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Direct approximation of fractional order systems as a reduced integer/fractional-order model by genetic algorithm

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Abstract. In this paper, a new method is proposed for the reduced-order model approximation of commensurate/incommensurate fractional order (FO) systems. For integer order approximation, the model order is determined via Hankel singular values of the original system; while the order of FO approximations is determined via optimization. Unknown parameters of the reduced model are obtained by minimizing a fitness function via the genetic algorithm (GA). This fitness function is the weighted sum of differences of Integral Square Error (ISE), steady-state errors, maximum overshoots, and ISE of the magnitude of the frequency response of the FO system and the reduced-order model. Therefore, both time and frequency domain characteristics of the system considered in obtaining the reduced-order model. The stability criteria of the reduced-order systems were obtained in various cases and added to the cost function as constraints. Three fractional order systems were approximated by the proposed method and their properties were compared with famous approximation methods to show the out-performance of the proposed method.

Keywords. Model order reduction; fractional order system; genetic Algorithm; constrained optimization; commensurate; incommensurate; Hankel singular value.

1. Introduction

In the recent two decades, fractional calculus has received increasing attention in the description and modeling of natural and real-world phenomena [1]. Consequently, related issues such as the fractional-order system stability analysis [2–4], fractional order system identification [5, 6], and fractional-order system approximation [7–9], have been investigated, vastly, in the literature. Also, many controllers have been designed for real-world dynamic processes through fractional-order systems [10, 11]. However, the main disadvantages of the fractional-order models and controllers are their complicated analysis, difficult implementation, and mismatching with conventional science. Therefore, several methods have been proposed for the integer-order approximation of fractional order systems and controllers such as Oustaloup's approximation algorithm [12], Matsuda's approximation algorithm [13], and many more. Although these integer order approximations may be accurate enough for system description, they are not usually effective for controller design due to their high order. That is, accurate analysis and controller design for such high order systems are very difficult and time-consuming. Also, the implementation of these high order models and controllers is very difficult and costly, and perhaps impractical.

To overcome this problem, researchers have suggested a model order reduction technique to improve efficiency. The first time, Davison [14] introduced the order reduction of the high order system in 1966. Since 1966, various model order reduction methods have been developed. The Balanced Truncation (BT) method was proposed by Moore [15] which is based on analysis of controllability and observability Gramians by computing the Lyapunov equations. This method is more popular because it is easy to use, preserves stability, yields good approximation, and has computable explicit error bounds. However, the balanced truncation method suffers from a steady-state error. In 1984, Glover introduced an optimal Hankel norm approximation method, whose advantages and disadvantages were almost similar to the balanced truncation method [16]. Model order reduction based on Krylov subspace methods is another important method in order reduction [17]. The main advantage of using a Krylov model order reduction is its significantly shorter computational time while the induced error is almost negligible or very low [18]. However, they do not necessarily preserve the stability of the model. In the last decade, evolutionary algorithms such as the Harmony Search algorithm and cuckoo search algorithm have been used for model order reduction [19–22].

A high integer order model obtained by conventional approximation methods such as Oustaloup's method or Matsuda's method from fractional-order systems can be

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reduced by different model order reduction approaches. Furthermore, some reduction methods such as Padé approximation and BT method have been used to approximate the fractional order systems directly as integer or non-integer reduced-order models [23–26]. Obtained results by these methods are accurate enough and applicable to large-scale fractional-order systems and descriptor systems, but their disadvantages are similar to the corresponding integer order model reduction methods. However, the analysis and implementation of the obtained reduced integer order model are simple and cheap.

In this paper, the fractional-order system is approximated with reduced order models directly, without using conventional integer approximation methods. Reduced-order model approximation can be an integer or non-integer and therefore, three reduced order models as integer order model, commensurate order model, and incommensurate order model could be assumed. For integer order approximation, the model order is determined via the Hankel singular values (HSV) of either the original system or the Oustaloup approximation of the fractional-order system. The order of fractional order approximations is, on the other hand, determined via optimization. Unknown parameters of the reduced model are obtained by minimizing a fitness function via a genetic algorithm (GA) [27–30]. This fitness function is a weighted summation of differences of Integral Sum of Errors of step responses, steady-state errors, maximum overshoots, and ISE of frequency responses of the fractional-order system and the reduced-order model. In this context, both the time domain and frequency domain properties of the system are considered in obtaining the reduced-order model. To preserve the stability of the reduced-order systems, the stability criteria would be obtained via the Routh-Hurwitz stability criterion for integer-order approximation [31] and some fractional stability theorems for fractional-order approximations [32–34]. These obtained criteria are added to the cost function as constraints. Therefore, the main contribution of this paper is the representation of the problem of approximation of fractional order via such an optimization problem that considers various performance indices, and its stability issues are guaranteed via some constraints. Especially, determining the proper degree of reduced fractional-order approximation via optimization is of interest. Via this approach, derivation, and reduction of the intermediate high order integer approximations are removed, which in turn, reduces the approximation errors. Finally, three fractional-order systems, two Single-Input, Single-Output (SISO) and one Multi-Input, Multi-Output (MIMO), would be approximated by the proposed method and their properties would be compared with famous approximation methods to show the out-performance of the proposed method.

The paper is organized as follows: The basics of fractional calculus is explained in section 2. The proposed method is introduced in section 3. The ability of the proposed approach is shown in section 4 through three test systems and the conclusions are in section 5.

2. Mathematical background

2.1 *The basics of fractional order calculus*

Various definition of fractional order derivative has been proposed by mathematicians. Three commonly-used definitions for fractional integral/derivative are Grünwald–Letnikov, Riemann–Liouville, and Caputo definitions [32]. The Caputo fractional derivative is more popular than the others. The Caputo fractional derivative of order q of a continuous function $f : R^+ \to R$ is defined as:

$$D_t^q f(t) = \begin{cases} \dfrac{1}{\Gamma(l-q)} \displaystyle\int_0^t \dfrac{f^{(l)}(\tau)}{(t-\tau)^{q-l+1}} d\tau & l-1 < q < l \\ \dfrac{d^l}{dt^l} f(t) & q = l \end{cases} \tag{1}$$

Where $l-1 \leq q < l, l \in N$, and l is the first integer which is not less than q and $\Gamma(.)$ is the well-known Gamma function. Laplace transform of the Caputo fractional derivative is then given by:

$$L\{D_t^q f(t)\} = s^q F(s) - \sum_{k=0}^{l-1} s^{q-1-k} f^{(k)}(0), \quad l-1 < q \leq l \tag{2}$$

A standard fractional-order model can be presented by the generalized fractional order differential equation as follows:

$$y(t) + \sum_{i=1}^{n} a_i D^{\alpha_i} y(t) = \sum_{j=0}^{m} b_j D^{\beta_j} u(t) \tag{3}$$

where, $u(t)$ and $y(t)$ are the system input and output, respectively. Also a_i, b_j are real number coefficients of system model and α_i, β_j are arbitrary positive real numbers $(\alpha_i, \beta_j \in R^+)$.

A general fractional-order system can be described by the following transfer function:

$$G_f(s) = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}} = \frac{Q(s^{\beta_j})}{P(s^{\alpha_i})} \tag{4}$$

2.2 *Stability of fractional order systems*

Consider the generalized linear time-invariant fractional-order MIMO system as [33]:

$$D_t^{\alpha}x(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \tag{5}$$

where, α represents the fractional commensurate order.

Theorem 2.1 (Matignon's stability theorem) *If the triplet (A, B and C) is minimal, system of Equation (5) is Bounded-Input Bounded-Output (BIBO) stable if and only if* $|\arg(\lambda)| > \alpha\frac{\pi}{2}$ *where* λ *is the eigenvalue of matrix A and* $\arg(\lambda) \in (-\pi, \pi]$ [23]. *In figures 1(a) and 1(b) the stability regions of fractional order systems are shown for* $0 < \alpha \le 1$ *and* $1 \le \alpha < 2$*, respectively* [32].

Theorem 2.2 *Eigenvalues of a* $n \times n$ *matrix A lie within the region Ω (Figure 1(b))* $\Omega = \{\lambda | \mathrm{Re}(\lambda)\cos\delta \pm \mathrm{Im}(\lambda) \sin\delta \le 0; 0 \le \delta < \pi/2\}$ *if and only if eigenvalues of the* $2n \times 2n$ *matrix* A^* *defined as below have negative real parts, where* $\delta = (\alpha - 1)\pi/2$ [32].

