^{-K_1}|U_0)(\vec{U}_2^{-K_2}|U_0).$$

2. For the second distribution $(P_2)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}K_1K_2}$, choose first a sample u_0 according to the marginal P^{U_0}. Let

$$\vec{U}_1|u_0 := (U_1(1) \times \cdots \times U_1(2^{r_1}))|u_0$$

be 2^{r_1} independent copies of the random variable $U_1|(U_0 = u_0)$ conditioned on the sample from U_0. Similarly, define

$$\vec{U}_2|u_0 := (U_2(1) \times \cdots \times U_2(2^{r_2}))|u_0$$

to be 2^{r_2} independent copies of the random variable $U_2|(U_0 = u_0)$. A pair $(k_1, k_2) \in [2^{r_1}] \times [2^{r_2}]$ is now chosen conditioned on the other random variables with exactly the same conditioning as in the distribution $(P_1)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}K_1K_2}$. We shall denote the complete distribution so obtained by $(P_2)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}K_1K_2}$ and denote it in brief by

$$(P_2)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}K_1K_2} := U_0(\vec{U}_1|U_0)(\vec{U}_2|U_0)((K_1, K_2)|U_0\vec{U}_1\vec{U}_2).$$

Then

$$\left\|(P_1)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}K_1K_2} - (P_2)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}K_1K_2}\right\|_1 \leq 4\epsilon.$$

Proof First, condition on a sample u_0 from the marginal P^{U_0}. Consider now the distributions $(P_1)^{(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}}|(U_0 = u_0)$, $(P_2)^{(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}}|(U_0 = u_0)$. Suppose one can show that

$$\left\|(P_1)^{(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}}|(U_0 = u_0) - (P_2)^{(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}}|(U_0 = u_0)\right\|_1 \leq 4\epsilon.$$

This will imply that

$$\left\|(P_1)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}} - (P_2)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}}\right\|_1 \leq 4\epsilon.$$

Now observe that the conditioning of (K_1, K_2) on the other random variables is exactly the same in the two distributions $(P_1)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}K_1K_2}$, $(P_2)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}K_1K_2}$. This implies that

$$\left\|(P_1)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}K_1K_2} - (P_2)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}K_1K_2}\right\|_1 = \left\|(P_1)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}} - (P_2)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}}\right\|_1 \leq 4\epsilon.$$

It only remains to show that

$$\left\|(P_1)^{(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}}|(U_0 = u_0) - (P_2)^{(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}}|(U_0 = u_0)\right\|_1 \leq 4\epsilon.$$

For this apply the bipartite convex split lemma of Anshu, Jain and Warsi with the observation that for classical probability distributions the 'smoothing' subdistribution $(P')^{U_0U_1U_2}$ of Definition 5 satisfies $(P')^{U_0U_1U_2} \leq P^{U_0U_1U_2}$, which implies that $\delta = 0$ in Lemma 3 of [18]. The aforementioned inequality then follows easily.

This completes the proof of our one-shot mutual covering lemma. □

We shall use the so-called *pretty good measurement (PGM)* [20, 21], also known as square root measurement, in order to construct our decoders. Given a set of POVM elements Π_m, $m \in [M]$, the PGM is a POVM defined as follows:

$$\Lambda_m := \left(\sum_{m'} \Pi_{m'}\right)^{-1/2} \Pi_m \left(\sum_{m'} \Pi_{m'}\right)^{-1/2}.$$

We use the famous Hayashi–Nagaoka [22] operator inequality in order to analyse the decoding error of the PGM POVM.

Fact 3

$$\mathbb{1} - \Lambda_m \leq 2(\mathbb{1} - \Pi_m) + 4 \sum_{m': m' \neq m} \Pi_{m'}.$$

3. The quantum joint typicality lemma

We now state the versions of the cq joint typicality lemma from [1] that suffice for the applications in this paper.

Fact 4 (cq joint typ. lem., intersec. case) Let $\mathcal{H}$, $\mathcal{L}$ be two Hilbert spaces and $\mathcal{X}$ be a finite set. We also use $\mathcal{X}$ to denote the Hilbert space with computational basis elements indexed by the set $\mathcal{X}$. Let c be a non-negative integer. Let A denote a quantum register with Hilbert space $\mathcal{H}$. For every $\mathbf{x} \in \mathcal{X}^c$, let $\rho_{\mathbf{x}}$ be a quantum state in A. Consider the extended quantum system

$$A' := (\mathcal{H} \otimes \mathbb{C}^2) \oplus \bigoplus_{S: \{\} \neq S \subseteq [c]} (\mathcal{H} \otimes \mathbb{C}^2) \otimes \mathcal{L}^{\otimes |S|}.$$

Also define the *augmented* classical system $\mathcal{X}' := \mathcal{X} \otimes \mathcal{L}$.

Here, $\mathbf{x}$, $\mathbf{l}$ denote computational basis vectors of $\mathcal{X}^{[c]}$, $\mathcal{L}^{\otimes [c]}$. Let $p(\cdot)$ be a probability distribution on the vectors $\mathbf{x}$. Define the cq-state

$$\rho^{\mathcal{X}[c]A} := \sum_{\mathbf{x}} p(\mathbf{x}) |\mathbf{x}\rangle\langle \mathbf{x}|^{\mathcal{X}_{[c]}} \otimes \rho_{\mathbf{x}}^A.$$

Let $\frac{\mathbb{1}^{\mathcal{L}^{\otimes c}}}{|\mathcal{L}|^c}$ denote the completely mixed state on c tensor copies of $\mathcal{L}$. View $\rho_{\mathbf{x}}^A \otimes (|0\rangle\langle 0|)^{\mathbb{C}^2}$ as a state in A' under the natural embedding, viz. the embedding into the first summand of A' defined earlier. Similarly, view $\rho^{\mathcal{X}_{[c]}A} \otimes (|0\rangle\langle 0|)^{(\mathbb{C}^2)} \otimes \frac{\mathbb{1}^{\mathcal{L}^{\otimes c}}}{|\mathcal{L}|^c}$ as a state in $\mathcal{X}'_{[c]}A'$ under the natural embedding.

Let $0 \leq \epsilon, \delta \leq 1$. Let (S_1, S_2, S_3) be disjoint subsets of $[c]$ such that $S_1 \cup S_2 \cup S_3 = [c]$. We allow S_1 or S_3 or both to be empty, and denote the triple by $(S_1, S_2, S_3) \dashv [c]$. Choose $\mathcal{L}$ to have dimension $|\mathcal{L}| = \frac{3^{13}|\mathcal{H}|^4}{2^4(1-\epsilon)^6}$. Then, there is a state ρ' and a POVM element Π' in $\mathcal{X}'_{[c]}A'$ such that

1. The state ρ' and POVM element Π' are classical on $\mathcal{X}^{\otimes [c]} \otimes \mathcal{L}^{[c]}$ and quantum on A'. More precisely, ρ', Π' can be expressed as

$$(\rho')^{\mathcal{X}'_{[c]}A'} = |\mathcal{L}|^{-c} \sum_{\mathbf{x},\mathbf{l}} p(\mathbf{x}) |\mathbf{x}\rangle\langle \mathbf{x}|^{\mathcal{X}_{[c]}} \otimes |\mathbf{l}\rangle\langle \mathbf{l}|^{\mathcal{L}_{[c]}} \otimes (\rho')_{\mathbf{x},\mathbf{l},\delta}^{A'},$$

$$(\Pi')^{\mathcal{X}'_{[c]}A'} = \sum_{\mathbf{x},\mathbf{l}} |\mathbf{x}\rangle\langle \mathbf{x}|^{\mathcal{X}_{[c]}} \otimes |\mathbf{l}\rangle\langle \mathbf{l}|^{\mathcal{L}_{[c]}} \otimes (\Pi')_{\mathbf{x},\mathbf{l},\delta}^{A'},$$

where $(\rho')_{\mathbf{x},\mathbf{l},\delta}^{A'}$, $(\Pi')_{\mathbf{x},\mathbf{l},\delta}^{A'}$ are quantum states and POVM elements, respectively, for all computational basis vectors $\mathbf{x} \in \mathcal{X}^{\otimes [c]}$, $\mathbf{l} \in \mathcal{L}^{\otimes [c]}$;

2.

$$\left\| (\rho')^{\mathcal{X}'_{[c]}A'} - \rho^{\mathcal{X}_{[c]}A} \otimes (|0\rangle\langle 0|)^{\mathbb{C}^2} \otimes \frac{\mathbb{1}^{\mathcal{L}^{\otimes c}}}{|\mathcal{L}|^c} \right\|_1 \leq 2^{\frac{c+1}{2}+1}\delta;$$

3.

$$\mathrm{Tr}\ [(\Pi')^{\mathcal{X}'_{[c]}A'} (\rho')^{\mathcal{X}'_{[c]}A'}] \geq 1 - \delta^{-2} 2^{2^{c+5}} 3^c \epsilon - 2^{\frac{c+1}{2}+1}\delta;$$

4. Let $S \subseteq [c]$. Let $\mathbf{x}_S$, $\mathbf{l}_S$ be computational basis vectors in $\mathcal{X}^{\otimes S}$, $\mathcal{L}^{\otimes S}$. In the following definition, let $\mathbf{x}'_{\bar{S}}$, $\mathbf{l}'_{\bar{S}}$ range over all computational basis vectors of $\mathcal{X}^{\otimes([c]\setminus S)}$, $\mathcal{L}^{\otimes([c]\setminus S)}$. Define a state in A':

$$(\rho')_{\mathbf{x}_S,\mathbf{l}_S,\delta}^{A'} := |\mathcal{L}|^{-|\bar{S}|} \sum_{\mathbf{x}'_{\bar{S}},\mathbf{l}'_{\bar{S}}} p(\mathbf{x}'_{\bar{S}}|\mathbf{x}_S) (\rho')_{\mathbf{x}_S\mathbf{x}'_{\bar{S}},\mathbf{l}_S\mathbf{l}'_{\bar{S}},\delta}^{A'}.$$

Analogously define

$$\rho_{\mathbf{x}_S}^A := \sum_{\mathbf{x}'_{\bar{S}}} p(\mathbf{x}'_{\bar{S}}|\mathbf{x}_S) \rho_{\mathbf{x}_S\mathbf{x}'_{\bar{S}}}^A.$$

Let $(S_1, S_2, S_3) \dashv [c]$. Define

$$\begin{aligned}(\rho')_{(S_1,S_2,S_3)}^{\mathcal{X}'_{[c]}A'} &:= |\mathcal{L}|^{-c} \sum_{\mathbf{x}_{S_1}} p(\mathbf{x}_{S_1}) |\mathbf{x}_{S_1}\rangle\langle \mathbf{x}_{S_1}|^{\mathcal{X}_{S_1}} \otimes |\mathbf{l}_{S_1}\rangle\langle \mathbf{l}_{S_1}|^{\mathcal{L}_{S_1}} \\ &\otimes \left(\sum_{\mathbf{x}_{S_2}} p(\mathbf{x}_{S_2}|\mathbf{x}_{S_1}) |\mathbf{x}_{S_2}\rangle\langle \mathbf{x}_{S_2}|^{\mathcal{X}_{S_2}} \otimes |\mathbf{l}_{S_2}\rangle\langle \mathbf{l}_{S_2}|^{\mathcal{L}_{S_2}} \right) \\ &\otimes \left(\sum_{\mathbf{x}_{S_3}} p(\mathbf{x}_{S_3}|\mathbf{x}_{S_1}) |\mathbf{x}_{S_3}\rangle\langle \mathbf{x}_{S_3}|^{\mathcal{X}_{S_3}} \otimes |\mathbf{l}_{S_3}\rangle\langle \mathbf{l}_{S_3}|^{\mathcal{L}_{S_3}} \otimes (\rho')_{\mathbf{x}_{S_1 \cup S_3},\mathbf{l}_{S_1 \cup S_3},\delta}^{A'} \right), \\ \rho_{(S_1,S_2,S_3)}^{\mathcal{X}_{[c]}A} &:= \sum_{\mathbf{x}_{S_1}} p(\mathbf{x}_{S_1}) |\mathbf{x}_{S_1}\rangle\langle \mathbf{x}_{S_1}|^{\mathcal{X}_{S_1}} \\ &\otimes \left(\sum_{\mathbf{x}_{S_2}} p(\mathbf{x}_{S_2}|\mathbf{x}_{S_1}) |\mathbf{x}_{S_2}\rangle\langle \mathbf{x}_{S_2}|^{\mathcal{X}_{S_2}} \right) \\ &\otimes \left(\sum_{\mathbf{x}_{S_3}} p(\mathbf{x}_{S_3}|\mathbf{x}_{S_1}) |\mathbf{x}_{S_3}\rangle\langle \mathbf{x}_{S_3}|^{\mathcal{X}_{S_3}} \otimes \rho_{\mathbf{x}_{S_1 \cup S_3}}^A \right).\end{aligned}$$

Then

$$\mathrm{Tr}\ [(\Pi')^{\mathcal{X}'_{[c]}A'}(\rho')^{\mathcal{X}'_{[c]}A'}_{(S_1,S_2,S_3)}] \le 2^{-I_H^\epsilon(X_{S_2}:AX_{S_3}|X_{S_1})_\rho},$$

where

$$I_H^\epsilon(X_{S_2}:AX_{S_3}|X_{S_1})_\rho := D_H^\epsilon(\rho^{\mathcal{X}_{[c]}A}\|\rho^{\mathcal{X}_{[c]}A}_{(S_1,S_2,S_3)}).$$

Informally speaking, this lemma guarantees the existence of a *single* POVM element Π' with robust properties that serves as an 'intersection' of the individual POVM elements achieving the hypothesis testing relative entropy quantities arising from the state $\rho^{\mathcal{X}_{[c]}A}$ by considering all possible collections of subsets of $[c]$.

We next state a more general cq joint typicality lemma that guarantees the existence of a *single* POVM element Π' with robust properties that serves as a 'union of intersection' of individual POVM elements.

Fact 5 (cq joint typ. lem., gen. case) Let $\mathcal{H}$, $\mathcal{L}$ be Hilbert spaces and $\mathcal{X}$ be a finite set. We will also use $\mathcal{X}$ to denote the Hilbert space with computational basis elements indexed by the set $\mathcal{X}$. Let c be a non-negative integer. Let A denote a quantum register with Hilbert space $\mathcal{H}$. For every $\mathbf{x} \in \mathcal{X}^c$, let $\rho_\mathbf{x}$ be a quantum state in A. Let t be a positive integer. Let $\mathbf{x}^t$ denote a t-tuple of elements of $\mathcal{X}^c$; we shall denote its ith element by $\mathbf{x}^t(i)$. Consider the extended quantum system $\hat{A}$ where $\hat{A} \cong A' \otimes \mathbb{C}^2 \otimes \mathbb{C}^{t+1}$, and A' is defined as

$$A' := (\mathcal{H}\otimes\mathbb{C}^2) \oplus \bigoplus_{S:i\in S\subseteq[c]\cup[k]} (\mathcal{H}\otimes\mathbb{C}^2)\otimes\mathcal{L}^{\otimes|S|}.$$

Also define the *augmented* classical system $\hat{\mathcal{X}} := \mathcal{X}\otimes\mathcal{L}$.

