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Design of the crank–rocker mechanism for various design cases based on the closed-form solution

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Abstract. The problem of motion transformation from a complete rotation into an oscillatory motion is seen in many applications, and the simplest solution to this problem is possible with the design of a crank–rocker mechanism. In this study the design of the crank–rocker mechanism in the analytical method, based on a closed-form solution, has been achieved according to several requirements that may arise by the designers for specific design cases, considering all the effective design parameters of the crank–rocker mechanism. Besides, numerical examples are presented with the help of a computer program developed using the obtained design equations.

Keywords. Crank–rocker mechanism; kinematic synthesis; closed-form solution; transmission angle; time ratio.

1. Introduction

The simplest solution to the problem of transformation from a full rotation motion into an oscillatory motion is possible by designing a crank–rocker mechanism. This is also the solution of the kinematic equations of a four-bar crank–rocker mechanism under certain restrictive conditions. Therefore, we deal with a kinematic synthesis problem. The most important constraint in the crank–rocker mechanism design is the minimum deviation of transmission angle from 90 degrees, which determines the load-bearing characteristic of the mechanism.

In general, three different approaches, can be used to solve these kinematic synthesis problems: graphical methods, methods that require the use of numerical techniques (sequential methods) and analytical methods. Graphical methods are limited in terms of coverage and drawing accuracy and are time-consuming for calculating each solution [1–3]. The methods that require the use of numerical techniques (sequential recalculation methods) have common problems such as the lack of convergence guarantee, the need to select appropriate initial values and the acquisition of a single solution value in response to overloaded computation. The studies by Gosselin and Angeles [4], Dhingra *et al* [5], Innocenti [6], Wampler *et al* [7, 8] and Roth and Freudenstein [9] can be used for sequential method applications applied to kinematic synthesis and analysis problems. However, in analytical methods (based on closed-form solutions), the system of equations is generally reduced to a polynomial equation, as shown in the studies by Auzinger and Stetter [10], Albert [11], Waldron *et al* [12], Innocenti and Parenti-Castelli [13], Dhingra *et al* [14, 15] and Balli and Chand [16]. Thus, all possible solutions can be obtained with lesser computation times. In this study, eight different design processes (or design cases) for designing the crank–rocker mechanism in accordance with the designer's different demands have been demonstrated. In some of these design cases the design is performed directly, whereas in other design cases the solution is reduced to a polynomial equation because of the nonlinear nature of these design equations.

The first seven design cases, regardless of the transmission angle, were designed based on the assumption that two unknown parameters of the kinematic equations obtained from the two different limit positions of the crank–rocker mechanism are given. However, in the eighth case, the maximum and minimum transmission angles of the crank–rocker mechanism are pre-determined. As known, uneven motion transmissions of mechanisms are characterized by constantly changing rates of motion transmission [17]. One of the important criteria of these mechanisms is the transmission angle. In design, it is desirable that the transmission angle variation from 90° is as less as possible. The recommended transmission angle is 90° ± 40° or 90° ± 50° [17, 18]. In literature, optimization techniques have often been used to find the mechanism with the best transmission angle [17–21]. The book of Söylemez [21], where the classical transmission angle problem is discussed, shows the problem in two parts. In the first part, the dimensions of the crank–rocker mechanism are calculated via the swing angle and crank rotation. From the infinite solution set obtained in the first part, the special dimensions of the

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crank–rocker mechanism for the best transmission angle with minimum deviation from 90° are found in the second part. The system of equations in the second part is reduced to a third-degree polynomial equation. Similarly, in the study by Brodell and Soni [22], a planar crank–rocker mechanism is designed for unit time ratio (*T.R*) using the design criterion of the minimum transmission angle. For the optimum transmission angle, a design chart was used in both studies.

In this work, instead of the optimal design of the crank–rocker mechanism with the best transmission angle, a closed-solution-based method is presented for designing the crank–rocker mechanism given the designer's maximum and minimum values of the transmission angle. In this way, the critical values of the transmission angle will be precisely controlled. Moreover, this study did not need a design chart as in studies by Söylemez [21] and Brodell and Soni [22] and there is no such comprehensive study about this topic in the kinematics literature. The cases discussed in this study, especially the cases in which the equations system is reduced to a polynomial, are unique.

2. Design of the crank–rocker mechanism

The crank–rocker mechanism is, basically, a four-bar mechanism in which the output link oscillates when the input link completely rotates, consistent with Grashof's Law. Determining the parameters that define the two limit positions occurring at two different positions of the input link, as shown in figure 1, is of great importance in terms of design.

When the input link, MA = q, is completely rotating, the output link, QB = u, oscillates between the limit positions QB_1 and QB_2. Thus, the parameters (α, β, η, Φ) that

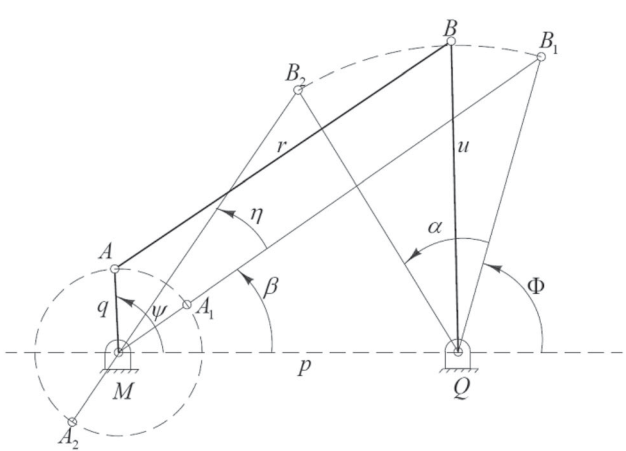

Figure 1. The parameters of the two limit positions and also the intermediate position of the crank–rocker mechanism.

determine the limit positions are utilized to design a crank–rocker mechanism.

Here, α is the amount of the output link oscillation, β is the angle of the input link at the first limit position measured with respect to the horizontal, η is the angle between the two limit positions of the coupler link and Φ is the angle of the output link at the first limit position measured with respect to the horizontal. The general configuration of the crank–rocker mechanism is represented by the angle ψ of the input link with respect to the horizontal.