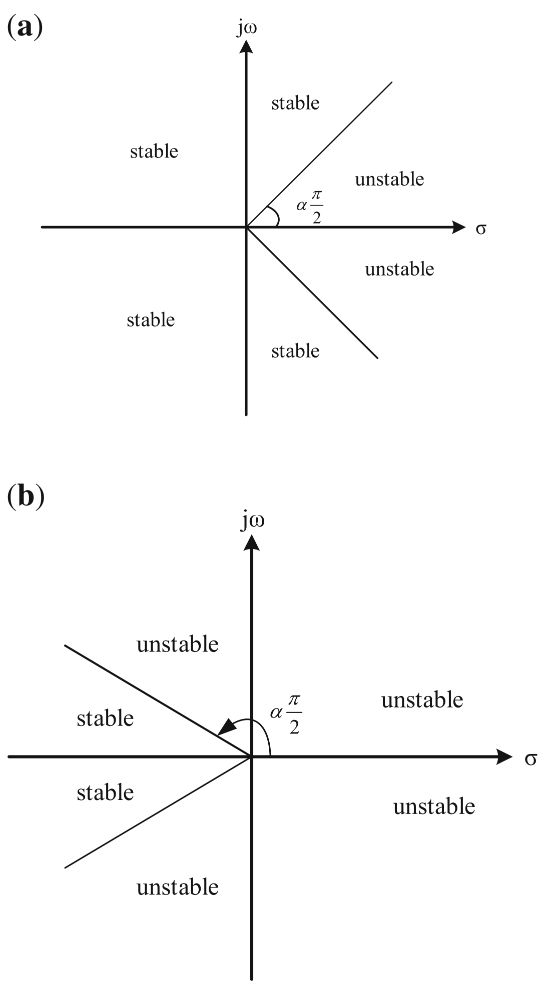

Figure 1. (**a**) Stability region of fractional order LTI system with order $0 \le \alpha \le 1$. (**b**) Stability region of fractional order LTI system with order$1 \le \alpha \le 2$.

$$A^* = \begin{bmatrix} A\cos\delta & -A\sin\delta \\ A\sin\delta & A\cos\delta \end{bmatrix} \tag{6}$$

Theorems 2.1 and 2.2 are applicable for commensurate fractional order systems.

Theorem 2.3 *Let the transfer function of an incommensurate order system can be represented in the form*

$$G_{incommensurate}(s) = \sum_{i=1}^{n}\sum_{j=1}^{v_i} \frac{A_{ij}}{(s^{\alpha_i} + \lambda_i)^j} \tag{7}$$

Where A_{ij} *are complex numbers.* λ_i *and* v_i *are roots and their multiplicity number of* $P(s^{\alpha})$*, respectively. Then, system of Equation (7) is BIBO stable if and only if* α_i *and the argument of* λ_i *denoted by* $\arg(\lambda_i)$ *satisfy the following inequalities* [34]:

$$0 < \alpha_i < 2 \text{ and } |\arg(\lambda_i)| < \pi\left(1 - \frac{\alpha_i}{2}\right) \text{ for all } i \tag{8}$$

Theorem 2.4 *Consider the general fractional order system described in Equation (4). Suppose that* $G_f(s)$ *is a strictly proper transfer function. The incommensurate system of Equation (4) can be described in commensurate form with common factor* $\alpha \in (0 < \alpha < 1)$ *and the transfer function has the following form* [35]:

$$G_{commensurate}(s) = \frac{K_0 \sum_{j=0}^{m} b_j (s^{\alpha})^j}{\sum_{i=0}^{n} a_i (s^{\alpha})^i} = \frac{Q(s^{\alpha})}{p(s^{\alpha})} \tag{9}$$

Where, $\alpha_i = \alpha i, \beta_j = \alpha j, (0 < \alpha, 1), \forall i, j \in Z$ and $n > m$.

2.3 *Stability of integer order systems*

Consider a reduced integer order system described by the following transfer function:

$$G_r(s) = \frac{C(s)}{D(s)} = \frac{c_1 s^{r-1} + c_2 s^{r-2} + \cdots + c_r}{s^r + d_1 s^{r-1} + d_2 s^{r-2} + \cdots + d_n} \tag{10}$$

By considering the first two rows of Routh–Hurwitz array and the first column of Routh–Hurwitz array, $D(s)$, the denominator of (10) can be represented as [31, 36]:

$$\begin{aligned} D(s) = {} & s^r + h_1 s^{r-1} + (h_2 + h_3 + \cdots + h_r)s^{r-2} \\ & + h_1(h_3 + h_4 + \cdots + h_r)s^{r-3} \\ & + \begin{bmatrix} h_2(h_4 + h_5 + \cdots + h_r) + \\ h_3(h_5 + h_6 + \cdots + h_r) + \\ h_4(h_6 + h_7 + \cdots + h_r) + \cdots h_{r-2}h_r \end{bmatrix} s^{r-4} \\ & + \cdots + h_{1+\eta}h_{3+\eta}\cdots h_{r-2}h_r \end{aligned} \tag{11}$$

In [36] have been proved that the first column of Routh-Hurwitz array consists of the sequence elements as:

$$1, h_1, h_2, h_1h_3, h_2h_4, h_1h_3h_5, \ldots h_{1+\eta}h_{3+\eta} \ldots h_{r-2}h_r \quad (12)$$

where, η is equal to 1 for even r and η is equal to 0 for odd r. Comparing the entries of the first row with $1, d_2, d_4, \ldots$ and those of the second row with $d_1, d_3, d_5, \ldots$ the relations defined in Equation (13) is obtained:

$$\begin{cases} d_1 = h_1 \\ d_2 = (h_2 + h_3 + \cdots + h_r) \\ d_3 = h_1(h_3 + h_4 + \cdots + h_r) \\ \vdots \\ d_r = \left(h_{1+\eta}h_{3+\eta} \cdots h_{r-2}h_r\right) \end{cases} \quad (13)$$

By substituting the above relations in the denominator of (10), Equation (11) is obtained. Therefore, the necessary and sufficient condition for all the poles of the reduced system to be strictly in the left-half plane is

$$\begin{cases} h_1 > 0 \\ h_2 > 0 \\ \vdots \\ h_r > 0 \end{cases} \quad (14)$$

and subsequently

$$\begin{cases} d_1 > 0 \\ d_2 > 0 \\ \vdots \\ d_r > 0 \end{cases} \quad (15)$$

Therefore, the reduced integer model described by (10) is stable if and only if a condition set of (15) is satisfied.

3. Proposed method

Consider a stable linear time-invariant fractional-order system described by Equation (4) numerator, denominator, and order of the fractional system are known constants. Also, consider a fixed structure transfer function as a reduced order model approximation in which their coefficients and orders are unknown. The aim is to determine the unknown parameters of reduced-order model approximation such that the important characteristic of the fractional-order system and reduced-order model approximation is almost identical. This fixed structure transfer function is given by (10) where r, c_i, and, d_i are unknown constants. Based on the desired integer or non-integer reduced-order model approximation, r can be a positive integer or a positive real number.

To obtain the reduced-order model, three main stages should be performed, as:

Stage 1: The order of the reduced-order model should be determined.
Stage 2: The unknown coefficients of the reduced-order model should be calculated.
Stage 3: The stability properties of the reduced-order model should be investigated.

To perform these steps, two cases of integer order and fractional-order reduced models are considered, separately.

3.1 *Integer order reduced model approximation*

Stage 1: As the first stage, the integer order, r, of reduced-order model approximation must be determined. For determining this integer order, two approaches have been presented in the literature. In the first approach, the Hankel singular values of the fractional-order system are calculated directly [25], and then based on strength of Hankel singular values, the order of reduced model approximation is determined. In the second approach, the Oustaloup approximation of the fractional system is determined, and then Hankel singular values of the Oustaloup approximation model is drawn. The order of reduced approximation equals to the number of large Hankel singular values [37]. As the basics of these two methods are the same, the resulting order would be the same for both approaches. However, as a rule of thumb, if the sum of ignored Hankel singular values is significant, the best integer order for the reduced-order model would be the number of large Hankel singular values plus one.
Stage 2: After determining the order, the unknown coefficients of the reduced-order model should be calculated, optimally. Here, GA is used for determining the unknown coefficients. To this end, the step response and magnitude of the frequency response of the fractional-order system are calculated as two vectors, named y_f and mag_f respectively. The unknown parameters of Equation (10) are then determined by minimizing the following fitness function:

$$J' = w_1 \int_0^{t_f} \left(y_f - y_r\right)^2 dt + w_2 \int_{\omega_1}^{\omega_2} \left(mag_f - mag_r\right)^2 d\omega + w_3\left(OS_f - OS_r\right) + w_4\left(e_{ss_f} - e_{ss_r}\right) \quad (16)$$

where, y_r is the step response of the reduced-order model approximation, t_f is the final time, *mag* is the amplitude of the frequency response, $\omega_1 \le \omega \le \omega_2$ is the desired frequency range of system, *OS* is the overshoot, e_{ss} is the steady-state error, w_1, w_2, w_3 and w_4 are weighted coefficients.