Here, $\mathbf{x}$, $\mathbf{l}$ denote, respectively, computational basis vectors of $\mathcal{X}^{[c]}$, $\mathcal{L}^{\otimes[c]}$. Let $p(\cdot)$ denote a probability distribution on the vectors $\mathbf{x}$. Let $p(1;\cdot),\ldots,p(t;\cdot)$ denote probability distributions on $\mathbf{x}^t$ such that the marginal of $p(i;\mathbf{x}^t)$ on the ith element is $p(\mathbf{x}^t(i))$. For $i \in [t]$, define the cq-states

$$\rho^{(\mathcal{X}_{[c]})^t A}(i) := \sum_{\mathbf{x}^t} p(i;\mathbf{x}^t)|\mathbf{x}^t\rangle\langle\mathbf{x}^t|^{(\mathcal{X}_{[c]})^t}\otimes\rho^A_{\mathbf{x}^t(i)}.$$

Let $\frac{\mathbb{1}^{\mathcal{L}^{\otimes c}}}{|\mathcal{L}|^c}$ denote the completely mixed state on c tensor copies of $\mathcal{L}$. View $\rho^A_\mathbf{x}\otimes(|0\rangle\langle 0|)^{\mathbb{C}^2}\otimes(|0\rangle\langle 0|)^{\mathbb{C}^2}\otimes(|0\rangle\langle 0|)^{\mathbb{C}^{t+1}}$ as a state in $\hat{A}$ under the natural embedding, viz. the embedding is into the first summand of A' defined earlier, tensored with $\mathbb{C}^2\otimes\mathbb{C}^{t+1}$. Similarly, view $\rho^{(\mathcal{X}_{[c]})^t A}(i) \otimes(|0\rangle\langle 0|)^{\mathbb{C}^2}\otimes\frac{\mathbb{1}^{\mathcal{L}^{\otimes ct}}}{|\mathcal{L}|^{ct}}\otimes(|0\rangle\langle 0|)^{\mathbb{C}^2}\otimes(|0\rangle\langle 0|)^{\mathbb{C}^{t+1}}$ as a state in $(\hat{\mathcal{X}}_{[c]})^{\otimes t}\hat{A}$ under the natural embedding.

Let $0\le\alpha,\epsilon,\delta\le 1$. Choose $\mathcal{L}$ to have dimension $|\mathcal{L}| = \frac{3^{13}|\mathcal{H}|^4}{2^4(1-\epsilon)^6}$. Then, there are states $\rho'(1),\ldots,\rho'(t)$ and a POVM element $\hat{\Pi}$ in $(\hat{\mathcal{X}}_{[c]})^{\otimes t}\hat{A}$ such that

1. The states $\rho'(1),\ldots,\rho'(t)$ and POVM element $\hat{\Pi}$ are classical on $\mathcal{X}^{\otimes[ct]}\otimes\mathcal{L}^{[ct]}$ and quantum on $\hat{A}$. More precisely, $\rho'(i)$, $i\in[t]$, $\hat{\Pi}$ can be expressed as

$$\begin{aligned}(\rho'(i))^{(\hat{\mathcal{X}}_{[c]})^t\hat{A}} &= |\mathcal{L}|^{-ct}\sum_{\mathbf{x}^t,\mathbf{l}^t} p(i;\mathbf{x}^t)|\mathbf{x}^t\rangle\langle\mathbf{x}^t|^{(\mathcal{X}_{[c]})^{\otimes t}}\otimes|\mathbf{l}^t\rangle\langle\mathbf{l}^t|^{(\mathcal{L}_{[c]})^{\otimes t}}\\ &\otimes(\rho')^{A'}_{\mathbf{x}^t(i),\mathbf{l}^t(i),\delta}\otimes(|0\rangle\langle 0|)^{\mathbb{C}^2}\otimes(|0\rangle\langle 0|)^{\mathbb{C}^{t+1}},\\ (\hat{\Pi})^{(\hat{\mathcal{X}}_{[c]})^t\hat{A}} &= \sum_{\mathbf{x}^t,\mathbf{l}^t}|\mathbf{x}^t\rangle\langle\mathbf{x}^t|^{(\mathcal{X}_{[c]})^{\otimes t}}\otimes|\mathbf{l}^t\rangle\langle\mathbf{l}^t|^{(\mathcal{L}_{[c]})^{\otimes t}}\otimes(\hat{\Pi})^{\hat{A}}_{\mathbf{x}^t,\mathbf{l}^t,\delta},\end{aligned}$$

   where $(\rho')^{A'}_{\mathbf{x},\mathbf{l},\delta}$ are quantum states for all computational basis vectors $\mathbf{x}\in\mathcal{X}^{\otimes[c]}$, $\mathbf{l}\in\mathcal{L}^{\otimes[c]}$ and $(\hat{\Pi})^{\hat{A}}_{\mathbf{x}^t,\mathbf{l}^t,\delta}$ are POVM elements for all computational basis vectors $\mathbf{x}^t\in\mathcal{X}^{\otimes[ct]}$, $\mathbf{l}^t\in\mathcal{L}^{\otimes[ct]}$;
2. For all $i\in[t]$

$$\begin{aligned}&\Big\|(\rho'(i))^{(\hat{\mathcal{X}}_{[c]})^t\hat{A}}\\ &-(\rho(i))^{(\mathcal{X}_{[c]})^t A}\otimes(|0\rangle\langle 0|)^{\mathbb{C}^2}\otimes\frac{\mathbb{1}^{\mathcal{L}^{\otimes ct}}}{|\mathcal{L}|^{ct}}\\ &\otimes(|0\rangle\langle 0|)^{\mathbb{C}^2}\otimes(|0\rangle\langle 0|)^{\mathbb{C}^{t+1}}\Big\|_1 \le 2^{\frac{c+1}{2}+1}\delta;\end{aligned}$$

3. For all $i\in[t]$

$$\mathrm{Tr}[(\hat{\Pi})^{(\hat{\mathcal{X}}_{[c]})^t\hat{A}}(\rho'(i))^{(\hat{\mathcal{X}}_{[c]})^t\hat{A}}]\ge 1-\delta^{-2}2^{2^{c+5}}3^c\epsilon-2^{\frac{c+1}{2}+1}\delta-\alpha;$$

4. Let $S\subseteq[c]$. Let $\mathbf{x}_S$, $\mathbf{l}_S$ be computational basis vectors in $\mathcal{X}^{\otimes S}$, $\mathcal{L}^{\otimes S}$. In the following definition, let $\mathbf{x}'_{\bar{S}}$, $\mathbf{l}'_{\bar{S}}$ range over all computational basis vectors of $\mathcal{X}^{\otimes([c]\setminus S)}$, $\mathcal{L}^{\otimes([c]\setminus S)}$. Define states in A'

$$(\rho')^{A'}_{\mathbf{x}_S,\mathbf{l}_S,\delta} := |\mathcal{L}|^{-|\bar{S}|}\sum_{\mathbf{x}'_{\bar{S}},\mathbf{l}'_{\bar{S}}} p(\mathbf{x}'_{\bar{S}}|\mathbf{x}_S)(\rho')^{A'}_{\mathbf{x}_S\mathbf{x}'_{\bar{S}},\mathbf{l}_S\mathbf{l}'_{\bar{S}},\delta}.$$

   Analogously define

$$\rho^A_{\mathbf{x}_S} := \sum_{\mathbf{x}'_{\bar{S}}} p(\mathbf{x}'_{\bar{S}}|\mathbf{x}_S)\rho^A_{\mathbf{x}_S\mathbf{x}'_{\bar{S}}}.$$

   For $i\in[t]$, $S\subseteq[c]$, let $q_{i;S}(\cdot)$ be a probability distribution on $\mathbf{x}^t$. Define

$$(\rho')^{(\hat{\mathcal{X}}_{[c]})^t\hat{A}}_{i;S} := |\mathcal{L}|^{-ct}\sum_{\mathbf{x}^t} q_{i;S}(\mathbf{x}^t)|\mathbf{x}^t\rangle\langle\mathbf{x}^t|^{\mathcal{X}^{\otimes[ct]}} \otimes |\mathbf{l}^t\rangle\langle\mathbf{l}^t|^{\mathcal{L}^{\otimes[ct]}} \otimes (\rho')^{A'}_{\mathbf{x}^t(i)_S,\mathbf{l}^t(i)_S,\delta},$$

$$\rho^{(\mathcal{X}_{[c]})^t A}_{i;S} := \sum_{\mathbf{x}^t} q_{i;S}(\mathbf{x}^t)|\mathbf{x}^t\rangle\langle\mathbf{x}^t|^{\mathcal{X}^{\otimes[ct]}} \otimes \rho^{A}_{\mathbf{x}^t(i)_S}.$$

Then

$$\mathrm{Tr}\ [(\hat{\Pi})^{(\hat{\mathcal{X}}_{[c]})^t\hat{A}}(\rho')^{(\hat{\mathcal{X}}_{[c]})^t\hat{A}}_{i;S}] \le \frac{1-\alpha}{\alpha}\sum_{j=1}^{t} 2^{-D^{\epsilon}_H(\rho(j)^{(\mathcal{X}_{[c]})^t A}\|\rho^{(\mathcal{X}_{[c]})^t A}_{i;S})}.$$

4. Broadcast channel

We now prove a one-shot Marton inner bound with common message for sending classical information through a q-BC. Such a result was not known earlier for a q-BC even in the asymptotic iid setting. The analogous inner bound in the one-shot classical setting was proved by Radhakrishnan, Sen and Warsi [19] (see also Liu *et al* [23]). Radhakrishnan, Sen and Warsi also proved Marton's inner bound, but without common message, in the one-shot quantum setting. The version with common message subsumes the version without, as well as the *superposition coding technique* for a broadcast channel [24–26]. This problem was also studied earlier by Hirche and Morgan [27] for a two-user binary input cq broadcast channel. Recently, Anshu, Jain and Warsi [28] proved nearly matching one-shot inner and outer bounds for the q-BC without common message. However, their bounds are not known to reduce to the standard Marton bounds in the asymptotic iid limit.

In the problem of sending classical information with common message through a q-BC the sender Alice has three classical messages $m_0 \in [2^{R_0}]$, $m_1 \in [2^{R_1}]$, $m_2 \in [2^{R_2}]$, and she wants to send (m_0, m_1) to Bob and (m_0, m_2) to Charlie. The parties have at their disposal a quantum channel $\mathfrak{C} : X \to Y_1Y_2$ with input Hilbert space $\mathcal{X}$ and output Hilbert spaces $\mathcal{Y}_1$, $\mathcal{Y}_2$. Alice encodes (m_0, m_1, m_2) into a quantum state $\sigma^X_{m_0,m_1,m_2} \in \mathcal{X}$ and inputs it to $\mathfrak{C}$. The channel $\mathfrak{C}$ applies a superoperator to σ_{m_0,m_1,m_2} and outputs a quantum state $\rho^{Y_1Y_2}_{m_0,m_1,m_2} := (\mathfrak{C}^{X\to Y_1Y_2}(\sigma^X_{m_0,m_1,m_2}))^{Y_1Y_2}$ jointly supported in the Hilbert space $\mathcal{Y}_1 \otimes \mathcal{Y}_2$. Bob and Charlie apply their respective decoding superoperators independently on $\rho^{YZ}_{m_0,m_1,m_2}$ in order to produce their respective guesses $(\hat{m}_0, \hat{m}_1)$, $(\hat{\hat{m}}_0, \hat{\hat{m}}_2)$ of the messages m_0, m_1, m_2. See Figure 1. Let $0 \le \epsilon \le 1$. Consider the uniform probability distribution over the message sets. We want that

$$\Pr[(\hat{m_0}, \hat{m}_1, \hat{\hat{m}}_0, \hat{\hat{m}}_2) \ne (m_0, m_1, m_0, m_2)] \le \epsilon,$$

where the probability is over the choice of the messages and actions of the encoder, channel and decoders. If there exists such encoding and decoding schemes for a particular channel $\mathfrak{C}$, we say there exists an $(R_0, R_1, R_2, \epsilon)$-q-BC code for sending classical information through $\mathfrak{C}$.

It is possible to extend the classical proof of the one-shot Marton's inner bound with common message of Radhakrishnan, Sen and Warsi [19] to the quantum setting using Fact 4 to obtain the quantum analogues of intersection operations used to define the sets $\mathcal{A}_{13}$, $\mathcal{A}_{24}$ just before Equation (42) of their paper. In this paper however we give a different proof following the style of Anshu, Jain and Warsi [18], which we believe is more transparent and intuitive.

We now state our one-shot Marton's inner bound with common message for transmitting classical information over a q-BC.

Theorem 1 **(One-shot Marton, common message)** *Let $\mathfrak{C}$ be a q-BC. Let $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2$ be three new sample spaces and (U_0, U_1, U_2) be a jointly distributed random variable on the sample space $\mathcal{U}_0 \times \mathcal{U}_1 \times \mathcal{U}_2$. For every element $(u_0, u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$, let $\sigma^X_{u_0,u_1,u_2}$ be a quantum state in the input Hilbert space $\mathcal{X}$ of $\mathfrak{C}$. Consider the cq-state*

$$\rho^{U_0U_1U_2Y_1Y_2} := \sum_{(u_0,u_1,u_2)\in\mathcal{U}_0\times\mathcal{U}_1\times\mathcal{U}_2} p_{U_0U_1U_2}(u_0,u_1,u_2) |u_0,u_1,u_2\rangle\langle u_0,u_1,u_2|^{U_0U_1U_2} \otimes \mathfrak{C}(\sigma^X_{u_0,u_1,u_2})^{Y_1Y_2}.$$

Let R_0, R_1, R_2, ϵ be such that

$$\begin{aligned}
R_0 + R_1 &\le I^{\epsilon}_H(U_0U_1 : Y_1) - 2 - \log\frac{1}{\epsilon}\\
R_0 + R_2 &\le I^{\epsilon}_H(U_0U_2 : Y_2) - 2 - \log\frac{1}{\epsilon}\\
R_0 + R_1 + R_2 &\le I^{\epsilon}_H(U_0U_2 : Y_2) + I^{\epsilon}_H(U_1 : Y_1|U_0)\\
&\quad - I^{\epsilon}_{\infty}(U_1 : U_2|U_0) - 4 - 4\log\frac{1}{\epsilon}\\
R_0 + R_1 + R_2 &\le I^{\epsilon}_H(U_0U_1 : Y_1) + I^{\epsilon}_H(U_2 : Y_2|U_0)\\
&\quad - I^{\epsilon}_{\infty}(U_1 : U_2|U_0) - 4 - 4\log\frac{1}{\epsilon}\\
2R_0 + R_1 + R_2 &\le I^{\epsilon}_H(U_0U_1 : Y_1) + I^{\epsilon}_H(U_0U_2 : Y_2)\\
&\quad - I^{\epsilon}_{\infty}(U_1 : U_2|U_0) - 4 - 4\log\frac{1}{\epsilon},
\end{aligned}$$

where the afore-mentioned mutual information quantities are computed with respect to the cq-state $\rho^{U_0U_1U_2Y_1Y_2}$. Then there exists an $(R_0, R_1, R_2, 2^7\epsilon^{1/6})$-q-BC code for sending classical information through $\mathfrak{C}$.