Based on the limit positions in the crank–rocker mechanism the Sine Law is applied to triangles ΔMB_1Q and ΔMB_2Q in figure 1, and thus the following design equations can be obtained after the necessary algebraic operations:

$$\frac{u}{p} = \frac{1}{2}\left[\frac{\sin\beta}{\sin(\Phi-\beta)}\right] \quad (1)$$

$$\frac{r}{p} = \frac{1}{2}\left[\frac{\sin\Phi}{\sin(\Phi-\beta)} + \frac{\sin(\Phi+\alpha)}{\sin(\Phi+\alpha-\beta-\eta)}\right] \quad (2)$$

$$\frac{q}{p} = \frac{1}{2}\left[\frac{\sin\Phi}{\sin(\Phi-\beta)} - \frac{\sin(\Phi+\alpha)}{\sin(\Phi+\alpha-\beta-\eta)}\right] \quad (3)$$

and the Φ angle is given by

$$\Phi = \mathrm{Tan}^{-1}\left[\frac{\mathrm{Tan}\beta[\mathrm{Sin}(\beta+\eta)+\mathrm{Sin}(\alpha-\beta-\eta)]}{\mathrm{Sin}(\beta+\eta)-\mathrm{Tan}\beta\mathrm{Cos}(\alpha-\beta-\eta)}\right] \quad (4)$$

In expressions (1)–(4) the basic design parameters, namely input link length (q), connecting rod length (r), output link length (u) and ground link length (p), are involved as proportional quantities. The way by which the expressions are written indicates that the length of the ground link (p) can be arbitrarily chosen to be ineffective in the design process. However, the ground link length can be desirably used to scale the obtained crank–rocker mechanism dimensions at the end of the design process. It is also often more appropriate to determine the η angle between the limit positions in the design process according to the round-trip T.R [22–24] of the crank–rocker mechanism. Accordingly, the equation between *T.R* and η angle is as follows:

$$\eta = 180\left[\frac{T.R-1}{T.R+1}\right] \quad (5)$$

From among the quantities existing in the expressions (1)–(4), the magnitude of the swing angle (α) is always given in advance. The other four parameters can be determined as a result of specifying two of the remaining six angular and dimensional parameters (φ, β, η, u/p, r/p, q/p). Accordingly the various cases that allow the determination of the angular and dimensional quantities of a crank–rocker mechanism, based on the given swing angle (α) and the additionally specified designer's requirements, are shown here.

2.1 *Case one*

If the *T.R* and the first limit position angle (β) are given, the following steps are taken to calculate the other quantities:

i) The given *T.R* is substituted in equation (5), and the angle between the limit positions (η) is calculated.
ii) Knowing α, η and β, the angle Φ is solved from equation (4).
iii) As a result of substituting the values of α, η, Φ and β in the formulas (1)–(3) the link length ratios (*u*/*p*, *r*/*p*, *q*/*p*) are determined, and thus the design is completed.

2.2 *Case two*

When the *T.R* and the first limit position angle of the output link (Φ) are given, the remaining parameters are determined as follows:

i) The given *T.R* is substituted in equation (5), and the η angle between the two limit positions is calculated.
ii) α, η and Φ are substituted in equation (4), and thus the resulting trigonometric equation becomes $f(\beta)$=0 depending only on the unknown β. If the tangent of half angle formulas, taking $t_\beta = \mathrm{Tan}\left(\frac{\beta}{2}\right)$, are used in the resulting equation, it is transformed into the fourth-degree polynomial in equation (6):

$$w_0 + w_1 t_\beta + w_2 t_\beta^2 - w_1 t_\beta^3 + w_0 t_\beta^4 = 0 \tag{6}$$

The coefficients in equation (6) are calculated as follows:

$$\left.\begin{aligned} w_0 &= Cos(\eta - \Phi) - Cos(\eta + \Phi) \\ w_1 &= -4[Sin(\eta - \Phi) + Sin(\alpha - \eta + \Phi)] \\ w_2 &= 8Cos(\alpha - \eta + \Phi) - 6Cos(\eta - \Phi) \\ &\quad -2Cos(\eta + \Phi). \end{aligned}\right\} \tag{7}$$

Four real and complex roots of the polynomial equation (6) are easily found. With these found roots the solution set of β angle, which signifies the smallest limit position, is determined as follows:

$$\beta_i = 2\mathrm{Arctan}(t_{\beta i}), \quad i = 1, 2, 3, 4 \tag{8}$$

iii) The process is completed by finding the solution set $\{(u/p, r/p, q/p)_i; i = 1, 2, 3, 4\}$ for the remaining link lengths by substituting the determined values β_i (for i = 1, …, 4), Φ and η in equations (1)–(3).

2.3 *Case three*

The following steps are used to determine the other parameters when the *T.R* and the link length ratio (*u*/*p*) of the crank–rocker mechanism are given.

i) The given *T.R* is substituted in equation (5), and the η angle is calculated.
ii) If the (α, η, *u*/*p*) values are substituted in equations (1) and (4), and the Φ angle is eliminated from these two equations, the trigonometric expression depending on β is obtained as in equation (9):

$$\begin{aligned} &2 - (u/p)^2 - \mathrm{Cos}\ \alpha - \mathrm{Cos}2\beta \\ &\quad + \mathrm{Cos}(\alpha + 2\beta) - \mathrm{Cos}(\alpha - 2\eta) \\ &\quad + (u/p)^2\mathrm{Cos}(2\alpha - 2\eta) + \mathrm{Cos}(\alpha - 2\beta - 2\eta) \\ &\quad - \mathrm{Cos}(2\beta + 2\eta) = 0 \end{aligned} \tag{9}$$

If the variable transformation,$t_\beta = \mathrm{Tan}\left(\frac{\beta}{2}\right)$, is performed on equation (9) using the tangent of the half angle formulas, the fourth-degree polynomial dependent on t_β is obtained in equation (10):

$$w_0 + w_1 t_\beta + w_2 t_\beta^2 - w_1 t_\beta^3 + w_0 t_\beta^4 = 0 \tag{10}$$

The coefficients in equation (10) are calculated as follows:

$$\begin{aligned} w_0 &= 1 - (u/p)^2 + (u/p)^2\mathrm{Cos}(2\alpha - 2\eta) - \mathrm{Cos}2\eta \\ w_1 &= 16\mathrm{Sin}0.5\alpha\mathrm{Sin}(0.5\alpha - \eta)\mathrm{Sin}\eta \\ w_2 &= 2[5 - (u/p)^2 - 4\mathrm{Cos}\alpha - 4\mathrm{Cos}(\alpha - 2\eta) \\ &\quad - \left(u/p\right)^2\mathrm{Cos}(2\alpha - 2\eta) + 3\mathrm{Cos}2\eta\Big]. \end{aligned} \tag{11}$$

The polynomial equation (10) has four real and complex roots in terms of t_β, and the solutions set (β_i for i = 1, 2, 3, 4) is calculated from the expression (8).

iii) The determined β_i values for i = 1, …, 4 and the given values are substituted in expression (4), and thus the angles set $\Phi_{i,}$ (for i =1, 2, 3, 4) is calculated. The solution set belonging to these two angles is evaluated in equations (2)–(3) and the design is completed by finding the link lengths $\{(r/p, q/p)_i$ for i = 1, 2, 3, 4$\}$.