Stage 3: Since the fractional-order system is assumed stable, the reduced-order model approximation should be stable, too. Thus, the stability conditions of the integer model obtained by (15), are used.

Therefore, integer reduced-order model approximation is stable if and only if unknown parameters of Equation (10) is determined by minimizing Equation (16) subject to Equation (15). Hence, the optimization problem of Equation (16) is converted to the constrained optimization problem and consequently fitness function as follows:

$$J_{\text{integer}} = \begin{cases} J' = w_1 \int_0^{t_f} \left(y_f - y_r\right)^2 dt + w_2 \int_{\omega_1}^{\omega_2} \left(mag_f - mag_r\right)^2 d\omega + \\ w_3\left(OS_f - OS_r\right) + w_4\left(e_{ss_f} - e_{ss_r}\right) \\ \text{subject to } d_j > 0\, j = 1, 2, \ldots, r \end{cases} \tag{17}$$

By minimizing the J_{integer}, the reduced-order model approximation that is achieved, one has a reduced-order model persevering almost all important specifications of the fractional-order system. The proposed method can be summarized in the following steps:

Step 1: Determine the order of reduced model approximation by either one of the two following methods.

a) Compute the Hankel singular values of fractional order system directly and determine the order of reduced model approximation based on the largest eigenvalues.

b) Derive integer approximation of the fractional-order system by conventional methods such as the Oustaloup algorithm or Matsuda algorithm. Then, draw the Hankel singular values of integer approximation and determine the order of reduced model approximation based on the largest Hankel singular values. However, if the sum of ignored Hankel singular values is significant, the best integer order for the reduced-order model would be the number of large Hankel singular values plus one.

Step 2: Calculate the step response and magnitude of the frequency response of the fractional-order system as two independent vectors.

Step 3: Consider a desired fixed structure for reduced-order model approximation whose coefficients of numerator and denominator are unknown.

Step 4: Minimize the fitness function Equation (17) by GA to find the best parameters for reduced-order model approximation.

3.2 *Fractional order reduced model approximation*

In the second state, the order of reduced model approximation is assumed to be non-integer. Therefore, the obtained reduced-order model approximation is fractional and can be either in commensurate or incommensurate forms. Hence, the order of the reduced model and coefficients of numerator and denominator of Equation (10) are unknown and must be determined. The unknown parameters of the commensurate form are less than the incommensurate form. Also, instead of the Routh-Hurwitz criterion, theorems 2.1 through 2.3 presented in section 2.1 are used to ensure stability. For such systems, all the three stages for reduced-order model approximation would be done via an optimization procedure. For this purpose, the fitness functions of commensurate and incommensurate forms of reduced-order model approximation are considered as:

$$J_{\text{commensurate}} = \begin{cases} J' = w_1 \int_0^{t_f} \left(y_f - y_r\right)^2 dt + w_2 \int_{\omega_1}^{\omega_2} \left(mag_f - mag_r\right)^2 d\omega + \\ w_3\left(OS_f - OS_r\right) + w_4\left(e_{ss_f} - e_{ss_r}\right) \\ \text{subject to} \\ \{\operatorname{Re}(\lambda)\cos\delta \pm \operatorname{Im}(\lambda)\sin\delta \le 0\}, 0 \le \delta < \pi/2 \end{cases} \tag{18}$$

$$J_{\text{incommensurate}} = \begin{cases} J' = w_1 \int_0^{t_f} \left(y_f - y_r\right)^2 dt + w_2 \int_{\omega_1}^{\omega_2} \left(mag_f - mag_r\right)^2 d\omega + \\ w_3\left(OS_f - OS_r\right) + w_4\left(e_{ss_f} - e_{ss_r}\right) \\ \text{subject to} \\ |\arg(\lambda_i)| < \pi\left(1 - \frac{\alpha_i}{2}\right) \text{ for all } i, 0 < \alpha_i < 2 \end{cases} \tag{19}$$

By minimizing the fitness functions Equation (18) and Equation (19), unknown parameters of the commensurate and incommensurate form of reduced-order model approximation are obtained. The proposed method can be implemented via the following steps:

Step 1: Calculate the step response and magnitude of the frequency response of the fractional-order system as two independent vectors.

Step 2: Consider a desired fixed structure for either commensurate or incommensurate forms of reduced-order model approximation whose coefficients of and order of fixed structure are unknown.

Step 3: Depending on commensurate and incommensurate forms, minimize the proper fitness function by GA to find the best parameters for reduced-order model approximation. The employed GA procedure has been shown in figure 2, which can be explained as:

(i) An initial population matrix is created. Because search space is continuous, real-valued coded is most suitable. Also, the length of the chromosome is shorter than in binary coded. Therefore, a double vector type population is considered. The size of the initial population matrix is $V \times W$ in which, V and W are population size and number of unknown variables, respectively. The matrix of the initial

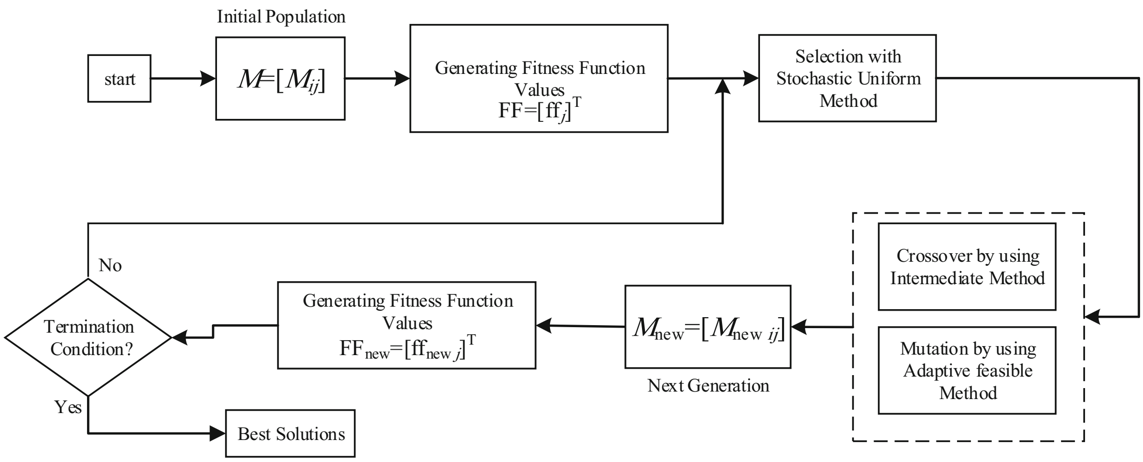

Figure 2. Procedure of employed genetic algorithm.

population is presented as follows:

$$M = \left[M_{i,j}\right] = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,W} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,W} \\ \vdots & \vdots & \ddots & \vdots \\ M_{V,1} & M_{V,2} & \cdots & M_{V,W} \end{bmatrix} \quad (20)$$

which $M_{i,j}$ is real values of reduced-order model parameters.

(ii) Next, the individuals of an initial population are evaluated with the fitness function of Equations (17) to (19).

(iii) The stochastic uniform method is applied for selecting the suitable individuals for the mating pool. In this method, the individuals are mapped to a line in which each parent corresponds to a section of length proportional to its expectation.

(iv) In the next step, new offspring for a new generation is produced by recombining the genes. For this purpose, the intermediate method is used. In this method, offspring are created by a random weighted average of the parents. The ratio parameter which controls the crossover has been considered 0.8. The intermediate method recombines the parents by using the following equation:

$$O_i = RM_i + (1 - R)E_i \quad (21)$$

where R, is ratio parameter, and M_i and E_i are parents.

(v) For covering the search space in the genetic algorithm, the mutation operator is used. The mutation operator changes the genes to prevent the failing GA into a local minimum. The mutation function is considered by the adaptive feasible function which is adaptive based on the last successful or unsuccessful generation.

(vi) Phases ii) to v) are repeated until termination criteria are met.

The above procedure is illustrated in figure 2.

Remark 1 According to the theorem 2.4, instead of using the fitness function of (18) for determining the commensurate reduced model, the incommensurate reduced model obtained by (19) can be converted to a commensurate model. Therefore, a unified cost function could be employed for both commensurate and incommensurate reduced models, as well.

4. Simulation results

In this section, three fractional-order test systems, including two SISO and one MIMO systems, are approximated by the proposed method to present the capabilities and advantages of the proposed method. In the proposed fitness function, ISE of step responses and magnitude of frequency responses are more significant compared to overshoot and steady-state error.