Proof We follow the structure of Marton's common message inner bound proof as in Radhakrishnan *et al* [19]

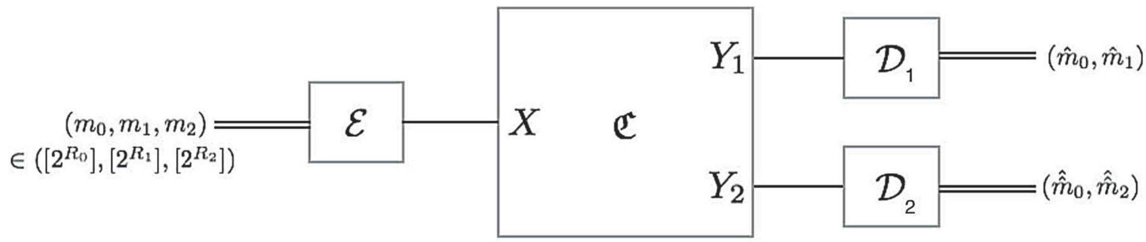

Figure 1. Quantum broadcast channel without entanglement assistance.

with the difference that we use the one-shot mutual covering lemma of Fact 2 instead. Let R_0, R_1, R_2, r_1, r_2, ϵ, be such that

$$\begin{aligned}
R_0 + R_1 + r_1 &\leq I_H^{\epsilon}(U_0U_1 : Y_1) - 2 - \log\frac{1}{\epsilon} \\
R_0 + R_2 + r_2 &\leq I_H^{\epsilon}(U_0U_2 : Y_2) - 2 - \log\frac{1}{\epsilon} \\
R_1 + r_1 &\leq I_H^{\epsilon}(U_1 : Y_1|U_0) - 2 - \log\frac{1}{\epsilon} \\
R_2 + r_2 &\leq I_H^{\epsilon}(U_2 : Y_2|U_0) - 2 - \log\frac{1}{\epsilon} \\
r_1 + r_2 &= I_{\infty}^{\epsilon}(U_1 : U_2|U_0) + 2\log\frac{1}{\epsilon}
\end{aligned}$$

where the afore-mentioned mutual information quantities are computed with respect to the cq-state $\rho^{U_0U_1U_2Y_1Y_2}$. Suppose we show that there is an $(R_0, R_1, R_2, 2^7\epsilon^{1/6})$-q-BC code for sending classical information through $\mathfrak{C}$. The standard Fourier–Motzkin elimination can be used to get rid of r_1 and r_2 and obtain the inner bound in the statement of the theorem.

Codebook

The codebook $\mathcal{C}$ has 2^{R_0} *pages*. Each page consists of a two-dimensional array of 'symbols' arranged in $2^{R_1+r_1}$ rows and $2^{R_2+r_2}$ columns. We will index 'entries' of $\mathcal{C}$ by the 4-tuple $(m_0, m_1, k_1, m_2, k_2)$ where $m_i \in 2^{R_i}$, $k_j \in 2^{r_j}$. The codebook is generated randomly as follows. First, sample $u_0(1), \ldots, u_0(2^{R_0})$ independently according to p_{U_0}. We will associate $u_0(m_0)$ with the m_0th page of $\mathcal{C}$. Now, to generate the contents of the m_0th page, sample

$$u_1(m_0, 1), \ldots, u_1(m_0, 2^{R_1+r_1}), u_2(m_0, 1), \ldots, u_2(m_0, 2^{R_2+r_2})$$

independently according to $p_{U_1|u_0(m_0)}$, $p_{U_2|u_0(m_0)}$. The codebook entry $\mathcal{C}(m_0, m_1, k_1, m_2, k_2)$ is the triple

$$(u_0(m_0), u_1(m_0, m_1, k_1), u_2(m_0, m_2, k_2),$$

where (m_i, k_i) can be thought of as an element in $[2^{R_i+r_i}]$. For $(m_0, m_1) \in [2^{R_0}] \times [2^{R_1}]$, define the 'row band' $\mathcal{C}(m_0, m_1)$ of samples $u_1(m_0, (m_1 - 1)2^{r_1} + 1), \ldots, u_1(m_0, m_1 2^{r_1})$. Similarly, one can define the 'column band' $\mathcal{C}(m_0, m_2)$ for each $(m_0, m_2) \in [2^{R_0}] \times [2^{R_2}]$. For a triple (m_0, m_1, m_2), we call the corresponding page, row and column bands together as the 'rectangle'. For each rectangle, we can now sample the 'indicator pair' $(k_1, k_2)(m_0, m_1, m_2) \in [2^{r_1}] \times [2^{r_2}]$ according to the random variable (K_1, K_2) conditioned on the contents of the rectangle as described in the distribution P_2 of Fact 2. The full description of the random codebook $\mathcal{C}$ consists of the pages, symbols and indicator pairs. Given the codebook $\mathcal{C}$, consider its *augmentation* $\mathcal{C}'$ obtained by additionally choosing independent and uniform samples l_0, l_1, l_2 of computational basis vectors of $\mathcal{L}$ to populate all the pages and the rows and columns of $\mathcal{C}$. We shall henceforth work with the augmented codebook $\mathcal{C}'$, which is revealed to Alice, Bob and Charlie.

Encoding

To send message triple (m_0, m_1, m_2), Alice picks up the entry $\mathcal{C}(m_0, m_1, k_1, m_2, k_2)$ where (k_1, k_2) is the indicator pair for the rectangle (m_0, m_1, m_2). She then inputs the quantum state $\sigma^X_{u_0(m_0), u_1(m_0, m_1, k_1), u_2(m_0, m_2, k_2)}$ into the channel $\mathfrak{C}$.

Decoding

Consider the marginal cq-state $\rho^{U_0U_1Y_1}$. Express it as

$$\rho^{U_0U_1Y_1} = \sum_{u_0, u_1} p(u_0, u_1)|u_0, u_1\rangle\langle u_0, u_1|^{U_0U_1} \otimes \rho^{Y_1}_{u_0, u_1}.$$

Define the cq-states $\rho^{U_0U_1Y_1}_{(\{U_0\}, \{U_1\}, \{\})}$, $\rho^{U_0U_1Y_1}_{(\{\}, \{U_0, U_1\}, \{\})}$ as in Claim 4 of Fact 4. Fix $0 < \delta < 1$. Fact 4 tells us that there is an augmentation of the classical systems U_0, U_1 to $U_0' := U_0 \otimes \mathcal{L}$, $U_1' := U_1 \otimes \mathcal{L}$, an extension Y_1' of the quantum system Y_1, i.e. $Y_1 \otimes \mathbb{C}^2 \leq Y_1'$, a cq-state $(\rho')^{U_0'U_1'Y_1'}$ and a cq-POVM element $(\Pi')^{U_0'U_1'Y_1'}$ such that

1. $(\rho')^{U_0'U_1'Y_1'} = |\mathcal{L}|^{-2} \sum_{u_0, u_1, l_0, l_1} p(u_0, u_1)|u_0, u_1\rangle \langle u_0, u_1|^{U_0U_1} \otimes |l_0, l_1\rangle\langle l_0, l_1|^{\mathcal{L}^{\otimes 2}} \otimes (\rho')^{Y_1'}_{u_0, l_0, u_1, l_1}$, for some quantum states $(\rho')^{Y_1'}_{u_0, l_0, u_1, l_1}$.
2. For some POVM elements $(\Pi')^{Y_1'}_{u_0, l_0, u_1, l_1, \delta}$

$$(\Pi')^{U_0'U_1'Y_1'} = \sum_{u_0,u_1,l_0,l_1} |u_0,u_1\rangle\langle u_0,u_1|^{U_0U_1} \otimes |l_0,l_1\rangle\langle l_0,l_1|^{\mathcal{L}^{\otimes 2}} \otimes (\Pi')^{Y_1'}_{u_0,l_0,u_1,l_1,\delta}.$$

3. $\left\|(\rho')^{U_0'U_1'Y_1'} - \rho^{U_0U_1Y_1} \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes \frac{\mathbb{1}^{\mathcal{L}^{\otimes 2}}}{|\mathcal{L}|^2}\right\|_1 \le 8\delta.$
4. $\mathrm{Tr}\ [(\Pi')^{U_0'U_1'Y_1'}(\rho')^{U_0'U_1'Y_1'}] \ge 1 - 2^8 \times 3 \times \delta^{-2}\epsilon - 8\delta.$
5. Define $(\rho')^{U_0'U_1'Y_1'}_{(\{U_0'\},\{U_1'\},\{\})}$, $(\rho')^{U_0'U_1'Y_1'}_{(\{\},\{U_0',U_1'\},\{\})}$ analogous to the states $\rho^{U_0U_1Y_1}_{(\{U_0\},\{U_1\},\{\})}$, $\rho^{U_0U_1Y_1}_{(\{\},\{U_0,U_1\},\{\})}$. Then

$$\mathrm{Tr}\ [(\Pi')^{U_0'U_1'Y_1'}(\rho')^{U_0'U_1'Y_1'}_{(\{U_0'\},\{U_1'\},\{\})}] \le 2^{-I_H^{\epsilon}(U_1:Y_1|U_0)_\rho},$$
$$\mathrm{Tr}\ [(\Pi')^{U_0'U_1'Y_1'}(\rho')^{U_0'U_1'Y_1'}_{(\{\},\{U_0',U_1'\},\{\})}] \le 2^{-I_H^{\epsilon}(U_0U_1:Y_1)_\rho}.$$

For $(m_0, m_1, k_1) \in [2^{R_0}] \times [2^{R_1+r_1}]$, define the POVM element $\Lambda^{Y_1}_{m_0,m_1,k_1}$ as follows: attach an ancilla of $|0\rangle\langle 0|^{\mathbb{C}^2}$ to register Y_1 and then apply POVM element $\Lambda^{Y_1'}_{(u_0,l_0)(m_0),(u_1,l_1)(m_0,m_1,k_1)}$. Here $\Lambda^{Y_1'}_{(u_0,l_0)(m_0),(u_1,l_1)(m_0,m_1,k_1)}$ is a POVM element from the PGM constructed, for the augmented codebook $\mathcal{C}'$, from the set of positive operators

$$\{(\Pi')^{Y_1'}_{(u_0,l_0)(m_0'),(u_1,l_1)(m_0',m_1',k_1'),\delta} : m_0' \in 2^{R_0}, (m_1',k_1') \in [2^{R_1+r_1}]\},$$

which in turn is provided by the earlier Claim 2. Observe that $\Lambda^{Y_1}_{m_0,m_1,k_1}$ depends only on $(u_0,l_0)(m_0')$, $(u_1,l_1)(m_0',m_1',k_1')$, $m_0' \in 2^{R_0}$, $(m_1',k_1') \in [2^{R_1+r_1}]$ of $\mathcal{C}'$. Similarly for $(m_0,m_2,k_2) \in [2^{R_0}] \times [2^{R_2+r_2}]$, we can define the POVM element $\Lambda^{Y_2}_{m_0,m_2,k_2}$. Bob applies his POVM to the contents of Y_1 and outputs the result $(\hat{m}_0, \hat{m}_1, \hat{k}_1)$ as his guess for (m_0,m_1,k_1). Similarly, Charlie outputs $(\hat{\hat{m}}_0, \hat{\hat{m}}_2, \hat{\hat{k}}_2)$ as his guess for (m_0,m_2,k_2). Bob and Charlie thus attempt to do the tougher job of decoding their respective actual symbols input into the channel instead of just 'decoding up to the band'.

Error probability
Suppose Alice transmits (m_0,m_1,m_2). We consider the expected decoding error of Bob over the choice of a random augmented codebook $\mathcal{C}'$. We first observe that by Fact 2, at the cost of an additive decoding error of 2ϵ, we can pretend that we have the distribution $(P_1)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}K_1K_2}$ instead of the actual distribution $(P_2)^{U_0(U_1)^{2^{r_1}}(U_2)^{2^{r_2}}K_1K_2}$ inside rectangle (m_1,m_2) of page m_0 of $\mathcal{C}$. In other words, we can pretend that we first choose a uniformly random $(k_1,k_2) \in [2^{r_1}] \times [2^{r_2}]$, put the cq-state $\rho^{U_0U_1U_2Y_1}$ between cell (k_1,k_2) of rectangle (m_1,m_2) of page m_0 and Bob's output register Y_1, and independent copies of $U_1|U_0$, $U_2|U_0$ in the other rows and columns of page m_0. In other pages, we continue to have independent samples from the random variables U_0, $U_1|U_0$, $U_2|U_0$. In all rectangles other than rectangle (m_1,m_2) of page m_0, we choose the indicator pairs as described earlier during the construction of the codebook $\mathcal{C}$. We call the modified construction of the codebook as $\mathcal{C}^{m_0,m_1,m_2,k_1,k_2}$ and its augmentation as $(\mathcal{C}')^{m_0,m_1,m_2,k_1,k_2}$. This explains the inequality in Step (a). Next, by Fact 4, at further cost of an additive decoding error of 4δ we shall pretend that we have the cq-state $(\rho')^{U_0'U_1'Y_1'}$ instead of $\rho^{U_0U_1Y_1}$ between cell (k_1,k_2) of rectangle (m_1,m_2) of page m_0 and register Y_1'. Combining this with Fact 3 explains the inequality in Step (b). The inequality in Step (c) follows by an application of Fact 4. We thus finally manage to bound Bob's expected decoding error.

