Sādhanā (2020)45:211
https://doi.org/10.1007/s12046-020-01468-1

On the solutions of the two preys and one predator type model approached by the fixed point theory

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MS received 14 December 2019; revised 30 March 2020; accepted 7 May 2020

Abstract. The purpose of this paper is to discuss a special type of functional equation that describes the relationship between the predator animals and their two choices of prey with their corresponding probabilities. Our aim is to find the existence and uniqueness results of the proposed functional equation using the Banach fixed point theorem. Finally, we give an illustrative example that supports our main results.

Keywords. Functional equations; mathematical biology; predator–prey model; fixed points; Banach fixed point theorem.

1. Introduction

The vigorous link of the predator and their prey is one of the major concerns in the ecosystem. Recently, researchers found that predation can impact the size of the prey population by going about as top-down control. Indeed, the interaction between these two types of population control provides opportunities to observe the changes in population over time [1, 2].

In 1973, Lyubich and Shapiro [3] studied the existence and uniqueness of a continuous solution $\varphi : [0,1] \to [0,1]$ of the following functional equation:

$$\varphi(x) = x\varphi((1-\mu)x+\mu) + (1-x)\varphi((1-\nu)x),\ x \in [0,1], \tag{1.1}$$

where $0<\mu\leq\nu<1$. The functional equation (1.1) appears in mathematical biology and the theory of learning to observe the nature of predator animals that hunt two kinds of prey. Such a conduct is defined by the Markov process in the state space [0, 1] with the probabilities of transition given by

$$\mathbf{P}(x \to (1-\mu)x+\mu) = x,\ \mathbf{P}(x \to (1-\nu)x) = 1-x.$$

In the mathematical model (1.1) the solution φ is the final probability of the event when the predator is fixed on one category of prey, knowing that the initial probability for this category to be chosen is equal to x. Also, Turab and Sintunavarat [4] used such a type of functional equation to observe the behaviour of the paradise fish in a two-choice situation.

In [3], Lyubich and Shapiro used Schauder's fixed point theorem to prove the existence of a solution of the functional equation (1.1) of the following form:

$$\varphi(x) = \sum_{i=1}^{\infty} \kappa_i x^i,\ \kappa_i \geq 0,$$

satisfying the conditions

$$\varphi(0) = 0,\ \varphi(1) = 1. \tag{1.2}$$

After this, Istrăţescu [5] proposed the existence and uniqueness result for the solution of the functional equation (1.1) with condition (1.2) using the Banach contraction mapping principle.

In this context, Dmitriev and Shapiro [6] found a solution of (1.1) by a direct method. They denoted $\lambda_1 = 1-\mu$ and $\lambda_2 = 1-\nu$ and used the substitution

$$\varphi(x) := x + (\lambda_2 - \lambda_1)x(1-x)\zeta(x), \tag{1.3}$$

to reduce the functional equation (1.1) with the unknown function φ to the following functional equation:

$$\zeta(x) = \lambda_1(1-\lambda_1(1-x))\zeta(1-\lambda_1(1-x)) + \lambda_2(1-\lambda_2 x)\zeta(\lambda_2 x) + 1, \tag{1.4}$$

where ζ is an unknown function. Thus, they proved that the solution of the functional equation (1.4) can be presented as

$$\zeta(x) = \sum_{i=0}^{\infty} \varphi_i(x), \tag{1.5}$$

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Published online: 28 August 2020

where

$$\varphi_0(x) = \sum_{j=0}^{\infty} \lambda_2^j \mu_j(x, \lambda_2),$$

$$\varphi_{i+1}(x) = \lambda_1 \sum_{j=0}^{\infty} \lambda_2^j \mu_j(x, \lambda_2)(1-(1-\lambda_2^j x))$$

$$\varphi_i(1-(1-\lambda_2^j x))$$

and

$$\begin{cases} \mu_0(x, \lambda_2) = 1 \text{ and} \\ \mu_j(x, \lambda_2) = (1-\lambda_2 x)\ldots(1-\lambda_2^j x), \; j \geq 1. \end{cases}$$