Therefore, they have relatively larger weighting coefficients, correspondingly. Hence, for all simulations, it is assumed that: $w_1 = 0.4, w_2 = 0.3, w_3 = w_4 = 0.15$. The desired frequency range has been also considered as $10^{-1} \leq \omega \leq 10^3$, for these examples. A step by step procedure is given for fractional-order test system 1 as:

Test system 1: consider the fractional-order system presented in [38] as:

$$\begin{bmatrix} D^{0.8}x_1(t) \\ D^{0.8}x_2(t) \\ D^{0.8}x_3(t) \\ D^{0.8}x_4(t) \\ D^{0.8}x_5(t) \\ D^{0.8}x_6(t) \end{bmatrix} = \begin{bmatrix} -6 & -6 & -4.4688 & -7.3047 & -6.1719 & -3.4688 \\ 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(t)$$

$$y(t) = [0.5 \quad 0.5625 \quad 0.2422 \quad 0.2266 \quad 0.1172 \quad 0.0313]x(t) \tag{22}$$

The transfer function of the above system is given by

$$G_{commensurate} = \frac{s^4 + 9s^{3.2} + 31.0016s^{2.4} + 58.0096s^{1.6} + 60.0064s^{0.8} + 16.0256}{s^{4.8} + 6s^4 + 48s^{3.2} + 286.0032s^{2.4} + 935.0016s^{1.6} + 1580.0064s^{0.8} + 888.0128} \tag{23}$$

The integer order reduced model approximation of test system 1 can be achieved using the proposed method by the following steps:

Step 1: The order of reduced model approximation is determined. For determining the integer-order two approaches are used. In the first approach, Hankel singular values of Equation (22) are calculated as $\{2.4 \times 10^{-1}, 6.68 \times 10^{-2}, 9.98 \times 10^{-5}, .4.99 \times 10^{-7}, 1.25 \times 10^{-10}, 1.71 \times 10^{-14}\}$ [25]. In the second approach, integer approximation of test system 1 is obtained by the Oustaloup algorithm and then Hankel singular values of integer approximation are drawn. Figure 3 shows the Hankel singular value chart of the integer approximation model. Due to the Hankel singular values of fractional-order system or Figure 3, the order of reduced model approximation can be 2, but, since the sum of ignored Hankel singular values is significant, it would be better to consider it 3.

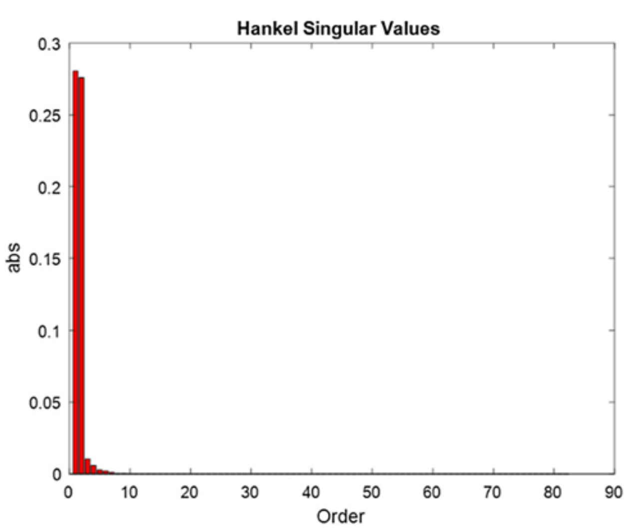

Figure 3. Hankel singular values of the Oustaloup integer approximation.

Step 2: The step response and magnitude of the frequency response of the fractional-order system Equation (22) are calculated and is noted by y_f and mag_f, respectively.
Step 3: According to the determined reduced model order, a fixed structure for reduced-order model approximation is considered as follows:

$$G_r = \frac{c_1 s^2 + c_2 s + c_3}{s^3 + d_1 s^2 + d_2 s + d_3} \tag{24}$$

where c_i and $d_i$are unknown parameters.
Step 4: The fitness function of Equation (17) is minimized for determining the numerator and denominator coefficients of Equation (24) via GA. The GA parameters used for this test system has been presented in Table 1. For selecting GA parameters, the following points are considered: Population size is effective on convergence speed and quality of optimization results. Too small population size degrades the obtained results, and too large population size too large makes the speed of convergence too slow. Therefore, a trade-off for selecting the population size is necessary. In this paper, population size is considered based on the number of unknown parameters as 50 and was encoded as real values. For producing the next generation, the *stochastic uniform* method is chosen as the selection function. For choosing the crossover *Intermediate* method is used in the GA solver of the Optimization Toolbox in MATLAB software. Intermediate creates children by a random weighted average of the parents. Intermediate crossover is controlled by a single parameter ratio. This parameter was considered as 0.8. With this choice, the offspring produced are within the hypercube defined by the parents' locations at opposite vertices. The mutation rate has been also selected by *adaptive feasible* function in optimization toolbox. By this method, the mutation rate is adaptive concerning the last successful or unsuccessful generation. Finally, the integer reduced-order model approximation of fractional order system Equation (22) is obtained as follows:

$$G_r(s) = \frac{2.713s^2 + 94.16s + 71.99}{s^3 + 46s^2 + 242.4s + 3810} \tag{25}$$

Table 1. GA parameters for test system1.

Parameter	Value
Population size	50
Selection function	Stochastic uniform
Crossover function	Intermediate Ratio = 0.8
Migration	0.2
Mutation function	Adaptive feasible

If the order of reduced model approximation is considered non-integer and the proposed procedure is performed, the commensurate and incommensurate forms of reduced fractional-order system are achieved as follows:

$$Gr_{commensurate}(s) = \frac{0.99609s^{0.8} + 0.71494}{s^{1.6} - 2.0946s^{0.8} + 37.4175} \quad (26)$$

$$Gr_{incommensurate}(s) = \frac{s^{0.93409} + 0.84153}{s^{1.7506} - 0.29524s^{1.5898} + 41.4194} \quad (27)$$

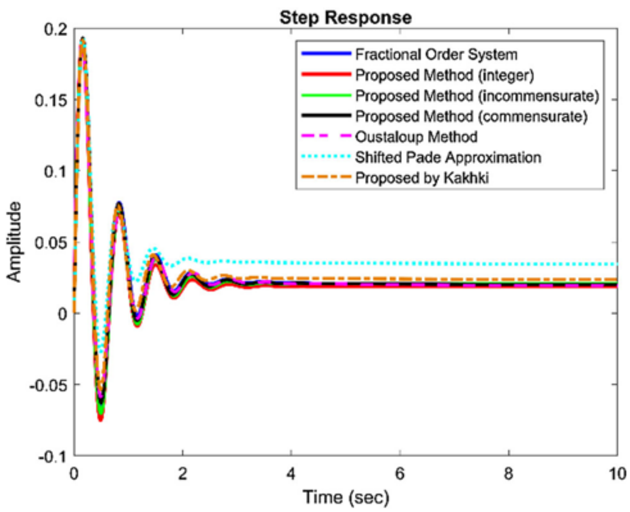

Figure 4. Comparison of step response of fractional order system and famous reduction methods by the proposed reduced order models approximation.

In figures 4 and 5, the step and frequency response of the obtained reduced models are compared with step and frequency responses of famous model order reduction techniques such as shifted Padé approximation [25] and the proposed method by Kakhki [25], respectively. The achieved results show the ability and efficiency of the proposed method.

Furthermore, for the quantitative comparison between the fractional-order system of Equation (22) and the obtained reduced-order models, some important specifications of the responses such as steady-state value, maximum overshoot, undershoot, and settling time are compared with those of achieved from some famous model reduction methods. The comparative results are shown in table 2. Also, the Integral Square Error (ISE) index is considered as an appropriate quantity for evaluating the approximation error. In computing ISE, the absolute error is considered as the difference between the step responses of the fractional-order system and that of reduced-order ones. According to table 2, all the integer/non-integer reduced-order models, obtained via the proposed method, match properly with the fractional-order system of Equation (22). For investigating