2.4 *Case four*

If the angle (β) and the link length ratio *u*/*p* are given, the following steps are taken to calculate the other parameters:

i) The angle Φ is solved by substituting the given *u*/*p* and β angles in equation (1).
ii) The angle η is calculated in equation (4) using the found parameters.
iii) The *T.R* is determined from equation (5) by the use of the η angle.

iv) The length ratios (r/p, q/p) are determined by substituting the values of η, Φ and β in equations (2)–(3).

2.5 *Case five*

In the case where the angle β and the angle Φ are different when the crank–rocker mechanism is at the first limit position, the following sequence is followed to calculate the other parameters:

i) The given β and Φ angles are substituted in equations (1), and the u/p length ratio is calculated.
ii) Since the angle Φ and β are known, the angle η is solved from equation (4).
iii) The $T.R$ is determined from the equation (5) with the knowledge of the η angle.
iv) The length ratios (r/p, q/p) are determined by substituting the (η,Φ, β) angles in equations (2)–(3).

2.6 *Case six*

The following steps are carried out for the design when the crank–rocker mechanism is at the first limit position and the Φ angle and the u/p length ratio of the output arm are given:

i) The given Φ and u/p ratio are substituted in expression (1) to find the β angle.
ii) With the knowledge of Φ and β, the angle η is solved from equation (4).
iii) The $T.R$ is determined by substituting the η angle in equation (5).
iv) The length ratios (r/p, q/p) are calculated by substituting the (η,Φ, β) angles in equations (2)–(3).

2.7 *Case seven*

The following steps are used to determine the other parameters when the $T.R$ and the ratio of the output link length to the connecting rod length (u/r) of the crank–rocker mechanism are given:

i) The given $T.R$ is substituted in equation (5) and the angle η is calculated.
ii) The expression (1) is divided by expression (2), side by side, to form an expression for u/r. With this expression, when the angle Φ is eliminated by the tangent of the half angle formula of equation (4), the following trigonometric equation related to β is obtained:

$$\begin{aligned}
&2-\left[\left(u/r\right)^2-1\right]\mathrm{Cos}\alpha-\mathrm{Cos}2\beta\\
&+\mathrm{Cos}(\alpha+2\beta)-\mathrm{Cos}(\alpha-2\eta)\\
&-\mathrm{Cos}(2\beta+2\eta)+\mathrm{Cos}(\alpha-2\beta-2\eta)\\
&-(u/r)^2\mathrm{Sin}^2 0.5\alpha[\mathrm{Cos}(\alpha+2\beta)+2\mathrm{Cos}(\alpha-\eta)\\
&+2\mathrm{Cos}(2\beta+\eta)+\mathrm{Cos}(\alpha-2\beta-2\eta)]=0
\end{aligned}\tag{12}$$

If the variable transformation, $t_\beta=\mathrm{Tan}\left(\frac{\beta}{2}\right)$, is performed using the tangent of the half angle formulas in equation (9), the following fourth-degree polynomial that depends on t_β is obtained for a closed-form solution:

$$w_0+w_1t_\beta+w_2t_\beta^2-w_1t_\beta^3+w_0t_\beta^4=0\tag{13}$$

The coefficients in equation (13) are calculated as follows:

$$\begin{aligned}
w_0=&-8\mathrm{Cos}^2 0.5\eta\left[-\left(\frac{u}{r}\right)\mathrm{Sin}(\alpha-0.5\eta)\right.\\
&\left.-2\mathrm{Sin}0.5\eta-\left(\frac{u}{r}\right)\mathrm{Sin}0.5\eta\right]\\
&\left[-\left(\frac{u}{r}\right)\mathrm{Sin}(\alpha-0.5\eta)\right.\\
&\left.+2\mathrm{Sin}0.5\eta-\left(\frac{u}{r}\right)\mathrm{Sin}0.5\eta\right],\\
w_1=&\,32\mathrm{Cos}0.5\eta\mathrm{Sin}0.5\alpha\mathrm{Sin}\,0.5\eta\\
&\left[2(u/r)^2\mathrm{Sin}0.5\alpha+4\mathrm{Sin}(0.5\alpha-\eta)\right.\\
&-(u/r)^2\mathrm{Sin}(0.5\alpha-\eta)\\
&\left.+(u/r)^2\mathrm{Sin}(3\alpha/2-\eta)\right]\\
w_2=&-2\left[20+7\left(\frac{u}{r^2}\right)+16\mathrm{Cos}\alpha\right.\\
&-10(u/r)^2\mathrm{Cos}\alpha+3(u/r)^2\mathrm{Cos}2\alpha\\
&+16\mathrm{Cos}(\alpha-2\eta)-6(u/r)^2\mathrm{Cos}(\alpha-2\eta)\\
&+3(u/r)^2\mathrm{Cos}(2\alpha-2\eta)+10(u/r)^2\mathrm{Cos}(\alpha-\eta)\\
&-2(u/r)^2\mathrm{Cos}(2\alpha-\eta)\\
&-14(u/r)^2\mathrm{Cos}\eta-12\mathrm{Cos}2\eta+3(u/r)^2\mathrm{Cos}\,2\eta\\
&\left.+6(u/r)^2\mathrm{Cos}(\alpha+\eta)\right]
\end{aligned}\tag{14}$$

The polynomial equation (13) has four real and complex roots in terms of t_β, and the solution set β_i (for i = 1, 2, 3, 4) is calculated from the expression (8).

iii) The found values and the calculated β_i values for i = 1, …, 4 are substituted in the expression (4), and (Φ_i; i =1, 2, 3, 4) is calculated. Then, the solution set of these two angles is computed in equations (2)–(3), and the design is completed by finding $\{(r/p, q/p)_i$ for i = 1, 2, 3, 4$\}$ length ratios.