For a state $\sigma^X_{u_0(m_0),u_1(m_0,m_1,k_1),u_2(m_0,m_2,k_2)}$ input to the channel $\mathfrak{C}$, let $\rho^{Y_1}_{u_0(m_0),u_1(m_0,m_1,k_1),u_2(m_0,m_2,k_2)}$ denote its output state at Bob's end. We can bound Bob's expected decoding error as follows:

$$\begin{aligned}
&\mathop{\mathbf{E}}_{\mathcal{C}'}[\Pr[\text{Bob's error}]] \\
&= \mathop{\mathbf{E}}_{\mathcal{C}'}[\mathrm{Tr}\ [(\mathbb{1}^{Y_1} - \Lambda^{Y_1}_{m_0,m_1,k_1})\rho^{Y_1}_{u_0(m_0),u_1(m_0,m_1,k_1),u_2(m_0,m_2,k_2)}]] \\
&\overset{a}{\le} 2\epsilon + 2^{-r_1-r_2}\sum_{k_1,k_2}\mathop{\mathbf{E}}_{(\mathcal{C}')^{m_0,m_1,m_2,k_1,k_2}}[\\
&\mathrm{Tr}\ [(\mathbb{1}^{Y_1} - \Lambda^{Y_1}_{m_0,m_1,k_1})\ \rho^{Y_1}_{u_0(m_0),u_1(m_0,m_1,k_1),u_2(m_0,m_2,k_2)}]] \\
&= 2\epsilon + 2^{-r_1-r_2}\sum_{k_1,k_2}\mathop{\mathbf{E}}_{(\mathcal{C}')^{m_0,m_1,m_2,k_1,k_2}}[\\
&\mathrm{Tr}\ [(\mathbb{1}^{Y_1} - \Lambda^{Y_1}_{m_0,m_1,k_1})\ \rho^{Y_1}_{u_0(m_0),u_1(m_0,m_1,k_1)}]] \\
&= 2\epsilon + 2^{-r_1-r_2}\sum_{k_1,k_2}\mathop{\mathbf{E}}_{(\mathcal{C}')^{m_0,m_1,m_2,k_1,k_2}}[\\
&\mathrm{Tr}\ [(\mathbb{1}^{Y_1'} - \Lambda^{Y_1'}_{(u_0,l_0)(m_0),(u_1,l_1)(m_0,m_1,k_1)}) \\
&(\rho^{Y_1}_{u_0(m_0),u_1(m_0,m_1,k_1)} \otimes |0\rangle\langle 0|^{\mathbb{C}^2})]] \\
&\le 2\epsilon + 2^{-r_1-r_2}\sum_{k_1,k_2}\mathop{\mathbf{E}}_{(\mathcal{C}')^{m_0,m_1,m_2,k_1,k_2}}[\\
&\mathrm{Tr}\ [(\mathbb{1}^{Y_1'} - \Lambda^{Y_1'}_{(u_0,l_0)(m_0),(u_1,l_1)(m_0,m_1,k_1)})\ \ (\rho')^{Y_1'}_{(u_0,l_0)(m_0),(u_1,l_1)(m_0,m_1,k_1)}] \\
&+ 2^{-r_1-r_2}\sum_{k_1,k_2}\mathop{\mathbf{E}}_{(\mathcal{C}')^{m_0,m_1,m_2,k_1,k_2}}[\\
&\frac{1}{2}\left\|(\rho')^{Y_1'}_{(u_0,l_0)(m_0),(u_1,l_1)(m_0,m_1,k_1)} - \rho^{Y_1}_{u_0(m_0),u_1(m_0,m_1,k_1)} \otimes |0\rangle\langle 0|^{\mathbb{C}^2}\right\|_1 \\
&= 2\epsilon + 2^{-r_1-r_2}\sum_{k_1,k_2}\mathop{\mathbf{E}}_{(\mathcal{C}')^{m_0,m_1,m_2,k_1,k_2}}[\\
&\mathrm{Tr}\ [(\mathbb{1}^{Y_1'} - \Lambda^{Y_1'}_{(u_0,l_0)(m_0),(u_1,l_1)(m_0,m_1,k_1)})\ \ (\rho')^{Y_1'}_{(u_0,l_0)(m_0),(u_1,l_1)(m_0,m_1,k_1)}] \\
&+ \frac{1}{2}\left\|(\rho')^{U_0'U_1'Y_1'} - \rho^{U_0U_1Y_1} \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes \frac{\mathbb{1}^{\mathcal{L}^{\otimes 2}}}{|\mathcal{L}|^2}\right\|_1
\end{aligned}$$

$$
\begin{aligned}
&\overset{b}{\leq} 2\epsilon + 4\delta \\
&\quad + 2^{-r_1-r_2} \times 2 \sum_{k_1,k_2} \mathop{\mathbf{E}}_{(\mathcal{C}')^{m_0,m_1,m_2,k_1,k_2}} [\\
&\quad \mathrm{Tr}\ [(\mathbb{1}^{Y_1'} - (\Pi')^{Y_1'}_{(u_0,l_0)(m_0),(u_1,l_1)(m_0,m_1,k_1)}) \quad (\rho')^{Y_1'}_{(u_0,l_0)(m_0),(u_1,l_1)(m_0,m_1,k_1)}] \\
&\quad + 2^{-r_1-r_2} \times 4 \sum_{k_1,k_2} \sum_{(m_1',k_1'):(m_1',k_1')\neq(m_1,k_1)} \\
&\quad \mathop{\mathbf{E}}_{(\mathcal{C}')^{m_0,m_1,m_2,k_1,k_2}} [\mathrm{Tr}\ [(\Pi')^{Y_1'}_{(u_0,l_0)(m_0),(u_1',l_1')(m_0,m_1',k_1')} \\
&\quad (\rho')^{Y_1'}_{(u_0,l_0)(m_0),(u_1,l_1)(m_0,m_1,k_1)}]] \\
&\quad + 2^{-r_1-r_2} \times 4 \sum_{k_1,k_2} \sum_{m_0',m_1',k_1':m_0'\neq m_0} \mathop{\mathbf{E}}_{(\mathcal{C}')^{m_0,m_1,m_2,k_1,k_2}} \\
&\quad [\mathrm{Tr}\ [(\Pi')^{Y_1'}_{(u_0',l_0')(m_0'),(u_1',l_1')(m_0',m_1',k_1')} (\rho')^{Y_1'}_{(u_0,l_0)(m_0),(u_1,l_1)(m_0,m_1,k_1)}]] \\
&\quad = 2\epsilon + 4\delta \\
&\quad\quad + 2|\mathcal{L}|^{-2} \sum_{u_0,u_1,l_0,l_1} p(u_0,u_1)\, \mathrm{Tr}\ [(\mathbb{1}^{Y_1'} - (\Pi')^{Y_1'}_{u_0,u_1,l_0,l_1}) \\
&\quad (\rho')^{Y_1'}_{u_0,u_1,l_0,l_1}] \quad + 4(2^{R_1+r_1} - 1)|\mathcal{L}|^{-3} \\
&\quad \sum_{u_0,l_0,u_1,l_1,u_1',l_1'} p(u_0)p(u_1|u_0)p(u_1'|u_0) \\
&\quad \mathrm{Tr}\ [(\Pi')^{Y_1'}_{u_0,u_1',l_0,l_1'} (\rho')^{Y_1'}_{u_0,u_1,l_0,l_1}] \quad + 4(2^{R_0} - 1)2^{R_1+r_1}|\mathcal{L}|^{-4} \\
&\quad \sum_{u_0,l_0,u_0',l_0',u_1,l_1,u_1',l_1'} p(u_0,u_1)p(u_0',u_1') \\
&\quad \mathrm{Tr}\ [(\Pi')^{Y_1'}_{u_0',u_1',l_0',l_1'} (\rho')^{Y_1'}_{u_0,u_1,l_0,l_1}] \\
&\quad = 2\epsilon + 4\delta + 2\,\mathrm{Tr}\ [(\mathbb{1}^{U_0'U_1'Y_1'} - (\Pi')^{U_0'U_1'Y_1'})(\rho')^{U_0'U_1'Y_1'}] \\
&\quad + 4(2^{R_1+r_1} - 1)\,\mathrm{Tr}\ [(\Pi')^{U_0'U_1'Y_1'} (\rho')^{U_0'U_1'Y_1'}_{(\{U_0'\},\{U_1'\},\{\})}] \\
&\quad + 4(2^{R_0} - 1)2^{R_1+r_1}\,\mathrm{Tr}\ [(\Pi')^{U_0'U_1'Y_1'} (\rho')^{U_0'U_1'Y_1'}_{(\{\},\{U_0',U_1'\},\{\})}] \\
&\quad \overset{c}{\leq} 2\epsilon + 4\delta + 2^9 \times 3 \times \delta^{-2}\epsilon + 16\delta \\
&\quad + 2^{R_1+r_1+2-I_H^{\epsilon}(U_1:Y_1|U_0)} + 2^{R_0+R_1+r_1+2-I_H^{\epsilon}(U_0U_1:Y_1)}.
\end{aligned}
$$

Setting $\delta := \epsilon^{1/3}$, we get that $\mathop{\mathbf{E}}_{\mathcal{C}'}[\Pr[\text{Bob's error}]] \leq 2^{11}\epsilon^{1/3}$.

Similarly, $\mathop{\mathbf{E}}_{\mathcal{C}'}[\Pr[\text{Charlie's error}]] \leq 2^{11}\epsilon^{1/3}$. Thus, there is an augmented codebook $\mathcal{C}'$ such that sum of Bob's and Charlie's average decoding errors is at most $2^{12}\epsilon^{1/3}$. The average probability that at least one of Bob or Charlie err for $\mathcal{C}'$ is thus seen to be at most $2^7\epsilon^{1/6}$ using Fact 1. This finishes the proof of one-shot Marton's inner bound with common message. □

A similar proof combined with *position-based coding* technique of Anshu, Jain and Warsi [18] can be used to obtain a one-shot Marton's inner bound with common message for sending classical information through an entanglement-assisted broadcast channel (see Figure 2). Earlier, Anshu, Jain and Warsi [18] had shown the achievability of a one-shot Marton's bound without common message.

Theorem 2 (Ent. assist. one-shot Marton, com. msg.) *Let $\mathfrak{C} : X \to Y_1Y_2$ be a q-BC. Let $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2$ be three new Hilbert spaces and $\psi^{U_0U_1U_2X}$ be a quantum state that is classical on U_0. Consider the cq-state*

$$
\begin{aligned}
&\rho^{U_0U_1U_2Y_1Y_2} \\
&:= \sum_{u_0} p(u_0)|u_0\rangle\langle u_0|^{U_0} \otimes \\
&((\mathfrak{C}^{X\to Y_1Y_2} \otimes \mathbb{1}^{U_1U_2})(\psi_{u_0}^{U_1U_2X}))^{U_1U_2Y_1Y_2}.
\end{aligned}
$$

Let R_0, R_1, R_2, ϵ be such that

$$
\begin{aligned}
R_0 + R_1 &\leq I_H^{\epsilon}(U_0U_1 : Y_1) - 2 - \log\frac{1}{\epsilon} \\
R_0 + R_2 &\leq I_H^{\epsilon}(U_0U_2 : Y_2) - 2 - \log\frac{1}{\epsilon} \\
R_0 + R_1 + R_2 &\leq I_H^{\epsilon}(U_0U_2 : Y_2) + I_H^{\epsilon}(U_1 : Y_1|U_0) \\
&\quad - I_{\infty}^{\epsilon,\epsilon^2}(U_1 : U_2|U_0) - 4 - 4\log\frac{1}{\epsilon} \\
R_0 + R_1 + R_2 &\leq I_H^{\epsilon}(U_0U_1 : Y_1) + I_H^{\epsilon}(U_2 : Y_2|U_0) \\
&\quad - I_{\infty}^{\epsilon,\epsilon^2}(U_1 : U_2|U_0) - 4 - 4\log\frac{1}{\epsilon} \\
2R_0 + R_1 + R_2 &\leq I_H^{\epsilon}(U_0U_1 : Y_1) + I_H^{\epsilon}(U_0U_2 : Y_2) \\
&\quad - I_{\infty}^{\epsilon,\epsilon^2}(U_1 : U_2|U_0) - 4 - 4\log\frac{1}{\epsilon}
\end{aligned}
$$

where the afore-mentioned mutual information quantities are computed with respect to the cq-state $\rho^{U_0U_1U_2Y_1Y_2}$. Then there exists an $(R_0, R_1, R_2, 2^7\epsilon^{1/10})$-q-BC code for sending classical information through $\mathfrak{C}$ with entanglement assistance.

Proof Let R_0, R_1, R_2, r_1, r_2, ϵ be such that

$$
\begin{aligned}
R_0 + R_1 + r_1 &\leq I_H^{\epsilon}(U_0U_1 : Y_1) - 2 - \log\frac{1}{\epsilon} \\
R_0 + R_2 + r_2 &\leq I_H^{\epsilon}(U_0U_2 : Y_2) - 2 - \log\frac{1}{\epsilon} \\
R_1 + r_1 &\leq I_H^{\epsilon}(U_1 : Y_1|U_0) - 2 - \log\frac{1}{\epsilon} \\
R_2 + r_2 &\leq I_H^{\epsilon}(U_2 : Y_2|U_0) - 2 - \log\frac{1}{\epsilon} \\
r_1 + r_2 &= I_{\infty}^{\epsilon,\epsilon^2}(U_1 : U_2|U_0) + 2\log\frac{1}{\epsilon}
\end{aligned}
$$

where the afore-mentioned mutual information quantities are computed with respect to the cq-state $\rho^{U_0U_1U_2Y_1Y_2}$. Suppose we show that there is a $(R_0, R_1, R_2, 2^7\epsilon^{1/10})$-q-BC code for sending classical information through $\mathfrak{C}$. The standard Fourier–Motzkin elimination can be used to get rid of r_1 and r_2 and obtain the inner bound in the statement of the theorem.

Codebook:

The codebook $\mathcal{C}$ is now cq. It has 2^{R_0} *pages* and is generated randomly as follows. First, sample

$$u_0(1), \ldots, u_0(2^{R_0})$$

independently according to ψ^{U_0}. We will associate $u_0(m_0)$ with the m_0th page of $\mathcal{C}$. Now, to generate the contents of the m_0th page, take independent copies

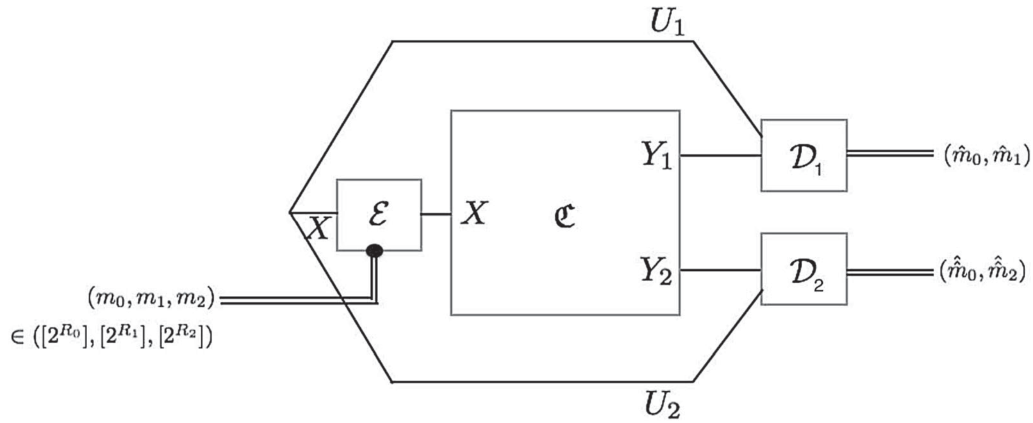

Figure 2. Quantum broadcast channel with entanglement assistance.

$$\psi^{U_1(m_0,1)U_1'(m_0,1)}|m_0, \ldots, \psi^{U_1(m_0,2^{R_1+r_1})U_1'(m_0,2^{R_1+r_1})}|m_0,$$
$$\psi^{U_2(m_0,1)U_2'(m_0,1)}|m_0, \ldots, \psi^{U_2(m_0,2^{R_2+r_2})U_2'(m_0,2^{R_2+r_2})}|m_0,$$

of the states $\psi^{U_1U_1'}|m_0$, $\psi^{U_2U_2'}|m_0$, where $\psi^{U_1U_1'}|m_0$ is a purification of $\psi^{U_1}|m_0$, the marginal state on U_1 obtained by conditioning the register U_0 in $\psi^{U_0U_1}$ to take the value m_0, and $\psi^{U_2U_2'}|m_0$ is defined similarly. The registers

$$U_1'(m_0, 1), \ldots, U_1'(m_0, 2^{R_1+r_1})$$

are with Bob,

$$U_2'(m_0, 1), \ldots, U_2'(m_0, 2^{R_2+r_2})$$

with Charlie, and

$$U_1(m_0, 1), \ldots, U_1(m_0, 2^{R_1+r_1}), U_2(m_0, 1), \ldots, U_2(m_0, 2^{R_2+r_2})$$

with Alice. The states

$$\psi^{U_1(m_0,1)U_1'(m_0,1)}|m_0, \ldots, \psi^{U_1(m_0,2^{R_1+r_1})U_1'(m_0,2^{R_1+r_1})}|m_0$$

form the prior entanglement between Alice and Bob, and the states

$$\psi^{U_2(m_0,1)U_2'(m_0,1)}|m_0, \ldots, \psi^{U_2(m_0,2^{R_2+r_2})U_2'(m_0,2^{R_2+r_2})}|m_0$$

form the prior entanglement between Alice and Charlie. For $(m_0, m_1) \in [2^{R_0}] \times [2^{R_1}]$, define the 'row band' $\mathcal{C}(m_0, m_1)$ to be the states

$$\psi^{U_1(m_0,(m_1-1)2^{r_1}+1)U_1'(m_0,(m_1-1)2^{r_1}+1)}|m_0,$$
$$\ldots, \psi^{U_1(m_0,m_1 2^{r_1})U_1'(m_0,m_1 2^{r_1})}|m_0$$

of page m_0. Similarly, one can define the 'column band' $\mathcal{C}(m_0, m_2)$ for each $(m_0, m_2) \in [2^{R_0}] \times [2^{R_2}]$. For a triple (m_0, m_1, m_2), we call the corresponding page, row and column bands together as the 'rectangle'.