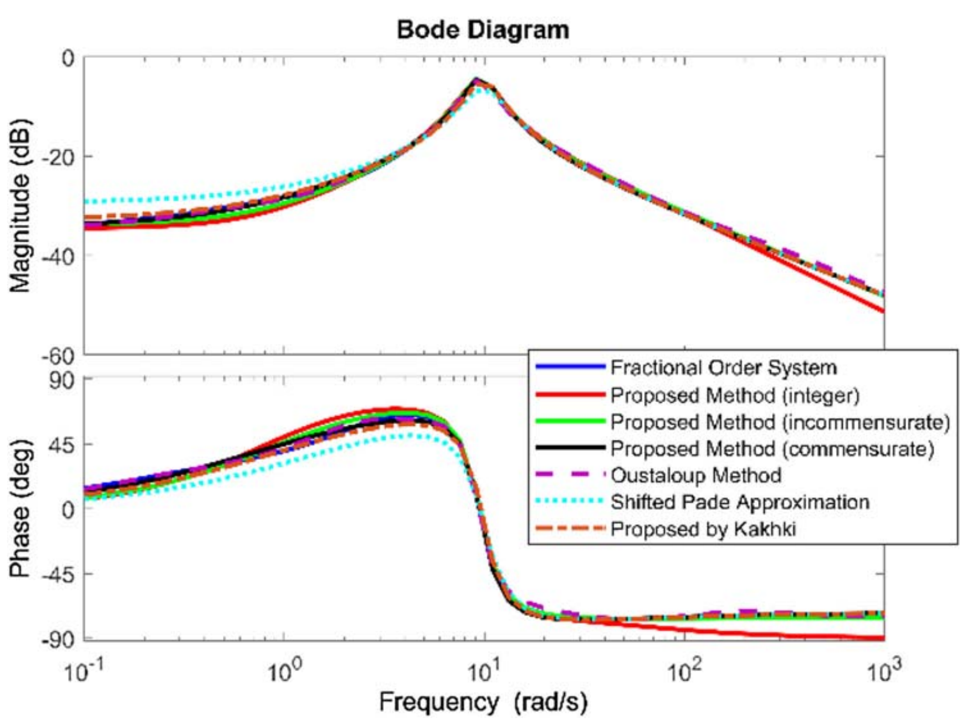

Figure 5. Comparison of frequency response of fractional order system and famous reduction methods by the proposed reduced order models approximation.

Table 2. Comparison of methods for test system 1.

	Order	Steady state	Overshoot (%)	Undershoot (%)	Rise time (sec)	Settling time (sec)	ISE
Fractional order system	4.8	0.0197	868.0697	333.5899	0.0083	2.9708	–
Proposed method (incommensurate)	1.7506	0.0205	831.4935	341.4131	0.0097	2.2941	4.1952e−05
Proposed method (commensurate)	1.6	0.0202	849.3519	309.2693	0.0086	2.3167	1.7191e−05
Proposed method (integer)	3	0.0189	915.1806	398.64	0.0058	2.2638	1.6409e−04
Oustaloup method	82	0.0194	877.8327	301.0735	0.0051	3.0169	5.0337e−05
Proposed by Kakhki	1.6	0.0238	702.8173	223.9387	0.0099	2.2593	1.6339e−04
Shifted Pade approximation	1.6	0.0347	453.1831	77.4061	0.0147	2.2176	0.0019

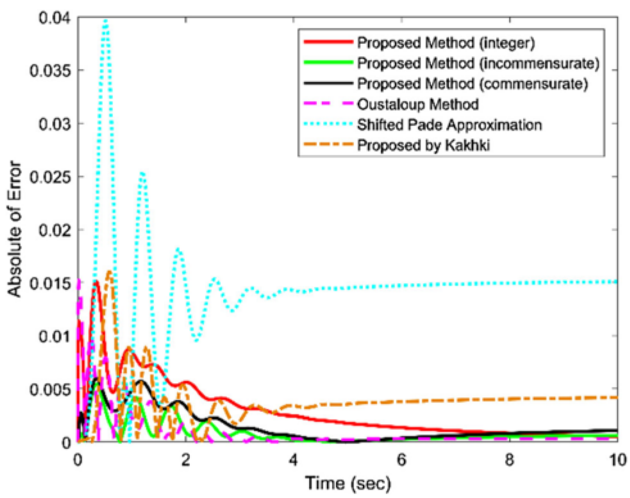

Figure 6. Time evolution of absolute error of various methods for test system 1.

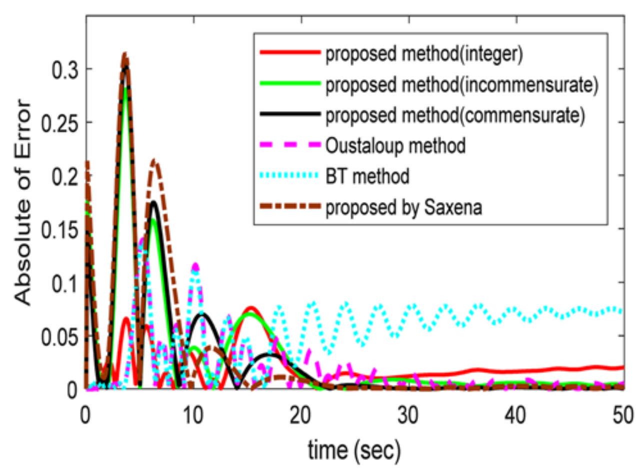

Figure 8. Absolute of error vs time for test system 2 and their approximations.

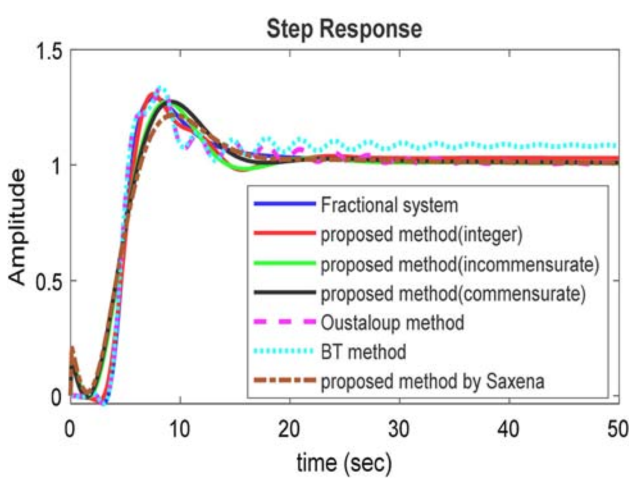

Figure 7. Step response of non-minimum fractional order system and their approximations.

the effectiveness of the proposed method, time evolutions of absolute errors of various methods have been shown in figure 6. According to this figure, it can be seen that the achieved error by the proposed method is far less than other reduction methods. It can be however seen that the proposed commensurate form performs superior to other methods in the steady-state value, overshoot, and ISE. The proposed integer reduced-order model has the best undershoot while rise time and settling time are better in the incommensurate model.

In general, the obtained models by the proposed method are more accurate than the other methods in the literature such as shifted Pade approximation [25] and the proposed method by Kakhki [25].

Test system 2: Let us consider the non-minimum phase fractional-order system introduced by Saxena in [39] as follows:

Table 3. The main specifications of test system 2 and their approximations.

	Order	Steady state	Overshoot (%)	Undershoot (%)	Rise time (sec)	Settling time (sec)	ISE
Fractional order system	12	1.0100	27.9595	2.9689	1.55	20.86	–
Proposed method (incommensurate)	2.39	1.0065	26.6292	0.6095	5.61	23.43	0.1774
Proposed method (commensurate)	2.4	1.0100	26.2437	0.5823	5.66	15.52	0.2000
Proposed method (integer)	6	1.0303	26.8283	2.2058	1.93	18.06	0.0342
Oustaloup method	95	1.0051	33.2752	3.4541	1.31	33.44	0.0546
Proposed by Saxena	2.4	1.0088	20.6641	0	5.81	17.97	0.2473
BT method	6	1.0818	23.4674	3.2029	1.41	24.21	0.1922

$$G_{commensurate} = \frac{-4000s^{4.8} - 26000s^{3.6} + 240000s^{2.4} - 690000s^{1.2} + 750000}{s^{12} + 75s^{10.8} + 2193s^{9.6} + 31914s^{8.4} + 251620s^{7.2} + 1167000s^{6} + 3357000s^{4.8} + 6032000s^{3.6} + 6433000s^{2.4} + 3563000s^{1.2} + 750000} \tag{28}$$

Following the same procedure as test system 1, the integer and non-integer reduced-order model approximations are achieved as:

$$G_r = \frac{0.02097s^5 - 0.06654s^4 - 0.01475s^3 + 0.5033s^2 - 1.72s + 1.606}{s^6 + 1.845s^5 + 7.956s^4 + 7.294s^3 + 10.1s^2 + 4.031s + 1.558} \tag{29}$$

$$Gr_{incommensurate} = \frac{0.18272s^{2.3993} - 0.18152s^{1.0344} + 0.18222}{s^{2.3979} + 0.63477s^{1.1762} + 0.18222} \tag{30}$$

$$Gr_{commensurate} = \frac{0.16759s^{2.4} - 0.16757s^{1.2} + 0.16675}{s^{2.4} + 0.67033s^{1.2} + 0.16649} \tag{31}$$

The step responses of the non-minimum phase fractional-order system of Equation (28), Oustaloup approximation model, proposed reduced order model approximations, and the BT method have been plotted in figure 7. Also, the time evolutions of absolute errors have been presented for test system 2 and its approximations in figure 8. Furthermore, the main properties of test system 2 and its approximations such as the proposed models, BT method, Oustaloup method, and proposed model by Saxena as well as ISE values of these methods have been compared in table 3. Based on the results, it can be pointed out that the proposed reduced-order model approximations are closer to the test system 2 in comparison with the other methods. The ISE index and overshoot of the proposed integer reduced order model are better than other methods; where, the steady-state value of the commensurate form is the best.