3. Case eight: design of transmission angle generating crank–rocker mechanism

The most important criterion determining the design quality of the crank–rocker mechanism is the transmission angle, which represents the load-bearing characteristic of the mechanism. The load-carrying characteristic is best if the condition by which the transmission angle is 90° is satisfied. However, due to the fact that the transmission angle is a function of the angle ψ (the amount of input link rotation), it takes different values at each position of the mechanism. Besides, it reaches the minimum (μ_{min}) and maximum (μ_{max}) values when the input link is at 0° and 180° positions, respectively, as shown in figure 2. In other positions, the transmission angle changes between these two values. For this reason, the criterion used to obtain the optimum transmission angle of the crank–rocker mechanism is the minimum of 90 degrees deviation of μ_{max} and μ_{min} [17, 18]. It is stated in the literature that these deviations do not exceed 40 or 50 degrees for a suitable load-carrying characteristic [17, 18].

The afore-mentioned seven different design methods of the crank–rocker mechanism do not involve the transmission angle. However, control over the minimum and maximum values of the transmission angle is possible if it is incorporated into the design process of the crank–rocker mechanism. If the transmission angle is not suitable, the appropriate transmission angle can be obtained by systematically changing the input parameters. However, this approach, which is based on trial and error, is not only long but it may not be possible to obtain the desired transmission angle, as well. For this reason, a pre-determination of the minimum and the maximum transmission angle has been introduced during the determination of the unknown kinematic parameters of a crank–rocker mechanism that oscillates up to the angle (α) of the output link. Thus, the force transfer characteristic of the crank–rocker mechanism will also be controlled as desired by precisely determining the transmission angle. For this purpose the following two equations (15)–(16), which give the minimum (μ_{min}) and maximum (μ_{max}) values of the transmission angle, can be obtained when the Cosine Law is applied to triangles A_1B_1Q and A_2B_2Q in figure 2:

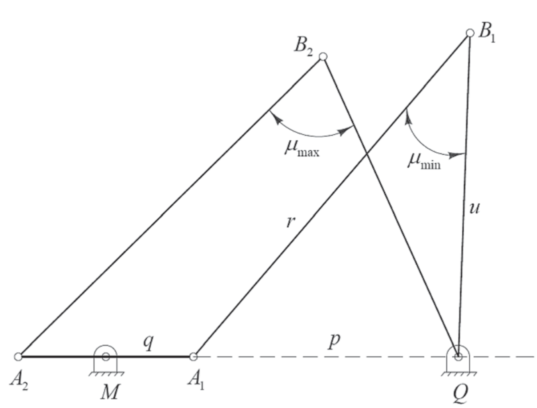

Figure 2. The minimum and maximum transmission angle configurations of crank–rocker mechanism.

$$\mathrm{Cos}\mu_{min} = \frac{u^2 + r^2 - p^2 - q^2 + 2pq}{2ur}, \tag{15}$$

$$\mathrm{Cos}\mu_{max} = \frac{u^2 + r^2 - p^2 - q^2 - 2pq}{2ur}. \tag{16}$$

Equations (1)–(4) and (15)–(16) form a set of nonlinear equations with six kinematic parameters of the crank–rocker mechanism, namely (φ, β, η, u/p, r/p, q/p). It is seen that when the values of μ_{min} and μ_{max} in equations (15)–(16) are determined, these equations can be solved for the six unknown parameters of the crank–rocker mechanism. Eventually, the following three non-linear equations with three unknowns are obtained after the proper algebraic manipulations in the expressions (1–3) and (15–16):

$$\begin{aligned} f_1(\beta,\eta,\phi) = &-\mathrm{Cos}(2\beta+\eta-\phi) \\ &-\mathrm{Cos}(\alpha-\eta+\phi)+\mathrm{Cos}(\alpha-2\beta-\eta+\phi) \\ &+\mathrm{Cos}(\eta+\phi)=0, \end{aligned} \tag{17}$$

$$\begin{aligned} f_2(\beta,\eta,\phi) = &-\mathrm{Cos}(\alpha+3\beta/2-\phi)+\mathrm{Cos}(\alpha-\beta/2+\phi) \\ &-\ \mathrm{Cos}(\alpha+\beta/2+\phi)-\mathrm{Cos}(\alpha-5\beta/2-\eta+\phi) \\ &-\ \mathrm{Cos}(\alpha+\beta/2-\eta+\phi)+\mathrm{Cos}(\alpha-3\beta/2+3\phi) \\ &+\ 2\mathrm{Sin}(3\beta/2-2\phi)\mathrm{Sin}(\alpha-\beta-\eta+\phi) \\ &+\ 2\mathrm{Cos}(3\beta/2)C\mu_{\mathrm{min}}[\mathrm{Cos}(\alpha+\beta)+\mathrm{Cos}(\alpha-\beta-\eta)] \\ &-\ 2\mathrm{Cos}(\eta/2)\mathrm{Cos}(\alpha-\beta-\eta/2+2\phi)=0 \end{aligned} \tag{18}$$

$$\begin{aligned} f_3(\beta,\eta,\phi) = &\ 4\mathrm{Sin}(\beta/2-\phi) \\ &\mathrm{Sin}(\beta-\phi)\mathrm{Sin}(\alpha+\phi)+2\mathrm{Sin}(\alpha-\beta-\eta+\phi) \\ &[-\mathrm{Cos}(3\beta/2)+2\mathrm{Cos}(3\beta/2-2\phi)] \\ &+\ 4\mathrm{Sin}(\beta/2)C\mu_{max}[\mathrm{Sin}(\beta-\phi)\mathrm{Sin}(\alpha+\phi) \\ &-\mathrm{Sin}\phi\mathrm{Sin}(\alpha-\beta-\eta+\phi)]=0. \end{aligned} \tag{19}$$

Here, the abbreviations Cos μ_{min} = $C\mu_{min}$ and Cos μ_{max} = $C\mu_{max}$ are used.

The use of tangent of the half angle formulas for the angle η in equations (17)–(19) transforms them to the following equations:

$$f_1(\beta,\eta,\phi) = aw_0 + aw_1 t_\eta - aw_0 t_\eta^2 = 0 \tag{20}$$

$$f_2(\beta,\eta,\phi) = bw_0 + bw_1 t_\eta + bw_2 t_\eta^2 = 0 \tag{21}$$

$$f_3(\beta,\eta,\phi) = cw_0 + cw_1 t_\eta + cw_2 t_\eta^2 = 0 \tag{22}$$

$$t_\eta = \mathrm{Tan}\frac{\eta}{2}. \tag{23}$$

The polynomial coefficients of equations (20)–(22) are as follows:

$$aw_0 = \mathrm{Cos}\left(\frac{\alpha}{2} - \beta + \phi\right)\mathrm{Sin}\left(\frac{\alpha}{2}\right)\mathrm{Sin}\ \beta \tag{24}$$