For each rectangle, we can now sample the 'indicator pair' $(k_1, k_2)(m_0, m_1, m_2) \in [2^{r_1}] \times [2^{r_2}]$ according to the random variable (K_1, K_2) arising from an application of the bipartite convex split lemma of [18] used with underlying state $\psi^{U_1U_2X}|m_0$. The sampling process creates as side effect a quantum register that we call a 'candidate channel input register' $X(m_0, m_1, k_1, m_2, k_2)$. The resulting state on the registers

$$X(m_0, m_1, k_1, m_2, k_2)$$
$$U_1(m_0, (m_1 - 1)2^{r_1} + 1)U_1'(m_0, (m_1 - 1)2^{r_1} + 1)$$
$$U_2(m_0, (m_2 - 1)2^{r_2} + 1)U_2'(m_0, (m_2 - 1)2^{r_2} + 1)$$

is (8ϵ)-close to a purification $\psi^{U_1U_1'U_2U_2'X}|m_0$ of $\psi^{U_1U_2X}|m_0$. For more details, see [18]. The full description of the random codebook $\mathcal{C}$ consists of the pages, prior entanglement, indicator pairs and candidate channel input registers. Given the codebook $\mathcal{C}$, consider its *augmentation* $\mathcal{C}'$ obtained by additionally choosing independent and uniform samples l_0, l_1, l_2 of computational basis vectors of $\mathcal{L}$ to populate all the pages and the rows and columns of $\mathcal{C}$. We shall henceforth work with the augmented codebook $\mathcal{C}'$, which is revealed to Alice, Bob and Charlie.

Encoding
To send message triple (m_0, m_1, m_2), Alice picks up the indicator pair (k_1, k_2) for the rectangle (m_0, m_1, m_2). She then inputs the register $X(m_0, m_1, k_1, m_2, k_2)$ into the channel $\mathfrak{C}$.

Decoding
Consider the marginal cq-state $\rho^{U_0U_1Y_1}$. Express it as

$$\rho^{U_0U_1Y_1} = \sum_{u_0} p(u_0)|u_0\rangle\langle u_0|^{U_0} \otimes \rho_{u_0}^{U_1Y_1}.$$

Define the cq-states

$$\rho^{U_0U_1Y_1}_{[\{U_0\},(\{U_1\},\{Y_1\})]} := \sum_{u_0} p(u_0)|u_0\rangle\langle u_0|^{U_0} \otimes \rho_{u_0}^{U_1} \otimes \rho_{u_0}^{Y_1},$$

$$\rho^{U_0U_1Y_1}_{[\{\},(\{U_0,U_1\},\{Y_1\})]} := \left(\sum_{u_0} p(u_0)|u_0\rangle\langle u_0|^{U_0} \otimes \rho_{u_0}^{U_1}\right) \otimes \rho^{Y_1}.$$

Fix $0 < \delta < 1$. The full version of the intersection case of the cq joint typicality lemma, viz. Lemma 1 from [1], tells us that there is an augmentation of the classical system U_0 to $U_0' := U_0 \otimes \mathcal{L}$, augmentations $U_1' := U_1 \otimes \mathbb{C}^2 \otimes \mathcal{L}$, $Y_1' := Y_1 \otimes \mathbb{C}^2 \otimes \mathcal{L}$ of the quantum systems U_1, Y_1, a cq-state $(\rho')^{U_0'U_1'Y_1'}$ and a cq-POVM element $(\Pi')^{U_0'U_1'Y_1'}$ such that

1.
$$(\rho')^{U_0'U_1'Y_1'} = |\mathcal{L}|^{-3} \sum_{u_0,l_0,l_1,\hat{l}_1} p(u_0)|u_0\rangle\langle u_0|^{U_0} \otimes |l_0,l_1,\hat{l}_1\rangle\langle l_0,l_1,\hat{l}_1|^{\mathcal{L}^{\otimes 3}} \otimes (\rho')^{U_1'Y_1'}_{u_0,l_0,l_1,\hat{l}_1}$$
for some quantum states $(\rho')^{U_1'Y_1'}_{u_0,l_0,l_1,\hat{l}_1}$;
2. for some POVM elements $(\Pi')^{Y_1'}_{u_0,l_0,l_1,\hat{l}_1,\delta}$,
$$(\Pi')^{U_0'U_1'Y_1'} = \sum_{u_0,l_0,l_1,\hat{l}_1} |u_0\rangle\langle u_0|^{U_0U_1} \otimes |l_0,l_1,\hat{l}_1\rangle\langle l_0,l_1,\hat{l}_1|^{\mathcal{L}^{\otimes 3}} \otimes (\Pi')^{Y_1'}_{u_0,l_0,l_1,\hat{l}_1,\delta};$$
3. $\left\| (\rho')^{U_0'U_1'Y_1'} - \rho^{U_0U_1Y_1} \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes \frac{\mathbb{1}^{\mathcal{L}^{\otimes 3}}}{|\mathcal{L}|^3} \right\|_1 \leq 8\delta$;
4. $\mathrm{Tr}\ [(\Pi')^{U_0'U_1'Y_1'}(\rho')^{U_0'U_1'Y_1'}] \geq 1 - 2^8 \times 3 \times \delta^{-4}\epsilon - 8\delta$;
5. define $(\rho')^{U_0'U_1'Y_1'}_{[\{U_0'\},(\{U_1'\},\{Y_1'\})]}$, $(\rho')^{U_0'U_1'Y_1'}_{[\{\},(\{U_0',U_1'\},\{Y_1'\})]}$ analogous to the states $\rho^{U_0U_1Y_1}_{[\{U_0\},(\{U_1\},\{Y_1\})]}$, $\rho^{U_0U_1Y_1}_{[\{\},(\{U_0,U_1\},\{Y_1\})]}$. Then
$$\mathrm{Tr}\ [(\Pi')^{U_0'U_1'Y_1'}(\rho')^{U_0'U_1'Y_1'}_{[\{U_0'\},(\{U_1'\},\{Y_1'\})]}] \leq 2^{-I_H^{\epsilon}(U_1:Y_1|U_0)_\rho},$$
$$\mathrm{Tr}\ [(\Pi')^{U_0'U_1'Y_1'}(\rho')^{U_0'U_1'Y_1'}_{[\{\},(\{U_0',U_1'\},\{Y_1'\})]}] \leq 2^{-I_H^{\epsilon}(U_0U_1:Y_1)_\rho}.$$

For $(m_0, m_1, k_1) \in [2^{R_0}] \times [2^{R_1+r_1}]$, define the POVM element $\Lambda^{Y_1}_{m_0,m_1,k_1}$ according to the position-based decoding strategy of [18] using the positive operators

$$\{(\Pi')^{Y_1'}_{(u_0,l_0)(m_0'),(l_1,\hat{l}_1)(m_0',m_1',k_1'),\delta} : m_0' \in 2^{R_0}, (m_1',k_1') \in [2^{R_1+r_1}]\},$$

which in turn is provided by afore-mentioned Claim 2. Observe that $\Lambda^{Y_1}_{m_0,m_1,k_1}$ depends only on $(u_0,l_0)(m_0')$, $(l_1,\hat{l}_1)(m_0',m_1',k_1')$, $m_0' \in 2^{R_0}$, $(m_1',k_1') \in [2^{R_1+r_1}]$ of $\mathcal{C}'$. Similarly for $(m_0,m_2,k_2) \in [2^{R_0}] \times [2^{R_2+r_2}]$, we can define the POVM element $\Lambda^{Y_2}_{m_0,m_2,k_2}$. Bob applies his POVM to the contents of Y_1 and outputs the result $(\hat{m}_0, \hat{m}_1, \hat{k}_1)$ as his guess for (m_0, m_1, k_1). Similarly, Charlie outputs $(\hat{\hat{m}}_0, \hat{\hat{m}}_2, \hat{\hat{k}}_2)$ as his guess for (m_0, m_2, k_2). Bob and Charlie thus attempt to do the tougher job of decoding their respective actual symbols input into the channel instead of just 'decoding up to the band'.

Error probability
The error probability calculation is very similar to that in the proof of Theorem 1. This is because the error analysis in the bipartite convex split lemma and position-based decoding of [18] is very similar to the error analysis in our mutual covering lemma (Fact 2) and in PGM-based decoding.

Setting $\delta := \epsilon^{1/5}$, we get $\mathbf{E}_{\mathcal{C}'}[\Pr[\text{Bob's error}]] \leq 2^{11}\epsilon^{1/5}$.

Similarly, $\mathbf{E}_{\mathcal{C}'}[\Pr[\text{Charlie's error}]] \leq 2^{11}\epsilon^{1/5}$. Thus, there is an augmented codebook $\mathcal{C}'$ such that sum of Bob's and Charlie's average decoding errors is at most $2^{12}\epsilon^{1/5}$. The average probability that at least one of Bob or Charlie err for $\mathcal{C}'$ is thus seen to be at most $2^7\epsilon^{1/10}$ using Fact 1. This finishes the proof of the entanglement-assisted one-shot Marton's inner bound with common message. □

Remark 1 The afore-mentioned theorem is unsatisfactory as the state $\psi^{U_0U_1U_2X}$ used therein is classical on U_0. This is because the inner bound expression in the theorem contains one-shot mutual information terms that condition on U_0. No proper definition of these terms is known when U_0 is quantum. This deficiency is further reflected in the statements of the cq joint typicality lemmas in Facts 4 and 5 as well as in their full versions in [1], all of which can only condition on classical registers. On a different vein, observe that the register U_0 captures the common message in the protocol. How to define a common message for a broadcast channel in the case of transmission of quantum information is unclear, whereas the personal messages have straightforward quantum analogues. This may also be another reason why we are unable to make U_0 quantum in the statement of the theorem. Making $\psi^{U_0U_1U_2X}$ fully quantum thus remains as an open problem.

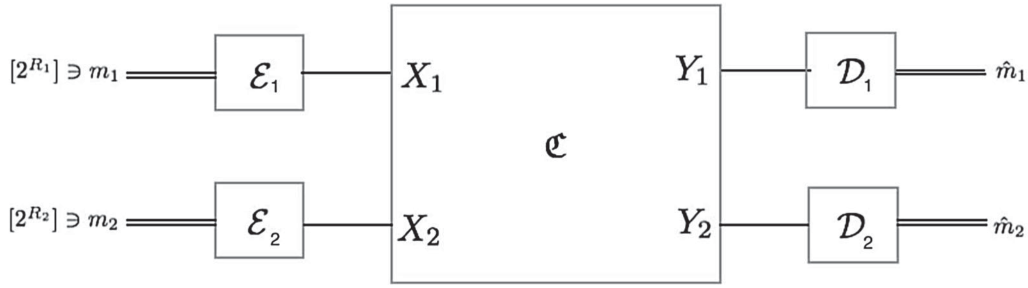

Figure 3. Quantum interference channel without entanglement assistance.

5. Interference channel

We now prove one-shot inner bounds for sending classical information through a q-IC. In this problem, there are two senders A_1, A_2 and their corresponding receivers B_1, B_2. Sender A_1 would like to send a classical message $m_1 \in [2^{R_1}]$ to B_1. Similarly, A_2 would like to send $m_2 \in [2^{R_2}]$ to B_2. The parties have at their disposal a quantum channel $\mathfrak{C} : X_1X_2 \to Y_1Y_2$ with input Hilbert spaces $\mathcal{X}_1$, $\mathcal{X}_2$ and output Hilbert spaces $\mathcal{Y}_1$, $\mathcal{Y}_2$. Sender A_1 encodes m_1 into a quantum state $\sigma_{m_0}^{X_1} \in \mathcal{X}_1$ and inputs it to $\mathfrak{C}$. Similarly, A_2 encodes m_2 into a quantum state $\sigma_{m_2}^{X_2} \in \mathcal{X}_2$ and inputs it to $\mathfrak{C}$. The channel outputs a quantum state $\rho_{m_1,m_2}^{Y_1Y_2} := (\mathfrak{C}^{X_1X_2 \to Y_1Y_2}(\sigma_{m_1}^{X_1} \otimes \sigma_{m_2}^{X_2}))^{Y_1Y_2}$ jointly supported in the Hilbert space $\mathcal{Y}_1 \otimes \mathcal{Y}_2$. Receivers B_1, B_2 apply their respective decoding superoperators independently on $\rho_{m_1,m_2}^{Y_1Y_2}$ in order to produce their respective guesses $\hat{m}_1$, $\hat{m}_2$ of the messages m_1, m_2. See Figure 3. Let $0 \leq \epsilon \leq 1$. Consider the uniform probability distribution over the message sets. We want $\Pr[(\hat{m}_1, \hat{m}_2) \neq (m_1, m_2)] \leq \epsilon$, where the probability is over the choice of the messages and actions of the encoder, channel and decoders. If there exist such encoding and decoding schemes for a particular channel $\mathfrak{C}$, we say that there exists an (R_1, R_2, ϵ)-q-IC code for sending classical information through $\mathfrak{C}$.

We now state and prove our one-shot Chong–Motani–Garg–El Gamal style inner bound for sending classical information through an unassisted q-IC. Our inner bound reduces to the standard Chong–Motani–Garg–El Gamal inner bound for the asymptotic iid setting, which is also known to be equivalent to the famous Han–Kobayashi inner bound [11]. However, in the one-shot setting it is unclear if the two inner bounds are the same.