Test system 3: Consider the fractional-order linear time-invariant MIMO system provided by Khanra in [40], given by Equation (32). According to the description given in the test system 1, and by applying the proposed method to each of the matrix elements, the reduced-order model approximations are reached as Equations (33-35) as:

$$G_{incommensurate}(s) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$
$$g_{11} = \frac{5s^{1.65} + 20}{s^{3.8} + 10s^{2.93} + 20s^{1.65} + 4}$$
$$g_{12} = \frac{0.8s^{2.15} + 3s^{1.65} + 8s^{1.28} + 27.2}{s^{3.8} + 10s^{2.93} + 20s^{1.65} + 4}$$
$$g_{21} = \frac{10s^{2.93} + 5s^{1.65} + 5}{s^{3.8} + 10s^{2.93} + 20s^{1.65} + 4}$$
$$g_{22} = \frac{0.5s^{3.8} + 11s^{2.93} + 0.2s^{2.15} + 13s^{1.65} + 0.4s^{1.28} + 8.2}{s^{3.8} + 10s^{2.93} + 20s^{1.65} + 4} \tag{32}$$

And so, the reduced integer order model would be:

Table 4. Comparison the main characteristics of test system 3.

	Order	Steady state	Overshoot (%)	Rise time (sec)	Settling time (sec)	ISE
Fractional order system	$\begin{bmatrix} 3.8 & 3.8 \\ 3.8 & 3.8 \end{bmatrix}$	$\begin{bmatrix} 5.0029 & 6.8026 \\ 1.2506 & 2.0508 \end{bmatrix}$	$\begin{bmatrix} 57.47 & 56.16 \\ 51.02 & 41.60 \end{bmatrix}$	$\begin{bmatrix} 2.75 & 2.73 \\ 4.06 & 3.81 \end{bmatrix}$	$\begin{bmatrix} 40.76 & 40.33 \\ 34.70 & 34.09 \end{bmatrix}$	–
Proposed method (incommensurate)	$\begin{bmatrix} 1.812 & 1.899 \\ 1.824 & 1.883 \end{bmatrix}$	$\begin{bmatrix} 4.9967 & 6.7785 \\ 1.2516 & 2.0602 \end{bmatrix}$	$\begin{bmatrix} 55.38 & 55.98 \\ 49.12 & 41.18 \end{bmatrix}$	$\begin{bmatrix} 2.80 & 2.74 \\ 2.96 & 2.98 \end{bmatrix}$	$\begin{bmatrix} 42.70 & 41.96 \\ 35.54 & 34.81 \end{bmatrix}$	$\begin{bmatrix} 0.2010 & 0.0613 \\ 0.4146 & 0.7072 \end{bmatrix}$
Proposed method (commensurate)	$\begin{bmatrix} 1.8 & 1.8 \\ 1.8 & 1.8 \end{bmatrix}$	$\begin{bmatrix} 5.0241 & 6.8254 \\ 1.2521 & 2.0465 \end{bmatrix}$	$\begin{bmatrix} 53.94 & 53.69 \\ 49.33 & 42.58 \end{bmatrix}$	$\begin{bmatrix} 2.82 & 2.79 \\ 2.95 & 2.81 \end{bmatrix}$	$\begin{bmatrix} 41.98 & 41.36 \\ 35.64 & 29.57 \end{bmatrix}$	$\begin{bmatrix} 0.2184 & 0.1831 \\ 0.4169 & 0.7037 \end{bmatrix}$
Proposed method (integer)	$\begin{bmatrix} 4 & 4 \\ 3 & 4 \end{bmatrix}$	$\begin{bmatrix} 5.0057 & 6.8125 \\ 1.2468 & 2.0478 \end{bmatrix}$	$\begin{bmatrix} 57.52 & 55.78 \\ 52.26 & 42.28 \end{bmatrix}$	$\begin{bmatrix} 2.84 & 2.77 \\ 4.23 & 3.91 \end{bmatrix}$	$\begin{bmatrix} 39.85 & 43.15 \\ 26.95 & 26.76 \end{bmatrix}$	$\begin{bmatrix} 0.0607 & 1.5536 \\ 0.0567 & 0.0449 \end{bmatrix}$
Oustaloup method	$\begin{bmatrix} 20 & 28 \\ 18 & 60 \end{bmatrix}$	$\begin{bmatrix} 5.0046 & 6.8096 \\ 1.2519 & 2.0501 \end{bmatrix}$	$\begin{bmatrix} 58.39 & 54.48 \\ 49.11 & 41.93 \end{bmatrix}$	$\begin{bmatrix} 2.63 & 2.63 \\ 3.89 & 3.74 \end{bmatrix}$	$\begin{bmatrix} 36.7 & 43.21 \\ 42.62 & 41.02 \end{bmatrix}$	$\begin{bmatrix} 3.5317 & 1.3885 \\ 0.0397 & 0.0856 \end{bmatrix}$
BT method	$\begin{bmatrix} 4 & 4 \\ 3 & 4 \end{bmatrix}$	$\begin{bmatrix} 5.0449 & 6.6286 \\ 0.9786 & 1.9356 \end{bmatrix}$	$\begin{bmatrix} 57.24 & 58.66 \\ 92.77 & 50.40 \end{bmatrix}$	$\begin{bmatrix} 2.64 & 2.61 \\ 3.54 & 3.64 \end{bmatrix}$	$\begin{bmatrix} 31.51 & 36.59 \\ 28.23 & 33.90 \end{bmatrix}$	$\begin{bmatrix} 3.5700 & 4.0705 \\ 5.0858 & 1.0935 \end{bmatrix}$

$$
\begin{aligned}
G_r(s) &= \begin{bmatrix} g_{r_{11}} & g_{r_{12}} \\ g_{r_{21}} & g_{r_{22}} \end{bmatrix} \\
g_{r_{11}} &= \frac{0.1556s^3 + 0.884s^2 + 1.49s + 0.2314}{s^4 + 1.699s^3 + 0.7517s^2 + 0.3091s + 0.04621} \\
g_{r_{12}} &= \frac{0.1588s^2 + 2.208s + 1.724}{s^3 + 1.804s^2 + 0.4234s + 0.2528} \\
g_{r_{21}} &= \frac{14.58s^3 - 1.647s^2 + 6.332s + 1.747}{s^4 + 17s^3 + 12.64s^2 + 5.235s + 1.401} \\
g_{r_{22}} &= \frac{25s^3 + 4.421s^2 + 13.4s + 3.328}{s^4 + 24.99s^3 + 15.64s^2 + 6.761s + 1.625}
\end{aligned}
\tag{33}
$$

Also, the reduced commensurate and incommensurate fractional-order models are as follows:

$$
\begin{aligned}
Gr_{\text{commensurate}}(s) &= \begin{bmatrix} gr_{comm11} & gr_{comm12} \\ gr_{comm21} & gr_{comm22} \end{bmatrix} \\
gr_{comm11} &= \frac{0.00025008s^{0.9} + 0.95792}{s^{1.8} + 0.050226s^{0.9} + 0.19053} \\
gr_{comm12} &= \frac{0.12712s^{0.9} + 1.3196}{s^{1.8} + 0.054267s^{0.9} + 0.19324} \\
gr_{comm21} &= \frac{0.00039085s^{0.9} + 0.23093}{s^{1.8} + 0.069904s^{0.9} + 0.18435} \\
gr_{comm22} &= \frac{0.18063s^{0.9} + 0.37263}{s^{1.8} + 0.1195s^{0.9} + 0.18208}
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
Gr_{\text{incommensurate}} &= \begin{bmatrix} gr_{inco11} & gr_{inco12} \\ gr_{inco21} & gr_{inco22} \end{bmatrix} \\
gr_{inco11} &= \frac{0.20735s^{0.2} + 0.76011}{s^{1.81} + 0.054584s^{0.48} + 0.16323} \\
gr_{inco12} &= \frac{0.36096s^{0.51} + 1.1082}{s^{1.89} + 0.11757s^{0.83} + 0.166} \\
gr_{inco21} &= \frac{0.02s^{0.36} + 0.20887}{s^{1.82} + 0.073992s^{0.73} + 0.16826} \\
gr_{inco22} &= \frac{0.22173s^{0.75} + 0.28599}{s^{1.88} + 0.13265s^{0.7} + 0.13785}
\end{aligned}
\tag{35}
$$

The main properties of the fractional-order MIMO system and their approximations have been compared in Table 4. Also, in figures 9 and 10, step responses and time evolution of absolute errors for the original system and its approximations have been illustrated, respectively.