$$\begin{aligned} aw_1 = &-\mathrm{Cos}(\alpha-\beta+\phi)\mathrm{Sin}\beta \\ &+ \mathrm{Cos}\beta\mathrm{Sin}(\beta-\phi) \end{aligned} \tag{25}$$

$$\begin{aligned} bw_0 = 0.5\mathrm{Cos}(\beta/2)[&-\mathrm{Cos}(\alpha+\beta-\phi) \\ &- \mathrm{Cos}(\alpha-2\beta+\phi) + \mathrm{Cos}(\alpha-\beta+\phi) \\ &+ \mathrm{Cos}(\alpha-2\beta+3\phi) + C\mu_{min}(\mathrm{Cos}(\alpha-\beta) \\ &+ \mathrm{Cos}(\alpha+\beta) - 2\mathrm{Cos}(\alpha-\beta+2\phi))] \end{aligned} \tag{26}$$

$$\begin{aligned} bw_1 = &\ 2\mathrm{Cos}(\alpha-\beta+\phi)\mathrm{Sin}\varphi \\ &\left[\mathrm{Cos}\left(\frac{3\beta}{2} - \phi\right) - \mathrm{Cos}\left(\frac{\beta}{2}\right)C\mu_{min}\right] \end{aligned} \tag{27}$$

$$\begin{aligned} bw_2 = 0.5\mathrm{Sin}(\beta/2)[&\mathrm{Sin}(\alpha+\beta-\phi) \\ &+ 2\mathrm{Sin}(\alpha+\phi) + \mathrm{Sin}(\alpha-2\beta+\phi) \\ &+ \mathrm{Sin}(\alpha-\beta+\phi) - \mathrm{Sin}(\alpha-2\beta+3\phi) \\ &+4\mathrm{Cos}^2\left(\frac{\beta}{2}\right)\mathrm{Sin}\alpha C\mu_{min}\Bigg] \end{aligned} \tag{28}$$

$$\begin{aligned} cw_0 = 0.5\mathrm{Sin}(\beta/2)[&\mathrm{Cos}(\alpha+\beta-\phi) \\ &- \mathrm{Cos}(\alpha-2\beta+\phi) - \mathrm{Cos}(\alpha-\beta+\phi) \\ &+ \mathrm{Cos}(\alpha-2\beta+3\phi) + C\mu_{max}(\mathrm{Cos}(\alpha-\beta) \\ &+\mathrm{Cos}(\alpha+\beta) - 2\mathrm{Cos}(\alpha-\beta+2\phi))] \end{aligned} \tag{29}$$

$$\begin{aligned} cw_1 = &\ 2\mathrm{Cos}(\alpha-\beta+\phi)\mathrm{Sin}\varphi \\ &\left[\mathrm{Sin}\left(\frac{3\beta}{2} - \varphi\right) - \mathrm{Cos}\left(\frac{\beta}{2}\right)C\mu_{max}\right] \end{aligned} \tag{30}$$

$$\begin{aligned} cw_2 = 0.5\mathrm{Cos}\left(\frac{\beta}{2}\right)[&\mathrm{Sin}(\alpha+\beta-\phi) \\ &- 2\mathrm{Sin}(\alpha+\phi) - \mathrm{Sin}(\alpha-2\beta+\phi) \\ &+ \mathrm{Sin}(\alpha-\beta+\phi) + \mathrm{Sin}(\alpha-2\beta+3\phi) \\ &+2(\mathrm{Cos}\beta - 1)\mathrm{Sin}\alpha C\mu_{max}]. \end{aligned} \tag{31}$$

Eliminating the angle η from equations (20)–(22) and using tangent of the half angle formulas for the angle β, algebraic manipulations first between f_1 and f_2 and then between f_1 and f_3 lead to the following equations:

$$g_2(\beta,\phi) = bp_0 + bp_1 t_\beta + bp_2 t_\beta^2 = 0 \tag{32}$$

$$t_\beta = \mathrm{Tan}\frac{\beta}{2} \tag{33}$$

The polynomial coefficients in equations (32)–(34) are as shown here:

$$\begin{aligned} ap_0 = &\ 2[\mathrm{Cos}\phi - C\mu_{min}][\mathrm{Cos}\alpha \\ &+ \mathrm{Cos}(\alpha+2\phi) - 2 \\ &+ 2C\mu_{min}(\mathrm{Cos}\phi - \mathrm{Cos}(\alpha+\phi))] \end{aligned} \tag{34}$$

$$\begin{aligned} ap_1 = &\ 4\mathrm{Sin}\phi[\mathrm{Cos}\alpha \\ &+ Cos(\alpha+2\phi) - 2 \\ &+2C\mu_{min}(\mathrm{Cos}\phi - \mathrm{Cos}(\alpha+\phi))] \end{aligned} \tag{35}$$

$$ap_2 = 4\mathrm{Sin}^2\phi[\mathrm{Cos}(\alpha+\phi) + C\mu_{min}] \tag{36}$$

$$bp_0 = 4\mathrm{Sin}^2\phi[C\mu_{max} - \mathrm{Cos}(\alpha+\phi)] \tag{37}$$

$$\begin{aligned} bp_1 = &\ 4\mathrm{Sin}\phi[\mathrm{Cos}\alpha + \mathrm{Cos}(\alpha+2\phi) \\ &-2 + 2C\mu_{max}(-\mathrm{Cos}\phi + \mathrm{Cos}(\alpha+\phi))] \end{aligned} \tag{38}$$

$$\begin{aligned} bp_2 = &-2[\mathrm{Cos}\phi + C\mu_{max}] \\ &[\mathrm{Cos}\alpha + \mathrm{Cos}(\alpha+2\phi) - 2 \\ &+2C\mu_{max}(-\mathrm{Cos}\phi - \mathrm{Cos}(\alpha+\phi))] \end{aligned} \tag{39}$$