Theorem 3 (One-shot Chong–Motani–Garg–El Gamal) *Let $\mathfrak{C} : X_1'X_2' \to Y_1Y_2$ be a q-IC. Let $\mathcal{Q}, \mathcal{U}_1, \mathcal{X}_1, \mathcal{U}_2, \mathcal{X}_2$ be four new sample spaces. Let the 4-tuple (Q, U_1, X_1, U_2, X_2) be a jointly distributed random variable with probability mass function $p(q)p(u_1, x_1|q)p(u_2, x_2|q)$. For every element $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$, let $\sigma_{x_1}^{X_1'}$, $\sigma_{x_2}^{X_2'}$ be quantum states in the input Hilbert spaces $\mathcal{X}_1'$, $\mathcal{X}_2'$ of $\mathfrak{C}$. Consider the cq-state*

$$\rho^{QU_1X_1U_2X_2Y_1Y_2} := \sum_{q,u_1,x_1,u_2,x_2} p(q)p(u_1,x_1|q)p(u_2,x_2|q) |q,u_1,x_1,u_2,x_2\rangle\langle q,u_1,x_1,u_2,x_2|^{QU_1X_1U_2X_2} \otimes (\mathfrak{C}(\sigma_{x_1}^{X_1'} \otimes \sigma_{x_2}^{X_2'}))^{Y_1Y_2}.$$

Let R_1, R_2, ϵ, be such that

$$\begin{aligned}
R_1 &\leq I_H^{\epsilon}(X_1 : Y_1|U_2Q) - 2 - \log\frac{1}{\epsilon}\\
R_1 &\leq I_H^{\epsilon}(X_1 : Y_1|U_1U_2Q) + I_H^{\epsilon}(X_2U_1 : Y_2|U_2Q) - 2 - \log\frac{1}{\epsilon}\\
R_2 &\leq I_H^{\epsilon}(X_2 : Y_2|U_1Q) - 2 - \log\frac{1}{\epsilon}\\
R_2 &\leq I_H^{\epsilon}(X_2 : Y_2|U_1U_2Q) + I_H^{\epsilon}(X_1U_2 : Y_1|U_1Q) - 2 - \log\frac{1}{\epsilon}\\
R_1 + R_2 &\leq I_H^{\epsilon}(X_1U_2 : Y_1|Q) + I_H^{\epsilon}(X_2 : Y_2|U_1U_2Q) - 2 - \log\frac{1}{\epsilon}\\
R_1 + R_2 &\leq I_H^{\epsilon}(X_2U_1 : Y_2|Q) + I_H^{\epsilon}(X_1 : Y_1|U_1U_2Q) - 2 - \log\frac{1}{\epsilon}\\
R_1 + R_2 &\leq I_H^{\epsilon}(X_1U_2 : Y_1|U_1Q) + I_H^{\epsilon}(X_2U_1 : Y_2|U_2Q) - 2 - \log\frac{1}{\epsilon}\\
2R_1 + R_2 &\leq I_H^{\epsilon}(X_1U_2 : Y_1|Q) + I_H^{\epsilon}(X_1 : Y_1|U_1U_2Q) + I_H^{\epsilon}(X_2U_1 : Y_2|U_2Q) - 2 - \log\frac{1}{\epsilon}\\
R_1 + 2R_2 &\leq I_H^{\epsilon}(X_2U_1 : Y_2|Q) + I_H^{\epsilon}(X_2 : Y_2|U_1U_2Q) + I_H^{\epsilon}(X_1U_2 : Y_1|U_1Q) - 2 - \log\frac{1}{\epsilon}
\end{aligned}$$

where the afore-mentioned mutual information quantities are computed with respect to the cq-state $\rho^{QU_1X_1U_2X_2Y_1Y_2}$.

Then there exists an $(R_1, R_2, 2^{2^{14}}\epsilon^{1/6})$*-q-IC code for sending classical information through* $\mathfrak{C}$.

Proof We follow the proof outline as given in El Gamal-Kim's book [2]. We use 'rate splitting' to divide A_1's message $m_1 \in [2^{R_1}]$ into a 'public part' $m_1' \in [2^{R_1'}]$ and a 'personal part' $m_1'' \in [2^{R_1-R_1'}]$. Similarly, we divide A_2's message $m_2 \in [2^{R_2}]$ into a 'public part' $m_2' \in [2^{R_2'}]$ and a 'personal part' $m_2'' \in [2^{R_2-R_2'}]$. The public messages must be recovered by both receivers whereas the personal messages need only to be recovered by the intended receiver. The messages are sent by a one-shot version of superposition coding whereby the 'cloud centres' u_1, u_2 carry the public messages m_1', m_2' and the 'satellite symbols' x_1, x_2, which will be decoded after first recovering u_1, u_2, and carry the personal messages m_1'', m_2''.

We now show that a rate quadruple $(R_1', R_1 - R_1', R_2', R_2 - R_2')$ is achievable if it satisfies the following inequalities:

$$\begin{aligned}
R_1 - R_1' &\le I_H^\epsilon(X_1 : Y_1|U_1U_2Q) - 2 - \log\frac{1}{\epsilon}\\
R_1 &\le I_H^\epsilon(X_1 : Y_1|U_2Q) - 2 - \log\frac{1}{\epsilon}\\
R_1 - R_1' + R_2' &\le I_H^\epsilon(X_1U_2 : Y_1|U_1Q) - 2 - \log\frac{1}{\epsilon}\\
R_1 + R_2' &\le I_H^\epsilon(X_1U_2 : Y_1|Q) - 2 - \log\frac{1}{\epsilon}\\
R_2 - R_2' &\le I_H^\epsilon(X_2 : Y_2|U_1U_2Q) - 2 - \log\frac{1}{\epsilon}\\
R_2 &\le I_H^\epsilon(X_2 : Y_2|U_1Q) - 2 - \log\frac{1}{\epsilon}\\
R_2 - R_2' + R_1' &\le I_H^\epsilon(X_2U_1 : Y_2|U_2Q) - 2 - \log\frac{1}{\epsilon}\\
R_2 + R_1' &\le I_H^\epsilon(X_2U_1 : Y_2|Q) - 2 - \log\frac{1}{\epsilon}
\end{aligned}$$

where the afore-mentioned mutual information quantities are computed with respect to the cq-state $\rho^{QU_1X_1U_2X_2Y_1Y_2}$. The standard Fourier–Motzkin elimination now gives us the rate region in the statement of the theorem. Note that in the one-shot case it is not clear if the second upper bounds on R_1 and R_2, in the rate region described in the theorem statement, can be eliminated, unlike the asymptotic iid case. This is because their elimination in the asymptotic iid case relies on the chain rule for Shannon mutual information, which is not known to hold for the hypothesis testing mutual information used in the one-shot setting.

Codebook
First, generate a sample q from the distribution $p(q)$. For each public message $m_1' \in [2^{R_1'}]$ independently generate a sample $u_1(m_1')$ from the distribution $p(u_1|q)$. Similarly, for each public message $m_2' \in [2^{R_2'}]$ independently generate a sample $u_2(m_2')$ from the distribution $p(u_2|q)$. Now for each public message m_1', independently generate samples $x_1(m_1', m_1'')$ from the distribution $p(x_1|u_1q)$ for all personal messages $m_1'' \in [2^{R_1-R_1'}]$. Similarly for each public message m_2', independently generate samples $x_2(m_2', m_2'')$ from the probability distribution $p(x_2|u_2q)$ for all personal messages $m_2'' \in [2^{R_2-R_2'}]$. These samples together constitute the random codebook $\mathcal{C}$. Given the codebook $\mathcal{C}$, consider its *augmentation* $\mathcal{C}'$ obtained by additionally choosing independent and uniform samples l_0, l_1', l_1'', l_2', l_2'' of computational basis vectors of $\mathcal{L}$ to populate all the entries of $\mathcal{C}'$. We shall henceforth work with the augmented codebook $\mathcal{C}'$, which is revealed to A_1, A_2, B_1, B_2.

Encoding
To send message $m_1 = (m_1', m_1'')$, A_1 picks up the symbol $x_1(m_1', m_1'')$ from the codebook $\mathcal{C}$ and inputs the state $\sigma^{X_1'}_{x_1(m_1', m_1'')}$ into the channel $\mathfrak{C}$. Similarly, to send message $m_2 = (m_2', m_2'')$, A_2 picks up the symbol $x_2(m_2', m_2'')$ from the codebook $\mathcal{C}$ and inputs the state $\sigma^{X_2'}_{x_2(m_2', m_2'')}$ into the channel $\mathfrak{C}$.

Decoding
The receiver B_1 decodes the tuple (m_1'', m_1') using simultaneous *non-unique* decoding. To do this he has to apply a 'union of intersection' of POVM elements, which in turn is provided by Fact 5. The 'union' is over all choices of $\hat{m}_2' \in [2^{R_2'}]$. In the asymptotic iid setting it turns out that non-unique decoding is not required in order to get the Chong–Motani–Garg–El Gamal rate region. Sen [4] showed that we can further require B_1 to recover m_2' and still obtain the same rate region. However the argument in [4] fails in the one-shot setting since it relies on chain rule of Shannon mutual information, which is not known to hold for the hypothesis testing mutual information. Chain rules for smooth one-shot mutual information quantities are typically inequalities and frequently involve two or more types of quantities in the same expression. Hence using them often leads to unsatisfactory bounds for channel coding problems. Therefore we use non-unique decoding in the one-shot setting as it possibly leads to a larger inner bound.

Consider the marginal cq-state $\rho^{QU_1X_1U_2Y_1}$. Express it as

$$\begin{aligned}
&\rho^{QU_1X_1U_2Y_1}\\
&\quad= \sum_{q,u_1,x_1,u_2} p(q)p(u_1,x_1|q)p(u_2|q)\\
&\quad|q,u_1,x_1,u_2\rangle\langle q,u_1,x_1,u_2|^{QU_1X_1U_2} \otimes \rho^{Y_1}_{q,u_1,x_1,u_2},
\end{aligned}$$

where in fact $\rho^{Y_1}_{q,u_1,x_1,u_2} = \rho^{Y_1}_{x_1,u_2}$, i.e. $\rho^{Y_1}_{q,u_1,x_1,u_2}$ is independent of q and u_1.

Let $t := 2^{R_2'}$. For $\tilde{m}_2 \in [t]$, define the cq-state

$$\rho^{QU_1X_1(U_2)^tY_1}(\tilde{m}_2) := \sum_{q,u_1,x_1,u_2^t} p(q)p(u_1,x_1|q)p(u_2^t|q) |q,u_1,x_1,u_2^t\rangle\langle q,u_1,x_1,u_2^t|^{QU_1X_1(U_2)^t} \otimes \rho^{Y_1}_{q,u_1,x_1,u_2^t(\tilde{m}_2)}.$$

These states will play the role of $\rho'(1),\ldots,\rho'(t)$ in Fact 5.

Define the cq-states

$$\rho^{QU_1X_1U_2Y_1}_{(\{QU_1U_2\},\{X_1\},\{\})}, \rho^{U_0U_1X_1U_2Y_1}_{(\{QU_2\},\{U_1X_1\},\{\})}, \rho^{U_0U_1X_1U_2Y_1}_{(\{QU_1\},\{U_2X_1\},\{\})}, \rho^{U_0U_1X_1U_2Y_1}_{(\{Q\},\{U_2U_1X_1\},\{\})},$$

as in Claim 4 of Fact 4. We can now define the cq-state

$$\rho^{QU_1X_1(U_2)^tY_1}_{(\{QU_1U_2\},\{X_1\},\{\})}(\tilde{m}_2) := \sum_{q,u_1,u_2^t} p(q)p(u_1|q)p(u_2^t|q)|q,u_1,u_2^t\rangle\langle q,u_1,u_2^t|^{QU_1(U_2)^t} \otimes \Big(\sum_{x_1} p(x_1|qu_1)|x_1\rangle\langle x_1|^{X_1}\Big) \otimes \rho^{Y_1}_{q,u_1,u_2^t(\tilde{m}_2)}.$$

In other words, $\rho^{QU_1X_1(U_2)^tY_1}_{(\{QU_1U_2\},\{X_1\},\{\})}(\tilde{m}_2)$ is the cq-state obtained by 'naturally extending' U_2 to $(U_2)^t$ and 'embedding' $\rho^{QU_1X_1U_2Y_1}_{(\{QU_1U_2\},\{X_1\},\{\})}$ at '$\tilde{m}_2$th position'. Similarly, we can define the quantum state $\rho^{QU_1X_1(U_2)^tY_1}_{(\{QU_2\},\{U_1X_1\},\{\})}(\tilde{m}_2)$.

Fix $0<\alpha,\delta<1$. Fact 5 tells us that there is an augmentation of the classical systems Q, U_1, X_1, U_2 to $\hat{Q} := Q \otimes \mathcal{L}$, $\hat{U}_1 := U_1 \otimes \mathcal{L}$, $\hat{X}_1 := X_1 \otimes \mathcal{L}$, $\hat{U}_2 := U_2 \otimes \mathcal{L}$, an extension $\hat{Y}_1$ of the quantum system Y_1, i.e. $Y_1 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{t+1} \leq \hat{Y}_1$, cq-states $(\rho')^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}(\tilde{m}_2)$ and a cq-POVM element $(\hat{\Pi})^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}$ such that

1.

$$(\rho')^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}(\tilde{m}_2) = |\mathcal{L}|^{-(t+3)} \cdot \sum_{q,u_1,x_1,u_2^t,\mathbf{l}} p(q)p(u_1,x_1|q)p(u_2^t|q) |q,u_1,x_1,u_2^t\rangle\langle q,u_1,x_1,u_2^t|^{QU_1X_1(U_2)^t} \otimes |\mathbf{l}\rangle\langle\mathbf{l}|^{\mathcal{L}^{\otimes(t+3)}} \otimes (\rho')^{Y_1'}_{q,u_1,x_1,u_2^t(\tilde{m}_2),\mathbf{l}^{\mathcal{L}^{\otimes 4}}} \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}},$$

for some quantum states $(\rho')^{Y_1'}_{q,u_1,x_1,u_2,\mathbf{l}^{\mathcal{L}^{\otimes 4}}}$;

2.

$$(\hat{\Pi})^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1} = \sum_{q,u_1,x_1,u_2^t,\mathbf{l}} |q,u_1,x_1,u_2^t\rangle\langle q,u_1,x_1,u_2^t|^{QU_1X_1(U_2)^t} \otimes |\mathbf{l}\rangle\langle\mathbf{l}|^{\mathcal{L}^{\otimes(t+3)}} \otimes (\hat{\Pi})^{\hat{Y}_1}_{q,u_1,x_1,u_2^t,\mathbf{l}},$$

for some POVM elements $(\hat{\Pi})^{\hat{Y}_1}_{q,u_1,x_1,(u_2)^t,\mathbf{l}}$;

3.