According to figures 9, 10, and table 4, it can be seen that proposed model approximations perform noticeably

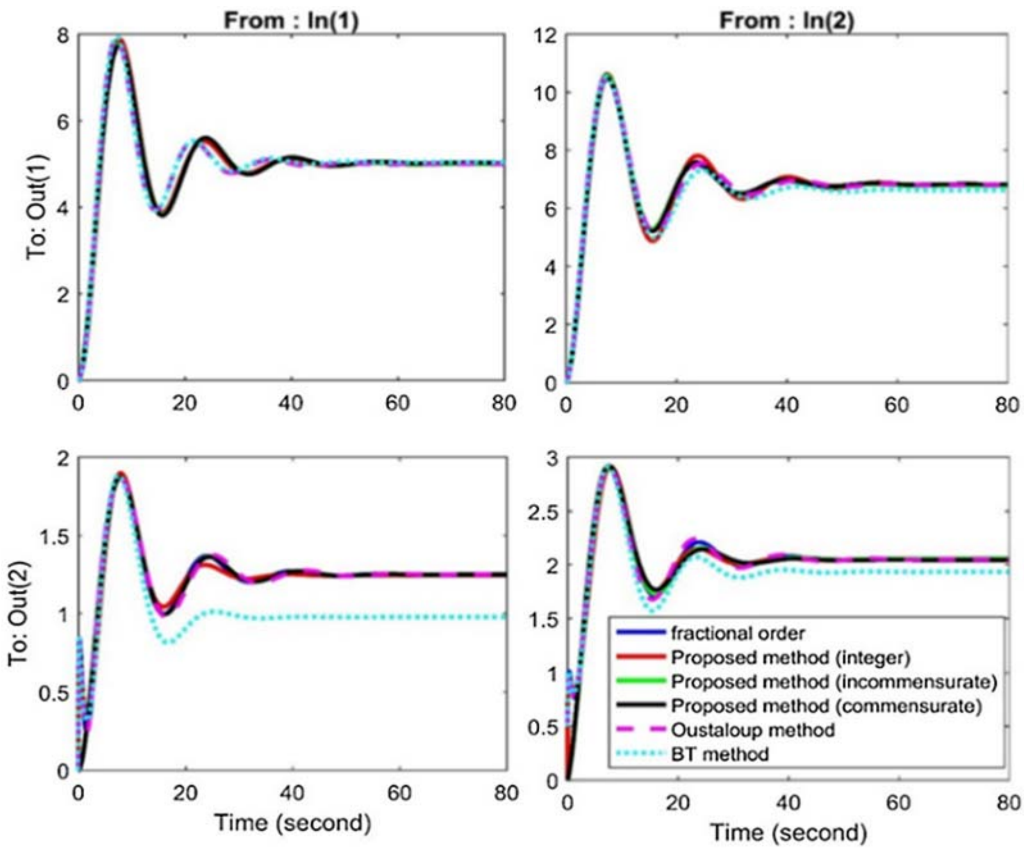

Figure 9. Step response of MIMO fractional order system and their approximations.

better than other methods for the case of the fractional-order MIMO system, too.

Remark 2 Because GA is a kind of population-based random search algorithm, it is expected that the obtained coefficients be slightly different over multiple runs. Therefore, to examine if these variations are in acceptable ranges, in table 5, the mean and standard deviations of reduced-order model parameters for the results of 21 independent runs have been brought.

It can be seen that the standard deviations of obtained parameters are relatively small (less than 7%) and so the results are reliable.

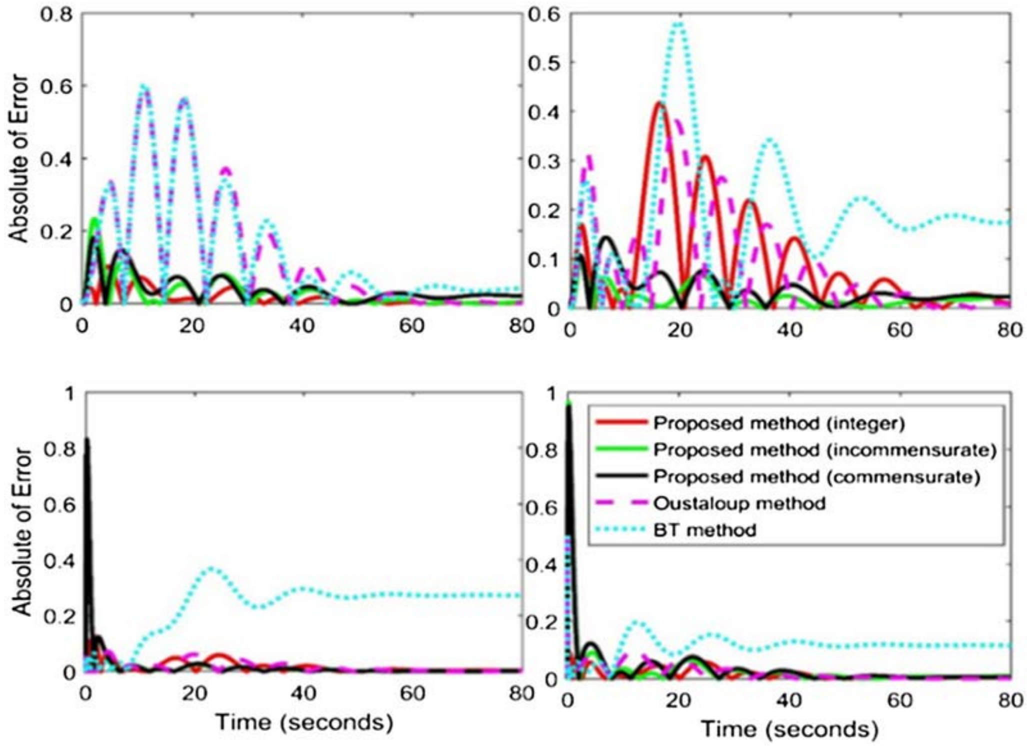

Figure 10. Absolute of error plot for test system 3.

Table 5. Mean and standard deviation of obtained result for 21 independent runs in test system 1.

		c_1	c_2	c_3	d_1	d_2	d_3	n_1	n_2	n_3
Integer model	Mean	2.71	94.52	71.22	46.05	243.86	3810.51	–	–	–
	standard deviation	0.0089	0.3436	1.1052	0.1821	1.0395	14.5265	–	–	–
Commensurate model	Mean	1.001	0.701	–	− 2.055	37.404	–	–	–	–
	Standard deviation	0.002	0.052	–	0.019	0.126	–	–	–	–
Incommensurate model	Mean	0.8475	–	–	− 0.276	42.893	–	0.944	1.7532	1.5515
	Standard deviation	0.0142	–	–	0.0135	1.067	–	0.0101	0.0175	0.1237

5. Conclusions

In this paper, a new method for reduced-order approximation of fractional-order system was presented. Reduced-order model approximation can be an integer or non-integer order. For determining the order of the integer reduced-order model, the Hankel singular values of the original system are used. For commensurate and incommensurate reduced-order approximations, the order was considered as an unknown parameter as well as the reduced-order model coefficients. A fitness function including the weighted summation of ISE index of the step response and the frequency responses, the absolute value of the difference of magnitude of frequency responses, differences of steady-state errors as well as maximum overshoots, was employed for optimally determining the unknown parameters/coefficients via GA. In this context, both the time domain and frequency domain properties of the system were considered in obtaining the reduced-order model. Depending on integer/non-integer reduced-order approximations, some stability conditions were added to the fitness function for ensuring stability. Accordingly, the optimization problems were converted to constrained optimization problems. Genetic Algorithm was applied to minimize the constrained fitness functions. Three fractional-order test systems were reduced by the proposed method. The results are representing that the proposed reduced order model approximations perform better than methods such as BT, shifted Pade approximation [25], and the proposed method by Kakhki [25]. That is the ISE index and overshoot of the proposed integer reduced-order model are better than other methods, the proposed commensurate form performs superior in the steady-state value, while the incommensurate is better in the rise time and settling time.