In recent times the result of Istrăţ escu [5] was expanded by Berinde and Khan [7], who discussed the existence and uniqueness of a solution of the proposed functional equation using the Banach fixed point theorem. They modelled the functional equation (1.1) in the following form:

$$\varphi(x) = x\varphi(\sigma_1(x)) + (1-x)\varphi(\sigma_2(x)), \tag{1.6}$$

where $\varphi : [0,1] \to \mathbb{R}$ is an unknown function, $\sigma_1, \sigma_2 : [0,1] \to [0,1]$ are contraction mappings satisfying

$$\begin{cases} \sigma_1(1) = 1 \text{ and} \\ \sigma_2(0) = 0. \end{cases} \tag{1.7}$$

In this work, they used the boundary conditions (1.7) to prove Theorem 2.2 in [7].

Our aim is to introduce new conditions and prove the existence and uniqueness of a solution of the functional equation (1.6) using one boundary condition in (1.7). Also, we present some examples to support our main results.

2. Preliminaries

Following definitions and the well-known fixed point results will be required in the continuation.

Definition 2.1 Let $\mathcal{X}$ and $\mathcal{Y}$ be two nonempty sets and $A : \mathcal{X} \to \mathcal{Y}$ be a single-valued mapping. A point $x \in \mathcal{X}$ is called a fixed point of A if and only if $x = Ax$.

Definition 2.2 Let $(\mathcal{X}, d)$ be a metric space. A mapping $A : \mathcal{X} \to \mathcal{X}$ is called a contraction on $\mathcal{X}$ if there is a non-negative real number $k < 1$ such that

$$d(Ax, Ay) \leq kd(x, y) \tag{2.1}$$

for all $x, y \in \mathcal{X}$. Here, the number k is called the contractive coefficient.

Definition 2.3 Let $(\mathcal{X}, d)$ be a metric space. A mapping $A : \mathcal{X} \to \mathcal{X}$ is said to be a Picard mapping if there exists $\bar{x} \in \mathcal{X}$ such that $F_A = \{\bar{x}\}$ and $\{A^n(x)\}_{n\in\mathbb{N}}$ converges to $\bar{x}$ for all $x \in \mathcal{X}$, where F_A denotes the set of all fixed points of A.

Theorem 2.4 (Banach fixed point theorem in [8]) *Consider a metric space $(\mathcal{X}, d)$, where $\mathcal{X} \neq \emptyset$. Suppose that $\mathcal{X}$ is complete and let $A : \mathcal{X} \to \mathcal{X}$ be a contraction mapping. Then A is a Picard mapping. Moreover, if $\bar{x}$ is a fixed point of A, then for each $x \in \mathcal{X}$ we get the following error estimation:*

$$d(A^n(x), \bar{x}) \leq \frac{k^n}{1-k} d(x, \bar{x})$$

for all $n \in \mathbb{N}$, where k is the contractive coefficient of A.

3. Main result

Let $\mathcal{X}$ be the collection of all continuous real-valued functions $\varphi : [0,1] \to \mathbb{R}$ such that $\varphi(0) = 0$ and

$$\sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|} < \infty. \tag{3.1}$$

If $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ is defined by

$$\|\varphi\| = \sup_{x\neq y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|} \tag{3.2}$$

for all $\varphi \in \mathcal{X}$, then $(\mathcal{X}, \|\cdot\|)$ is a Banach space. Throughout this paper, unless otherwise specified, $\|\cdot\|$ is a norm on X defined by (3.2). Furthermore, we shall be interested with the existence of a solution of the following functional equation:

$$\varphi(x) = x\varphi(\sigma_1(x)) + (1-x)\varphi(\sigma_2(x)), \; x \in [0,1], \tag{3.3}$$

where $\varphi : [0,1] \to \mathbb{R}$ is an unknown function, $\sigma_1, \sigma_2 : [0,1] \to [0,1]$ are given contraction mappings. We now turn to our main results in this paper.