$$\Big\| (\rho')^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}(\tilde{m}_2) - \rho^{QU_1X_1(U_2)^tY_1}(\tilde{m}_2) \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes \frac{\mathbb{1}^{\mathcal{L}^{\otimes(t+3)}}}{|\mathcal{L}|^{t+3}} \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}} \Big\|_1 \leq 2^4\delta;$$

4.

$$\mathrm{Tr}\ [(\hat{\Pi})^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}(\rho')^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}(\tilde{m}_2)] \geq 1 - 2^{2^9} \times 3^4 \times \delta^{-2}\epsilon - 2^4\delta - \alpha.$$

5. Define

$$(\rho')^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}_{(\{\hat{Q}\hat{U}_1\hat{U}_2\},\{\hat{X}_1\},\{\})}(\tilde{m}_2), (\rho')^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}_{(\{\hat{Q}\hat{U}_2\},\{\hat{U}_1\hat{X}_1\},\{\})}(\tilde{m}_2)$$

analogously as the corresponding quantities

$$\rho^{QU_1X_1(U_2)^tY_1}_{(\{QU_1U_2\},\{X_1\},\{\})}(\tilde{m}_2), \rho^{QU_1X_1(U_2)^tY_1}_{(\{QU_2\},\{U_1X_1\},\{\})}(\tilde{m}_2)$$

defined earlier. Then

$$\mathrm{Tr}\ [(\hat{\Pi})^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}(\rho')^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}_{(\{\hat{Q}\hat{U}_1\hat{U}_2\},\{\hat{X}_1\},\{\})}(\tilde{m}_2)] \leq \frac{1-\alpha}{\alpha}\Bigg(\sum_{j\neq\tilde{m}_2} 2^{-D_H^\epsilon(\rho^{QU_1X_1(U_2)^tY_1}(j)\|\rho^{QU_1X_1(U_2)^tY_1}_{(\{QU_1U_2\},\{X_1\},\{\})}(\tilde{m}_2))} + 2^{-D_H^\epsilon(\rho^{QU_1X_1(U_2)^tY_1}(\tilde{m}_2)\|\rho^{QU_1X_1(U_2)^tY_1}_{(\{QU_1U_2\},\{X_1\},\{\})}(\tilde{m}_2))}\Bigg) \leq \frac{1-\alpha}{\alpha}(2^{R_2'}2^{-I_H^\epsilon(X_1U_2:Y_1|QU_1)} + 2^{-I_H^\epsilon(X_1:Y_1|QU_1U_2)}),$$

where the hypothesis mutual information quantities are computed with respect to the cq-state $\rho^{QU_1X_1U_2X_2Y_1}$. The last inequality follows because the POVM element optimising the hypothesis testing mutual information quantity $I_H^\epsilon(X_1U_2 : Y_1|QU_1)$, when applied at the '$j$th position', $j \neq m_2$, accepts $\rho^{QU_1X_1(U_2)^tY_1}(j)$ with probability at least $1-\epsilon$ and accepts $\rho^{QU_1X_1(U_2)^tY_1}_{(\{QU_1U_2\},\{X_1\},\{\})}(\tilde{m}_2)$ with probability at most $2^{-I_H^\epsilon(X_1U_2:Y_1|QU_1)}$. Similarly, the

POVM element optimising the hypothesis testing mutual information quantity $I_H^{\epsilon}(X_1 : Y_1|QU_1U_2)$, when applied at the '$\tilde{m}_2$th position', accepts $\rho^{QU_1X_1(U_2)^tY_1}(\tilde{m}_2)$ with probability at least $1-\epsilon$ and accepts $\rho^{QU_1X_1(U_2)^tY_1}_{(\{QU_1U_2\},\{X_1\},\{\})}(\tilde{m}_2)$ with probability at most $2^{-I_H^{\epsilon}(X_1:Y_1|QU_1U_2)}$. Arguing along the same lines, we get

$$\begin{aligned}
&\mathrm{Tr}\ [(\hat{\Pi})^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}(\rho')^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}_{(\{\hat{Q}\hat{U}_2\},\{\hat{U}_1\hat{X}_1\},\{\})}(\tilde{m}_2)] \\
&\quad\leq \frac{1-\alpha}{\alpha}(2^{R_2'}2^{-I_H^{\epsilon}(X_1U_1U_2:Y_1|Q)} + 2^{-I_H^{\epsilon}(U_1X_1:Y_1|QU_2)}).
\end{aligned}$$

For $(m_1'', m_1') \in [2^{R_1-R_1'}] \times [2^{R_1'}]$, define the POVM element $\Lambda^{Y_1}_{m_1'',m_1'}$ as follows: attach an ancilla of $|0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}}$ to register Y_1 and then apply the POVM element $\Lambda^{\hat{Y}_1}_{(u_1,l_1')(m_1'),(x_1,l_1'')(m_1',m_1'')}$. Here $\Lambda^{\hat{Y}_1}_{(u_1,l_1')(m_1'),(x_1,l_1'')(m_1',m_1'')}$ is a POVM element from the PGM constructed, for the augmented codebook $\mathcal{C}'$, from the set of positive operators

$$\Big\{(\hat{\Pi})^{\hat{Y}_1}_{(q,l_0),(u_1,l_1')(\tilde{m}_1'),(x_1,l_1'')(\tilde{m}_1',\tilde{m}_1''),\{(u_2,l_2')(\tilde{m}_2'):\tilde{m}_2'\in[2^{R_2'}]\},\delta} : (\tilde{m}_1', \tilde{m}_1'') \in [2^{R_1-R_1'}] \times [2^{R_1'}]\Big\},$$

which in turn is provided by Fact 5. Observe that $\Lambda^{Y_1}_{m_1',m_1''}$ depends only on the entries q, $(u_1, l_1')(\tilde{m}_1')$, $(x_1, l_1'')(\tilde{m}_1', \tilde{m}_1'')$, $(u_2, l_2')(\tilde{m}_2')$, $\tilde{m}_1' \in [2^{R_1'}]$, $\tilde{m}_1'' \in [2^{R_1-R_1'}]$ and $\tilde{m}_2' \in [2^{R_2'}]$ of $\mathcal{C}'$. Similarly for $(m_2'', m_2') \in [2^{R_2-R_2'}] \times [2^{R_2'}]$, we can define the POVM element $\Lambda^{Y_2}_{m_2',m_2''}$. Receiver B_1 applies his POVM to the contents of Y_1 and outputs the result $\hat{m}_1 := (\hat{m}_1', \hat{m}_1'')$ as his guess for $m_1 = (m_1', m_1'')$. Similarly, receiver B_2 applies his POVM to the contents of Y_2 and outputs the result $\hat{m}_2 := (\hat{m}_2', \hat{m}_2'')$ as his guess for $m_2 = (m_2', m_2'')$.

Error probability

Suppose the senders A_1, A_2 transmit (m_1, m_2). For a state $\sigma^{X_1'}_{x_1(m_1',m_1'')} \otimes \sigma^{X_2'}_{x_2(m_2',m_2'')}$ input into the channel $\mathfrak{C}$, denote its output state at B_1 by $\rho^{Y_1}_{x_1(m_1',m_1''),x_2(m_2',m_2'')}$. We bound the expected decoding error of B_1 over the choice of a random augmented codebook $\mathcal{C}'$ as follows:

$$\begin{aligned}
&\mathop{\mathbf{E}}_{\mathcal{C}'}[\Pr[B_1'\text{s error}]] \\
&\quad= \mathop{\mathbf{E}}_{\mathcal{C}'}[\mathrm{Tr}\ [(\mathbb{1}^{Y_1} - \Lambda^{Y_1}_{m_1',m_1''})\rho^{Y_1}_{x_1(m_1',m_1''),x_2(m_2',m_2'')}]] \\
&\quad= \mathop{\mathbf{E}}_{\mathcal{C}'}[\mathrm{Tr}\ [(\mathbb{1}^{Y_1} - \Lambda^{Y_1}_{m_1',m_1''})\rho^{Y_1}_{q,u_1(m_1'),x_1(m_1',m_1''),u_2(m_2'),x_2(m_2',m_2'')}]] \\
&\quad= \mathop{\mathbf{E}}_{\mathcal{C}'}[\mathrm{Tr}\ [(\mathbb{1}^{Y_1} - \Lambda^{Y_1}_{m_1',m_1''})\rho^{Y_1}_{q,u_1(m_1'),x_1(m_1',m_1''),u_2(m_2')}]] \\
&\quad= \mathop{\mathbf{E}}_{\mathcal{C}'}[\mathrm{Tr}\ [(\mathbb{1}^{\hat{Y}_1} - \Lambda^{\hat{Y}_1}_{(u_1,l_1')(m_1'),(x_1,l_1'')(m_1',m_1'')}) \\
&\quad(\rho^{Y_1}_{q,u_1(m_1'),x_1(m_1',m_1''),u_2(m_2')} \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \\
&\quad\otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}})]] \\
&\quad\leq \mathop{\mathbf{E}}_{\mathcal{C}'}[\mathrm{Tr}\ [(\mathbb{1}^{\hat{Y}_1} - \Lambda^{\hat{Y}_1}_{(u_1,l_1')(m_1'),(x_1,l_1'')(m_1',m_1'')}) \\
&\quad((\rho')^{Y_1'}_{(q,l_0),(u_1,l_1')(m_1'),(x_1,l_1'')(m_1',m_1''),(u_2,l_2')(m_2')} \\
&\otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}})]] \\
&\quad+ \mathop{\mathbf{E}}_{\mathcal{C}'}\Big[\frac{1}{2}\Big\|(\rho')^{Y_1'}_{(q,l_0),(u_1,l_1')(m_1'),(x_1,l_1'')(m_1',m_1''),(u_2,l_2')(m_2')} \\
&\otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}} \\
&- \rho^{Y_1}_{q,u_1(m_1'),x_1(m_1',m_1''),u_2(m_2')} \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \\
&\quad\otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}}\Big\|_1\Big] \\
&\quad\overset{a}{\leq} 2\mathop{\mathbf{E}}_{\mathcal{C}'}[\mathrm{Tr}\ [(\mathbb{1}^{\hat{Y}_1} - \\
&\quad(\hat{\Pi})^{\hat{Y}_1}_{(q,l_0),(u_1,l_1')(m_1'),(x_1,l_1'')(m_1',m_1''),\{(u_2,l_2')(\tilde{m}_2):\tilde{m}_2\in[2^{R_2'}]\},\delta}) \\
&\quad((\rho')^{Y_1'}_{(q,l_0),(u_1,l_1')(m_1'),(x_1,l_1'')(m_1',m_1''),(u_2,l_2')(m_2')} \\
&\quad\otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}})]] \\
&\quad+ 4\sum_{\tilde{m}_1''\neq m_1''}\mathop{\mathbf{E}}_{\mathcal{C}'}[\\
&\quad\mathrm{Tr}\ [(\hat{\Pi})^{\hat{Y}_1}_{(q,l_0),(u_1,l_1')(m_1'),(x_1,l_1'')(m_1',\tilde{m}_1''),\{(u_2,l_2')(\tilde{m}_2):\tilde{m}_2\in[2^{R_2'}]\},\delta} \\
&\quad((\rho')^{Y_1'}_{(q,l_0),(u_1,l_1')(m_1'),(x_1,l_1'')(m_1',m_1''),(u_2,l_2')(m_2')} \\
&\quad\otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}})]] \\
&\quad+ 4\sum_{\tilde{m}_1'\neq m_1',\tilde{m}_2''}\mathop{\mathbf{E}}_{\mathcal{C}'}[\mathrm{Tr}\ [(\hat{\Pi})^{\hat{Y}_1}_{(q,l_0),(u_1,l_1')(\tilde{m}_1'),(x_1,l_1'')(\tilde{m}_1',\tilde{m}_1''),\{(u_2,l_2')(\tilde{m}_2):\tilde{m}_2\in[2^{R_2'}]\},\delta} \\
&\quad((\rho')^{Y_1'}_{(q,l_0),(u_1,l_1')(m_1'),(x_1,l_1'')(m_1',m_1''),(u_2,l_2')(m_2')} \\
&\quad\otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}})]] \\
&\quad+ |\mathcal{L}|^{-4} \times \\
&\quad\sum_{q,u_1,x_1,u_2,l_0,l_1',l_1'',l_2'} p(q)p(u_1,x_1|q)p(u_2|q) \\
&\quad\frac{1}{2}\Big\|(\rho')^{Y_1'}_{(q,l_0),(u_1,l_1'),(x_1,l_1''),(u_2,l_2')} \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}} \\
&\quad-\rho^{Y_1}_{q,u_1,x_1,u_2} \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}}\Big\|_1
\end{aligned}$$