List of symbols

Variables

A, B, C, D	State space model matrices of system
a_i, b_j	Coefficients of fractional order Transfer Function (TF)
c_i, d_i	Unknown coefficients of reduced TF
e_{ss}	Steady state error
E_i, M_i	Parent chromosomes
K_0	Gain of fractional commensurate TF
M	Number of terms in numerator of fractional TF
M	Matrix of initial population of GA
N	Number of terms in denominator of fractional TF
O_i	Recombined parents
Q	Order of fractional derivative
R	Ratio parameter in GA
R	Order of reduced model
t_f	Final time
U	Input vector
V	Population size in GA
w_1, w_2, w_3, w_4	Weighting coefficients
W	Number of unknown parameters in GA
X	State vector
Y	Output vector
y_f	Step response of fractional order system
y_r	Step response of reduced order model

Greek letters

α	Commensurate fractional order
α_i, β_j	Orders of fractional TF
Γ	Gamma function
λ	Eigenvalues of system
v_i	Positive integer number
ω	Frequency range

Operations and functions

D_t^q	Fractional derivative operator
f	Continuous function
$G_{commensurate}$	Commensurate fractional order TF
G_f	General fractional order TF
$G_{incommensurate}$	Incommensurate fractional TF
G_r	Reduced integer order TF
$Gr_{commensurate}$	Reduced fractional order commensurate TF
$Gr_{incommensurate}$	Reduced fractional order incommensurate TF
J_{integer}	Constrained fitness function for reduced integer model
$J_{\text{commensurate}}$	Constrained fitness function for reduced commensurate fractional model
$J_{\text{incommensurate}}$	Constrained fitness function for reduced incommensurate fractional model
J'	Unconstrained fitness function
L	Laplace operator
mag	Magnitude of frequency response
N	The set of natural number
OS	Overshoot
R^+	Positive real numbers
Z	The set of integer numbers

References

[1] Oldham K B and Spanier J 1974 *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*. Academic Press, New York

[2] Trigeassou J C, Maamri N, Sabatier J and Oustaloup A 2011 A Lyapunov approach to the stability of fractional differential equations. *Signal Process*. 91(3): 437–445

[3] [3] Alagoz B B 2017 Hurwitz stability analysis of fractional order LTI systems according to principal characteristics equations. *ISA Trans*. 70: 7–15

[4] Baleanu D, Wu G C, Bai Y R and Chen F L 2017 Stability analysis of Caputo–like discrete fractional systems. *Commun. Nonlinear Sci. Numer. Simul*. 48: 520–530

[5] Gabano J D and Poinot T 2011 Fractional modelling and identification of thermal systems. *Signal Process*. 91(3): 531–541

[6] Tang Y, Liu H, Wang W, Lian Q and Guan X 2015 Parameter identification of fractional order systems using block pulse functions. *Signal Process*. 127: 272–281

[7] Mansouri R, Bettayeb M and Djennoune S 2011 Comparison between two approximation methods of state space fractional systems. *Signal Process*. 91: 461–469

[8] Wei Y, Tse P W, Du B and Wang Y 2016 An innovative fixed-pole numerical approximation for fractional order systems. *ISA Trans*. 62: 94–102

[9] -Aoun M, Malti R, Levron F and Oustaloup A 2007 Synthesis of fractional Laguerre basis for system approximation. *Automatica*. 43: 1640–1648

[10] Tepljakov A 2017 *Fractional-order modelling and control of dynamic systems*, Springer International Publishing, Switzerland

[11] Monje C A, Chen Y Q, Vinagre B M, Xue D and Feliu V 2010 *Fractional-order systems and controls*. Springer-Verlag, London

[12] Oustaloup A, Levron F, Mathieu B and Nanot F M 2000 Frequency-band complex non integer differentiator: characterization and synthesis. *IEEE T. Circuits Syst. I*. 47(1): 25–39

[13] Matsuda K and Fuji H 1993 H_∞ optimized wave-absorbing control: Analytical and experimental results. *J. Guid Control Dynam*. 16(6): 1146–1153

[14] Davison E J 1966 A method for simplifying linear dynamic systems. *IEEE Trans. Automat. Contr*. 11: 93–101

[15] Moore B C 1981 Principal component analysis in linear systems: controllability, observability and model reduction. *IEEE Trans. Automat. Contr*. 26: 17–32

[16] Glover K 1984 All optimal Hankel norm approximation of linear multivariable systems and their L_∞ error bounds. *Int. J. Control*. 39: 1115–1193

[17] Baur U, Benner P and Feng L 2014 Model order reduction for linear and nonlinear systems: A system-theoretic perspective. *Arch. Computat. Methods Eng.* 21:331–358

[18] Freund R W 2003 Model reduction methods based on Krylov subspaces. *Acta Numerica*. 12: 267–319

[19] Nasiri Soloklo H and Farsangi M M 2013 Chebyshev rational functions approximation for model order reduction using harmony search. *Sci. Iran*. 20(3): 771–777

[20] Nasiri Soloklo H, Hajmohammadi R and Farsangi M M 2015 Model order reduction based on moment matching using Legendre wavelet and harmony search algorithm. *Iran J. Sci. Tech. Trans. Electr. Eng*. 39: 39–54

[21] Sikander A and Prasad R 2017 New technique for system simplification using Cuckoo search and ESA. *Sadhana Acad Proc Eng Sci* 42(9): 1453–1458

[22] Gupta A K, Kumar D and Samuel P 2018 A meta-heuristic cuckoo search and eigen permutation approach for model order reduction. *Sadhana Acad Proc Eng Sci* 43:65

[23] Jiang Y L and Xiao Z H 2013 Arnoldi-based model reduction for fractional order linear systems. *Int. J. Syst. Sci*. 46(8): 1411–1420

[24] Stanisławski R, Rydel M and Latawiec K J 2017 Modelling of discrete-time fractional-order state space systems using the balanced truncation method. *J. Franklin Inst*. 354: 3008–3020

[25] Tavakoli-Kakhki M and Haeri M 2009 Model reduction in commensurate fractional-order linear systems. *P I Mech. Eng. I. J. Syst. Contr. Eng*. 223(4): 493–505

[26] Garrappa R and Maione G 2013 Model order reduction on Krylov subspaces for fractional linear systems. In: *6th Workshop on Fractional Differentiation and Its Applications Part of 2013 IFAC Joint Conference* SSSC, FDA, TDS Grenoble, France. 4–6

[27] Edwards K, Edgar T F and Manousiouthakis V I 1998 Kinetic Model Reduction Using Genetic Algorithms. *Comput. Chem. Eng.* 22(1–2): 239–246

[28] Bunnag D and Sun M 2005 Genetic Algorithm for Constrained Global Optimization in Continuous Variables. *Appl. Math. Comput*. 171: 604–636

[29] Goldberg D 1989 *Genetic Algorithms in Search, Optimization and Machine Learning.* United State, MA: Addison-Wesley Longman Publishing

[30] Dey S, Saha I, Bhattacharyya S and Maulik U 2014 Multi-level Thresholding using Quantum Inspired Meta heuristics. *Knowl. Based Syst.* 67: 373–400, 2014

[31] Nasiri Soloklo H and Farsangi M M 2013 Multi-objective weighted sum approach model reduction by Routh-Pade approximation using harmony search. *Turk J. Elec. Eng. & Compit. Sci*. 21: 2283-2293

[32] Tavazoei M S and Haeri M 2009 A note on the stability of fractional order systems. *Math. Comput. Simul*. 79: 1566–1576

[33] Djamah T, Mansouri R, Bettayeb M and Djennoune S 2009 State space realization of fractional order systems. In: *2nd Mediterranean Conference on Intelligent Systems and Automation - CISA '09, American Institute of Physics (AIP) Conference Proceedings*, Zarzis, Tunisia, 37–42

[34] Rivero M, Rogosin S V, Machado J A T and Trujillo J J 2014 Stability of Fractional Order Systems. *Math. Probl. Eng.* 2013: 1–14

[35] Bayat F M, Afshar M and Ghartemani M K 2009 Extension of the root-locus method to a certain class of fractional-order systems. *ISA Trans.* 48: 48–53

[36] Parks P C 1962 A new proof of the Routh-Hurwitz criterion using the second method of Lyapunov. *Math. Proc. Cambridge Philos. Soc*. 694–702

[37] Skogestad S and Postethwaite I 1996 *Multivariable feedback control, analysis and design.* 2nd edition, John Wiley Press, New York

[38] Krajewski W and Viaro U 2014 A method for the integer-order approximation of fractional-order systems. *J. Franklin Inst*. 351(1): 555–564

[39] Saxena S, Hote Y V and Arya P P 2016 Reduced-Order Modelling of Commensurate Fractional-Order Systems. In: *14th International Conference on Control, Automation, Robotics & Vision*, Phuket, Thailand

[40] Khanra M, Pal J and Biswas K 2013 Reduced Order Approximation of MIMO Fractional Order Systems. *IEEE J. Emerg. Sel. Topic Power Electron*. 3(3): 451–458