If the tangent of the half angle formulas for angle Φ are used in the equations set (32)–(33), t_β terms are removed from equations (32)–(33) to lead to the following relationship:

$$z(\phi) = z_1(\phi)\ z_2(\phi) = 0. \tag{40}$$

The functions $z_1(\phi)$ and $z_2(\phi)$ in the expression (40) are as follows:

$$\begin{aligned} z_1(\phi) = &\ p_0 + p_1 t_\phi + p_2 t_\phi^2 \\ &+ p_3 t_\phi^3 + p_4 t_\phi^4 = 0 \end{aligned} \tag{41}$$

$$\begin{aligned} z_2(\phi) = &\ w_0 + w_1 t_\phi + w_2 t_\phi^2 \\ &+ w_3 t_\phi^3 + w_4 t_\phi^4 + w_5 t_\phi^5 \\ &+ w_6 t_\phi^6 + w_7 t_\phi^7 + w_8 t_\phi^8 = 0 \end{aligned} \tag{42}$$

$$t_\phi = \mathrm{Tan}\frac{\phi}{2} \tag{43}$$

and the polynomial coefficients in equations (41) and (42) are as follows:

$$\begin{aligned} p_0 &= (1-\mathrm{Cos}\alpha)(1+C\mu_{\max})(C\mu_{\min}-1) \\ p_1 &= 2\mathrm{Sin}\alpha(1+C\mu_{\max})(C\mu_{\min}-1) \\ p_2 &= 4\mathrm{Cos}^2\frac{\alpha}{2}(C\mu_{\min}-C\mu_{\max}) \\ p_3 &= 2\mathrm{Sin}\alpha(C\mu_{\max}-1)(C\mu_{\min}+1) \\ p_4 &= (\mathrm{Cos}\alpha-1)(C\mu_{\max}-1)(C\mu_{\min}+1) \end{aligned} \tag{44}$$

$$
\begin{aligned}
w_0 &= 8\mathrm{Sin}^4\frac{\alpha}{2}(1+C\mu_{\max})^2(C\mu_{\min}-1)^2\\
w_1 &= 32\mathrm{Cos}\frac{\alpha}{2}\mathrm{Sin}^3\frac{\alpha}{2}(1+C\mu_{\max})^2(C\mu_{\min}-1)^2\\
w_2 &= 16\mathrm{Sin}^2\frac{\alpha}{2}(C\mu_{\min}-1)(1+C\mu_{\max})\\
&\quad(-5\mathrm{Cos}\alpha+2C\mu_{\min}\mathrm{Cos}\alpha+\\
&\quad C\mu_{\max}(-2\mathrm{Cos}\alpha+(1+\mathrm{Cos}\alpha)C\mu_{\min}))\\
w_3 &= -8\mathrm{Sin}\alpha(C\mu_{\min}-1)(1+C\mu_{\max})(9-\mathrm{Cos}\alpha-\\
&\quad 3C\mu_{\min}(\mathrm{Cos}\alpha-1)-\\
&\quad 2\mathrm{Sin}^2\frac{\alpha}{2}C\mu_{\max}(3+C\mu_{\min}))\\
w_4 &= 2(33+36\mathrm{Cos}\alpha-5\mathrm{Cos}2\alpha+\\
&\quad 4\mathrm{Sin}^2\frac{\alpha}{2}(4(3+\mathrm{Cos}\alpha)(C\mu_{\max}C\mu_{\min}-\\
&\quad (1+7\mathrm{Cos}\alpha)C\mu_{\min}^2+\\
&\quad C\mu_{\max}^2(-1-7\mathrm{Cos}\alpha+(3+5\mathrm{Cos}\alpha)C\mu_{\min}^2)))\\
w_5 &= 8\mathrm{Sin}\alpha(C\mu_{\min}+1)(C\mu_{\max}-1)(9-\mathrm{Cos}\alpha-\\
&\quad 2\mathrm{Sin}^2\frac{\alpha}{2}C\mu_{\max}(C\mu_{\min}-3)+\\
&\quad 3(\mathrm{Cos}\alpha-1)C\mu_{\min})\\
w_6 &= 16\mathrm{Sin}^2\frac{\alpha}{2}(C\mu_{\min}+1)(C\mu_{\max}-1)\\
&\quad(-5-\mathrm{Cos}\alpha-2C\mu_{\min}\mathrm{Cos}\alpha+\\
&\quad C\mu_{\max}(2\mathrm{Cos}\alpha+(1+\mathrm{Cos}\alpha)C\mu_{\min}))\\
w_7 &= -32\mathrm{Cos}\frac{\alpha}{2}\mathrm{Sin}^3\frac{\alpha}{2}(C\mu_{\max}-1)^2(C\mu_{\min}+1)^2\\
w_8 &= 8\mathrm{Sin}^4\frac{\alpha}{2}(C\mu_{\max}-1)^2(C\mu_{\min}+1)^2
\end{aligned}
\tag{45}
$$

Using these design equations the following steps can be executed to design the crank–rocker mechanism, which generates the required maximum and minimum transmission angles as well as the desired swing angle. As a result of the numerical experiments, it is seen that the fourth-degree polynomial $z_1(\varphi)$ in equation (41) is the solution that is being sought. Therefore, the system of equations is reduced to a fourth-degree univariate polynomial.

i) Coefficients (p_i for i = 0, 1, 2, 3, 4) are calculated using the given swing angle (α), the minimum transmission angle (μ_{min}) and the maximum transmission angle (μ_{max}) by means of expressions (44).

ii) The obtained coefficients are substituted in equation (41) and a total of four roots ($t_{\phi i}$ for i = 1, 2, …, 4), complex or real values, are found.

iii) The ($t_{\phi i}$ for i =1, 2, …, 4) solutions set is substituted in expression (43) to determine the (ϕ_i for i =1, 2, …, 4) solutions set of the angle that determines the limit position of the output link of the crank–rocker mechanism.

iv) (ap_0, ap_1, ap_2) coefficients are calculated to determine the $\beta_{\pm}$ angle shown in equation (33) by means of equations (34)–(36) and the angles {α, μ_{min}, μ_{max} and ϕ_i (for i =1, 2, …, 4)}. Thus, a total of eight solution sets will be obtained since there are two different angles $\beta_{\pm}$ for each value of the (ϕ_i for i =1, 2, …, 4) angles set. The results can be checked by referring to equation (32), which is supposed to yield the same.

v) Using the sets (β_+ and ϕ_i for i =1, 2, …, 4) and (β_- and ϕ_i for i =1, 2, …, 4), any one of equations (20)–(22) can be made use of to determine the angle $\eta_{\pm}$.

vi) The values of the link length ratios (*u*/*p*, *r*/*p*, *q*/*p*) of the crank–rocker mechanism are calculated, referring to expressions (1)–(3), by utilizing the real solutions from a total of 16 solution sets obtained in (η_+, β_+ and ϕ_i for i =1, 2, …, 4), (η_-, β_+ and ϕ_i for i =1, 2, …, 4), (η_+, β_- and ϕ_i for i =1, 2, …, 4), (η_-, β_- and ϕ_i for i =1, 2, …, 4), and by executing the earlier steps ii, iii, iv and v. The suitability of the solutions can be understood by evaluating the obtained 16 solution sets in equations (15)–(16). Since the system equations are reduced to a fourth-degree polynomial equation, it has been observed that four solutions at most are suitable to provide the transmission angles as given in numerical experiments.