$$
\begin{aligned}
&= 2|\mathcal{L}|^{-(t+3)} \times \\
&\sum_{q,u_1,x_1,u_2^t,l_0,l_1',l_1'',(l_2')^t} p(q)p(u_1,x_1|q)p(u_2^t|q) \\
&\mathrm{Tr}\ [(\mathbb{1}^{\hat{Y}_1} - (\hat{\Pi})^{\hat{Y}_1}_{(q,l_0),(u_1,l_1'),(x_1,l_1''),(u_2^t,(l_2')^t),\delta}) \\
&\quad ((\rho')^{Y_1'}_{(q,l_0),(u_1,l_1'),(x_1,l_1''),(u_2^t,(l_2')^t)(m_2')} \\
&\quad \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}})] \\
&\quad + 4|\mathcal{L}|^{-(t+4)} \times \\
&\quad \sum_{\tilde{m}_1'' \neq m_1''} \sum_{q,u_1,x_1,\tilde{x}_1,u_2^t,l_0,l_1',l_1'',\tilde{l}_1'',(l_2')^t} \\
&\quad p(q)p(u_1|q)p(x_1|u_1q)p(\tilde{x}_1|u_1q)p(u_2^t|q) \\
&\quad \mathrm{Tr}\ [(\hat{\Pi})^{\hat{Y}_1}_{(q,l_0),(u_1,l_1'),(\tilde{x}_1,\tilde{l}_1''),u_2^t,(l_2')^t,\delta} \\
&\quad ((\rho')^{Y_1'}_{(q,l_0),(u_1,l_1'),(x_1,l_1''),(u_2^t,(l_2')^t)(m_2')} \\
&\quad \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}})] \\
&\quad + 4|\mathcal{L}|^{-(t+5)} \times \\
&\quad \sum_{\tilde{m}_1' \neq m_1', \tilde{m}_2''} \sum_{q,u_1,x_1,\tilde{u}_1,\tilde{x}_1,u_2^t,l_0,l_1',l_1'',\tilde{l}_1',\tilde{l}_1'',(l_2')^t} \\
&\quad p(q)p(u_1,x_1|q)p(\tilde{u}_1,\tilde{x}_1|q)p(u_2^t|q) \\
&\quad \mathrm{Tr}\ [(\hat{\Pi})^{\hat{Y}_1}_{(q,l_0),(\tilde{u}_1,\tilde{l}_1'),(\tilde{x}_1,\tilde{l}_1''),u_2^t,(l_2')^t,\delta} \\
&\quad ((\rho')^{Y_1'}_{(q,l_0),(u_1,l_1'),(x_1,l_1''),(u_2^t,(l_2')^t)(m_2')} \\
&\quad \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}})] \\
&\quad + \frac{1}{2}\Big\| (\rho')^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}(m_2') - \\
&\quad \rho^{QU_1X_1(U_2)^tY_1}(m_2') \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes \frac{\mathbb{1}^{\mathcal{L}^{\otimes(t+3)}}}{|\mathcal{L}|^{t+3}} \\
&\quad \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}} \Big\|_1 \\
&\quad = 2\,\mathrm{Tr}\ [(\mathbb{1}^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1} - (\hat{\Pi})^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}) \\
&\quad (\rho')^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}(m_2')] \\
&\quad + 4(2^{R_1-R_1'} - 1) \\
&\quad \times\ \mathrm{Tr}\ [(\hat{\Pi})^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}(\rho')^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}_{(\{\hat{Q}\hat{U}_1\hat{U}_2\},\{\hat{X}_1\},\{\})}(m_2')] \\
&\quad + 4(2^{R_1'} - 1)2^{R_1-R_1'} \times \\
&\quad \mathrm{Tr}\ [(\hat{\Pi})^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}(\rho')^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}_{(\{\hat{Q}\hat{U}_2\},\{\hat{U}_1\hat{X}_1\},\{\})} \\
&\quad (m_2')] + \frac{1}{2}\Big\| (\rho')^{\hat{Q}\hat{U}_1\hat{X}_1(\hat{U}_2)^t\hat{Y}_1}(m_2') - \rho^{QU_1X_1(U_2)^tY_1}(m_2') \\
&\quad \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \quad \otimes \frac{\mathbb{1}^{\mathcal{L}^{\otimes(t+3)}}}{|\mathcal{L}|^{t+3}} \otimes |0\rangle\langle 0|^{\mathbb{C}^2} \otimes |0\rangle\langle 0|^{\mathbb{C}^{t+1}} \Big\|_1 \\
&\quad \leq (2^{2^{12}}\delta^{-2}\epsilon + 2^5\delta + 2\alpha) \\
&\quad + \frac{1-\alpha}{\alpha} 2^{R_1-R_1'+2}(2^{R_2'-I_H^{\epsilon}(X_1U_2:Y_1|QU_1)} + 2^{-I_H^{\epsilon}(X_1:Y_1|QU_1U_2)}) \\
&\quad + \frac{1-\alpha}{\alpha} 2^{R_1+2}(2^{R_2'-I_H^{\epsilon}(X_1U_2:Y_1|Q)} + 2^{-I_H^{\epsilon}(X_1:Y_1|QU_2)}) \\
&\quad \leq (2^{2^{12}}\delta^{-2}\epsilon + 2^5\delta + 2\alpha) \\
&\quad + \frac{1-\alpha}{\alpha}(3^{R_1-R_1'+R_2'+2-I_H^{\epsilon}(X_1U_2:Y_1|QU_1)} \\
&\quad + 2^{R_1-R_1'+2-I_H^{\epsilon}(X_1:Y_1|QU_1U_2)}) \\
&\quad + \frac{1-\alpha}{\alpha}(2^{R_1+2+R_2'-I_H^{\epsilon}(X_1U_2:Y_1|Q)} + 2^{R_1+2-I_H^{\epsilon}(X_1:Y_1|QU_2)}),
\end{aligned}
$$

where Step (a) follows from Fact 3. Setting $\delta := \epsilon^{1/3}$, $\alpha := \epsilon^{2/3}$ we get $\mathop{\mathbf{E}}_{\mathcal{C}'}[\Pr[B_1' \text{s error}]] \leq 2^{2^{13}}\epsilon^{1/3}$.

Similarly, $\mathop{\mathbf{E}}_{\mathcal{C}'}[\Pr[B_2' \text{s error}]] \leq 2^{2^{13}}\epsilon^{1/3}$. Thus, there is an augmented codebook $\mathcal{C}'$ such that sum of average decoding errors of B_1 and B_2 is at most $2^{2^{14}}\epsilon^{1/3}$. The average probability that at least one of B_1 or B_2 err for $\mathcal{C}'$ is thus seen to be at most $2^{2^{14}}\epsilon^{1/6}$ using Fact 1. This finishes the proof of one-shot Chong–Motani–Garg–El Gamal inner bound. □

It is possible to give a one-shot Han–Kobayashi style inner bound for the interference channel also. To do so, we need to use the one-shot simultaneous decoder for the three-sender multiple access channel constructed in [1]. In contrast with the iid setting, in the one-shot setting it is not known if the Han–Kobayashi and Chong–Motani–Garg–El Gamal rate regions are the same or not. This is because we do not have good chain rules for the hypothesis testing mutual information.

An advantage of the Han–Kobayashi inner bound technique is that it can be easily extended to give a non-trivial inner bound for the interference channel with entanglement assistance (see Figure 4). The Chong–Motani–Garg–El Gamal inner bound technique does not seem to be suitable for this endeavour. We consider the case of an interference channel with independent prior entanglement between A_1 and B_1, and between A_2 and B_2, which seems to be the most natural scenario. We shall call this the interference channel with *cis entanglement*. For this channel, we can obtain the following inner bound.

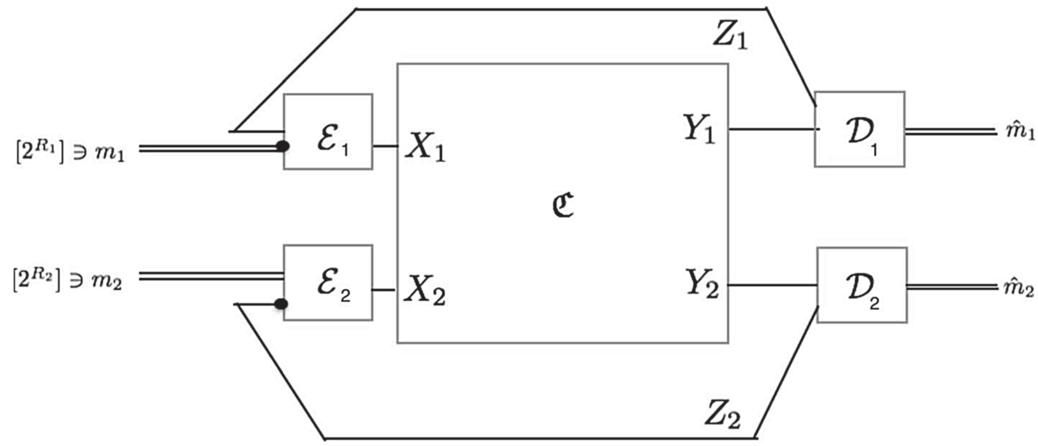

Figure 4. Quantum interference channel with cis entanglement assistance.

Theorem 4 (One-shot Han–Kobayashi, ent. assist.) *Let $\mathfrak{C} : X_1X_2 \to Y_1Y_2$ be a q-IC. We are allowed use of arbitrary amount of prior entanglement between A_1 and B_1, and A_2 and B_2. Let $\mathcal{Q}, \mathcal{U}_1, \mathcal{U}_2$ be three new sample spaces and (Q, U_1, U_2) be a jointly distributed random variable with probability mass function $p(q)p(u_1|q)p(u_2|q)$. Let $\mathcal{X}''_1$, $\mathcal{Z}_1$, $\mathcal{X}''_2$, $\mathcal{Z}_2$ be four new Hilbert spaces. Let $\psi_1^{X'_1Z_1} \otimes \psi_2^{X'_2Z_2}$ be a tensor product quantum state in $X'_1Z_1X'_2Z_2$. For every element $u_1 \in \mathcal{U}_1$, let $\mathcal{E}_{1,u_1}^{X'_1 \to X_1}$ be a fixed encoding superoperator; similarly for every $u_2 \in \mathcal{U}_2$. Consider the cq-state*

$$\begin{aligned}
&\rho^{QU_1U_2Y_1Z_1Y_2Z_2} \\
&\quad := \sum_{q,u_1,u_2} p(q)p(u_1|q)p(u_2|q)|q,u_1,u_2\rangle\langle q,u_1,u_2|^{QU_1U_2} \\
&\quad \otimes((\mathfrak{C}^{X_1X_2 \to Y_1Y_2} \otimes \mathbb{I}^{Z_1Z_2})(\\
&\quad ((\mathcal{E}_{1,u_1}^{X'_1 \to X_1} \otimes \mathbb{I}^{Z_1})(\psi_1^{X'_1Z_1}))^{X_1Z_1} \otimes \\
&\quad ((\mathcal{E}_{2,u_2}^{X'_2 \to X_2} \otimes \mathbb{I}^{Z_2})(\psi_1^{X'_2Z_2}))^{X_2Z_2}))^{Y_1Z_1Y_2Z_2}.
\end{aligned}$$

Let R'_1, R''_1, R'_2, R''_2, ϵ, be such that

$$\begin{aligned}
R'_1 &\le I_H^{\epsilon}(U_1 : Y_1U_2Z_1|Q) - 2 - \log\frac{1}{\epsilon} \\
R''_1 &\le I_H^{\epsilon}(Z_1 : Y_1U_1U_2|Q) - 2 - \log\frac{1}{\epsilon} \\
R'_1 + R''_1 &\le I_H^{\epsilon}(U_1Z_1 : Y_1U_2|Q) - 2 - \log\frac{1}{\epsilon} \\
R'_1 + R'_2 &\le I_H^{\epsilon}(U_1U_2 : Y_1Z_1|Q) - 2 - \log\frac{1}{\epsilon} \\
R''_1 + R'_2 &\le I_H^{\epsilon}(Z_1U_2 : Y_1U_1|Q) - 2 - \log\frac{1}{\epsilon} \\
R'_1 + R''_1 + R'_2 &\le I_H^{\epsilon}(U_1U_2Z_1 : Y_1|Q) - 2 - \log\frac{1}{\epsilon}
\end{aligned}$$

$$\begin{aligned}
R'_2 \le I_H^{\epsilon}(U_2 &: Y_2U_1Z_2|Q) - 2 - \log\frac{1}{\epsilon} R''_2 \le I_H^{\epsilon}(Z_2 \\
&: Y_2U_1U_2|Q) - 2 - \log\frac{1}{\epsilon} R'_2 + R''_2 \le I_H^{\epsilon}(U_2Z_2 \\
&: Y_2U_1|Q) - 2 - \log\frac{1}{\epsilon} R'_1 + R'_2 \le I_H^{\epsilon}(U_1U_2 \\
&: Y_2Z_2|Q) - 2 - \log\frac{1}{\epsilon} R''_2 + R'_1 \le I_H^{\epsilon}(Z_2U_1 \\
&: Y_2U_2|Q) - 2 - \log\frac{1}{\epsilon} R'_2 + R''_2 \\
&\quad + R'_1 \le I_H^{\epsilon}(U_1U_2Z_2 \\
&: Y_2|Q) - 2 - \log\frac{1}{\epsilon}
\end{aligned}$$

where the afore-mentioned mutual information quantities are computed with respect to the cq-state $\rho^{QU_1U_2Y_1Z_1Y_2Z_2}$. Define $R_1 := R'_1 + R''_1$, $R_2 := R'_2 + R''_2$. Then there exists an $(R_1, R_2, 2^{2^{14}}\epsilon^{1/6})$-q-IC code for sending classical information through $\mathfrak{C}$ with cis entanglement assistance.

Proof We employ rate splitting as in the traditional Han–Kobayashi inner bound proof. We split the message m_1 into a common message m'_1 and a personal message m''_1; similarly for m_2. The message triple (m''_1, m'_1, m'_2) is sent to B_1 by treating the interference channel as a three-sender multiple access channel. Message m''_1 is transmitted using the position-based coding technique, which requires the assistance of many independent copies of the state $\psi_1^{X'_1Z}$ with the X'_1 parts under the possession of A_1 and the Z_1 parts under the possession of B_1. Message m'_1 is transmitted without entanglement assistance by applying the encoding superoperator $\mathcal{E}_{1,u_1(m'_1)}$ to register $X'_1(u_1(m'_1))$, and feeding its output to the input register X_1 of the channel $\mathfrak{C}$. Similar statements hold for messages m''_2 and m'_2. An analogous consideration holds for receiver B_2.

We obtain the afore-mentioned region by employing simultaneous decoders for the two three-sender multiple access channels with receivers B_1 and B_2 induced by $\mathfrak{C}$. The simultaneous decoders have to handle both entanglement-assisted as well as unassisted messages, so in a sense they are the hybrid of the two simultaneous decoders described in [1]. Another important difference from the standard decoders for the multiple access channel is that we do not want the additional constraints

$$R_2' \leq I_H^{\epsilon}(U_2 : Y_1 U_1 Z_1 | Q) - 2 - \log \frac{1}{\epsilon},$$
$$R_1' \leq I_H^{\epsilon}(U_1 : Y_2 U_2 Z_2 | Q) - 2 - \log \frac{1}{\epsilon}$$

to appear in the rate region. For this we need to use 'union of intersection of POVM elements', which can be done by appealing to Fact 5. The 'union' expresses the observation that it is unnecessary for B_1 to decode m_2' if he has already successfully decoded (m_1', m_1''), or equivalently decoding m_2' wrongly is not a problem if B_1 has already successfully decoded (m_1', m_1''). A similar comment holds for B_2. In the asymptotic iid setting, presence of these constraints does not affect the rate region. In the one-shot setting it is not clear if this is true, simply because of the lack of chain rules for the hypothesis testing mutual information. In the interest of obtaining as large an inner bound as possible, we use the 'union' technique. □

Remark 2 It is possible to get another Han–Kobayashi style inner bound if we allow independent prior entanglement between all the four possible sender–receiver pairs, i.e. if we allow both cis and trans entanglements, where trans entanglement refers to prior entanglement between A_1 and B_2, and A_2 and B_1. In this scenario we can directly employ, as a subroutine, the entanglement-assisted one-shot inner bound for the three-sender quantum multiple access channel described in [1]. All the message parts will now be transmitted using entanglement assistance. However, we do not discuss this further in this paper because we feel that trans entanglement is an unnatural resource.

6. Conclusions

In this paper, we have fruitfully used the quantum joint typicality lemmas from [1] to prove some novel inner bounds for sending classical information through multiterminal quantum channels. All our inner bounds require us to construct simultaneous decoders, and hold in the one-shot setting. For some of these problems, one-shot inner bounds were hitherto unknown even in the classical setting. All our one-shot inner bounds are strong enough to reduce to the standard inner bounds in the asymptotic iid limit, and provide non-trivial second order rates.

The Han–Kobayashi inner bound for a q-IC with entanglement assistance given in this paper does not use the prior entanglement to send the common parts of the messages. It seems that there is scope for improvement in this regard, which is left for future work.

It will be interesting to find other applications of simultaneous decoders in quantum network information theory. Already, Ding, Gharibyan, Hayden and Walter [29] have used the joint typicality lemmas to construct a simultaneous decoder for a particular quantum relay channel.

The quantum joint typicality lemmas give us robust tools to handle union and intersection for 'packing type' problems. However, they fail for 'covering type' problems. Covering type problems often arise in source coding. Constructing simultaneous decoders for them remains a major open problem.

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