Minimum and maximum transmission angles are calculated by means of equations (15)–(16) using the link length ratios obtained. The calculated transmission angles are compared to the required transmission angle values; as a result, the total number of the desired solution set is determined.

4. Numerical examples

The methods for designing the crank–rocker mechanism in the direction of the designer's various demands have been shown earlier in eight different ways. In cases number 1, 4, 5 and 6 the solutions are reached directly, similar to the solution in the first part in the book of Söylemez [21], whereas in cases number 2, 3, 7 and 8 the solution is reduced to a polynomial equation. Some numerical results obtained by developing a PASCAL computer program including these design cases are shown here. Only the examples where direct solution is achievable are not included. In all the examples the ground link length (p), deemed as a scale factor, equals 1.

4.1 *Example one*

In this example, if the swing angle α and the angle of the first limit position ϕ are chosen to equal 90° and 65°, respectively, and the *T.R* is 1.4 (thus η = 30°) then the other unknown parameters involving the second case are obtained as given in table 1. From among the four solutions in table 1, only two unique solutions with different

Table 1. Numerical results for case 2.

Sol. no.	β (°)	u (unit)	r (unit)	q (unit)
1	− 150.06	0.86890	− 1.00103	− 0.57676
2	140.06	− 0.66446	− 1.28233	0.34431
3	− 39.94	− 0.66446	1.28233	− 0.34431
4	29.94	0.86890	1.00103	0.57676

Table 2. Numerical results for case 3.

Sol. no.	β (°)	ϕ (°)	r (unit)	q (unit)
1	− 156.87	52.53	− 1.11250	− 0.50410
2	126.87	37.47	− 1.11250	0.50410
3	− 53.13	37.47	1.11250	− 0.50410
4	23.13	52.53	1.11250	0.50410

dimensions exist. The other solutions are actually representations of the same solution in different angular notations. In figure 3(a) and (b), the geometric representations of the two unique solutions are presented.

4.2 *Example two*

In this example, the design results of the crank–rocker mechanism selected when the swing angle α = 90°, the output link length u = 0.8 units and $T.R$ = 1.4 (thus η = 30°) are as shown in table 2.

From table 2, all the four solutions obtained refer to a single crank–rocker mechanism. The geometric representation of this mechanism is as shown in figure 4.

4.3 *Example three*

Here the design results of the crank–rocker mechanism are shown in table 3, as related to the case where the swing angle α = 90°, $T.R$ = 1.4 (thus η = 30°) and the ratio of the output link length to the connecting rod length u/r = 0.9 are selected.

In table 3, all the solutions indicate a single crank–rocker mechanism. The geometric representation of this mechanism is as shown in figure 5.

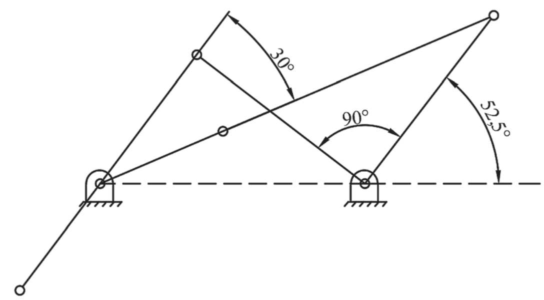

Figure 4. The crank–rocker mechanism and its limit positions obtained in case 3.

Table 3. Numerical results for case 7.

Sol. no.	β (°)	ϕ (°)	r (unit)	q (unit)
1	− 148.81	67.20	− 0.97897	− 0.58924
2	118.81	22.80	− 0.97897	0.58924
3	− 61.19	22.80	0.97897	− 0.58924
4	31.19	67.20	0.97897	0.58924

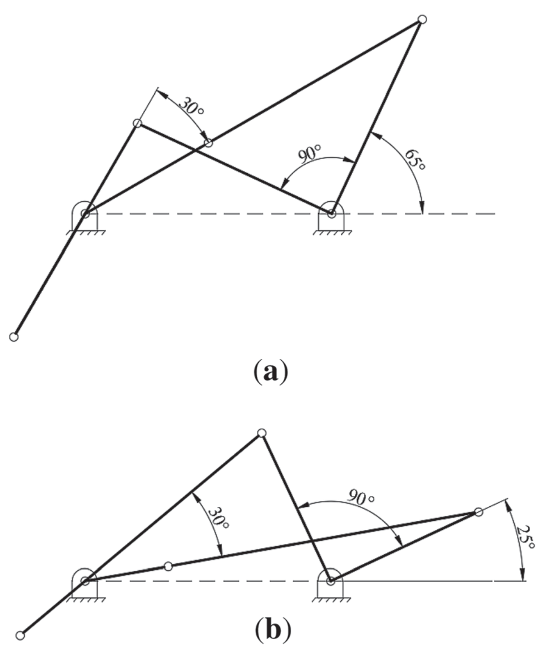

Figure 3. The crank–rocker mechanism and its limit positions obtained in case 2: (**a**) first solution and (**b**) second solution.

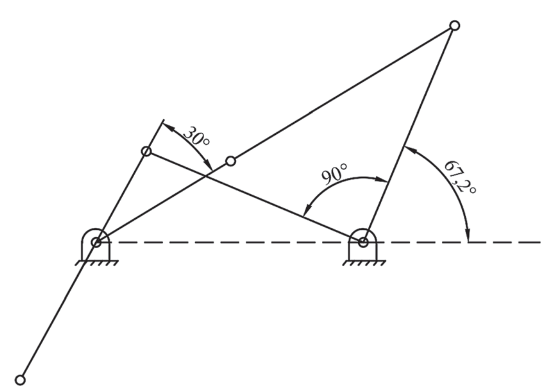

Figure 5. The crank–rocker mechanism and its limit positions obtained in case 7.

4.4 *Example four*

The design results of the crank–rocker mechanism obtained for the swing angle α = 40°, the maximum transmission angle μ_{max} = 115° and the minimum transmission angle μ_{min} = 65° are shown in table 4. By checking the solutions out of 16 solution sets obtained in the 8th design case, I ended up with four design results having the validation of the required values of the maximum and minimum transmission angle. However, the four solutions are obtained in different angular notations and indicate the same crank–rocker mechanism. Hence, one solution is given in table 4.

In order to compare the obtained result in this example to that of the technique 'The Classical Transmission Angle Problem' discussed in Example 4.3 by Söylemez [21], the crank–rocker mechanism was designed by selecting the swing angle α = 40° as in [21]. In [21] the maximum transmission angle deviation from 90° (critical transmission angle) was 58.15°, while in case 8 this value was found to be 25°. The reason behind this is that in [21] an angular dimension of the crank–rocker mechanism is considered as a known parameter besides the swing angle (α). In my approach of case 8, the design is made according to only one design parameter given: the swing angle (α). Therefore, in case 8, the number of design parameters unknown is higher than what is in the book of Söylemez [21]. Therefore, the system of equations is reduced to a fourth-degree univariate polynomial equation instead of third-degree polynomial equation as shown in the literature.

The kinematic dimensions and limit positions of the crank–rocker mechanism obtained according to both results are shown in figure 6.

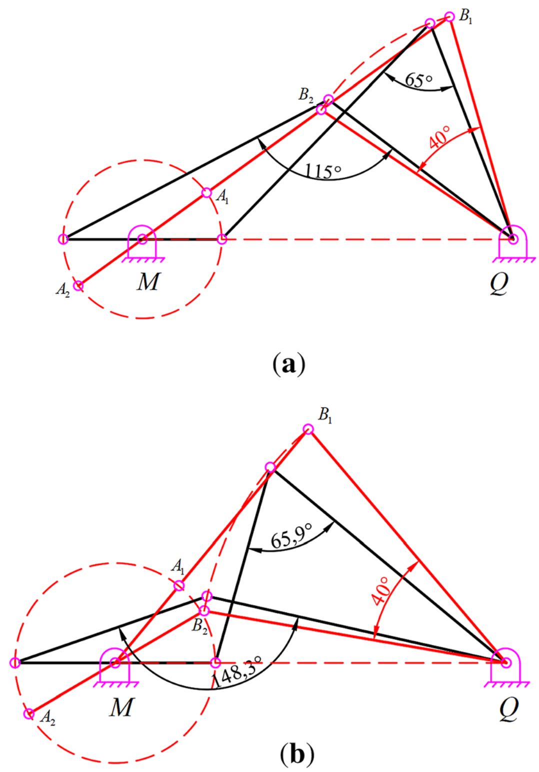

Figure 6. The crank–rocker mechanism and its limit positions obtained: (**a**) in case 8 and (**b**) in reference [21].

4.5 *Example five*

Another example is given to show the effectiveness of case 8. In this example, unlike Example 4, the transmission angle deviation from 90° is chosen as bigger and different. Here, the swing angle α, the maximum transmission angle μ_{max} and the minimum transmission angle μ_{min} are 40°, 150° and 50°, respectively. Accordingly, the maximum transmission angle deviation from 90° is 60° and the minimum transmission angle deviation from 90° is 40°. One of the four solutions obtained in different angular notations is given in table 5. Thus, the geometric representation of the mechanism is as shown in figure 7.

4.6 *Example six*

In this example, the smallest possible transmission angle deviation from 90° is chosen. The swing angle α, the maximum transmission angle μ_{max} and the minimum value μ_{min} are 40°, 112° and 68°, respectively. In this case, the maximum transmission angle deviation from 90° is 22°. If this value is given less than 22°, physically meaningful mechanism dimensions cannot be obtained from case 8 under the present conditions. For this example, one of the four solutions obtained in different angular notations but showing the same crank–rocker mechanism is given in

Table 4. Numerical results for case 8.

Sol. no.	1
β (°)	35.97
ϕ (°)	105.97
η (°)	0.00
u (unit)	0.62511
r (unit)	0.80929
q (unit)	0.21380

Table 5. Numerical results for case 8.

Sol. no.	1
β (°)	59.59
ϕ (°)	133.91
η (°)	18.57
u (unit)	0.89577
r (unit)	0.44648
q (unit)	0.30172

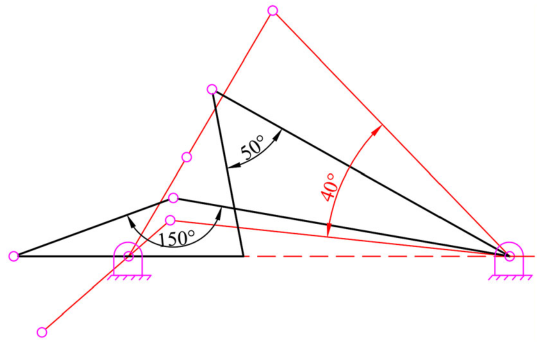

Figure 7. The crank–rocker mechanism of a maximum deviation from 90° equals 60° and a minimum deviation equals 40° with its limit positions obtained in case 8.

Table 6. Numerical results for case 8.

Sol. no.	1
β (°)	24.08
ϕ (°)	94.08
η (°)	0.00
u (unit)	0.43411
r (unit)	0.91301
q (unit)	0.14848

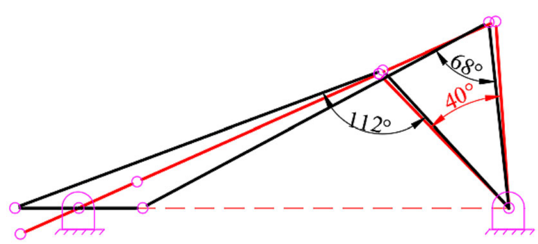

Figure 8. The crank–rocker mechanism maximum deviation from 90° equals 22° and its limit positions obtained in case 8.

table 6. The geometric representation of the mechanism is presented in figure 8.

5. Discussion and conclusion

In the afore-mentioned numerical examples it can be seen that the fourth-degree polynomial equation in the second and third cases can have up to four different real roots whereas in the end, as known, the crank–rocker mechanism with four different link lengths does not exist. For example two different link lengths are obtained for the second case, while only one mechanism exists for the third and seventh cases. A similar situation can be encountered in the eighth case. Here, after the polynomial solution at the fourth step, there are a total of 16 solution sets in addition to the other parameters; however, one crank–rocker mechanism design that is of practical value can be obtained. From table 4, it can be argued that there are additional solutions of the design equations depending on the elimination methods used during the solution process and that there are also unrealistic solutions. While, in case 8, four solutions are obtained as a maximum, all of these solutions are representations of a single crank–rocker mechanism in different angular notations. Moreover, as we have seen in Examples 4–6, the maximum and minimum transmission angle deviations from 90° can be equal. This situation is different from those in the examples made with ‘The Classical Transmission Angle Problem’ in the book of Söylemez [21].

New cases can be added to the eight different design cases analysed in this paper by pre-designating different parameters and thus reducing the number of unknown parameters from among the whole design parameters in the design equations of the crank–rocker mechanism. Eventually, as shown in this study, it is possible to achieve the design of the crank–rocker mechanism for different design cases according to the designer’s needs in practice.

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