Theorem 3.1 *Consider the functional equation (3.3). Suppose that $\sigma_1, \sigma_2 : [0,1] \to [0,1]$ are contraction mappings with contractive coefficients a and b, respectively, such that $\sigma_2(0) = 0$ and*

$$|\sigma_1(x) - \sigma_2(y)| \leq c|x-y| \tag{3.4}$$

for all $x, y \in [0,1]$ with $x \neq y$, where $c \in [0,1)$ such that

$$2a + b + c < 1.$$

Then (3.3) has a unique solution. Moreover, the sequence $\{\varphi_n\}$ in $\mathcal{X}$ defined by

$$\varphi_n(x) = x\varphi_{n-1}(\sigma_1(x)) + (1-x)\varphi_{n-1}(\sigma_2(x))$$

for all $n \in \mathbb{N}$, where φ_0, given in $\mathcal{X}$, converges to a unique solution of (3.3).

Proof Let $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a metric induced by $\|\cdot\|$ on $\mathcal{X}$. Then $(\mathcal{X}, d)$ is a complete metric space. We consider the operator A from $\mathcal{X}$ defined by

$$(A\varphi)(x) = x\varphi(\sigma_1(x)) + (1-x)\varphi(\sigma_2(x)), \; x \in [0,1],$$

for all $\varphi \in \mathcal{X}$. For each $\varphi \in \mathcal{X}$, we obtain

$$(A\varphi)(0) = \varphi(\sigma_2(0)) = \varphi(0) = 0.$$

Also, $A\varphi$ is continuous and $\|A\varphi\| < \infty$ for all $\varphi \in \mathcal{X}$. Therefore, A is a self-operator on X. Furthermore, it is clear that the solution of the functional equation (3.3) is equivalent to the fixed point of an operator A.

As A is a linear mapping, for $\varphi, \phi \in \mathcal{X}$, we have

$$\|A\varphi - A\phi\| = \|A(\varphi - \phi)\|.$$

Thus, to estimate $\|A\varphi - A\phi\|$, we marked the following framework:

$$\Omega_{x,y} := \frac{A(\varphi-\phi)(x) - A(\varphi-\phi)(y)}{x-y}, \; x, y \in [0,1], \; x \neq y.$$

For each $x, y \in [0,1]$ with $x \neq y$, we get

$$\begin{aligned}\Omega_{x,y} &= \frac{1}{x-y}\Big[x(\varphi-\phi)(\sigma_1(x)) + (1-x)(\varphi-\phi)(\sigma_2(x)) \\ &\quad - y(\varphi-\phi)(\sigma_1(y)) - (1-y)(\varphi-\phi)(\sigma_2(y))\Big] \\ &= \frac{1}{x-y}\Big[x(\varphi-\phi)(\sigma_1(x)) - x(\varphi-\phi)(\sigma_1(y)) \\ &\quad + (1-x)(\varphi-\phi)(\sigma_2(x)) - (1-x)(\varphi-\phi)(\sigma_2(y)) \\ &\quad + x(\varphi-\phi)(\sigma_1(y)) - y(\varphi-\phi)(\sigma_1(y)) \\ &\quad + (1-x)(\varphi-\phi)(\sigma_2(y)) - (1-y)(\varphi-\phi)(\sigma_2(y))\Big].\end{aligned}$$

Then we have

$$\begin{aligned}&|\Omega_{x,y}| \\ &\leq x\frac{|(\varphi-\phi)(\sigma_1(x)) - (\varphi-\phi)(\sigma_1(y))|}{|\sigma_1(x)-\sigma_1(y)|} \\ &\quad \frac{|\sigma_1(x)-\sigma_1(y)|}{|x-y|} \\ &\quad + (1-x)\frac{|(\varphi-\phi)(\sigma_2(x)) - (\varphi-\phi)(\sigma_2(y))|}{|\sigma_2(x)-\sigma_2(y)|} \\ &\quad \frac{|\sigma_2(x)-\sigma_2(y)|}{|x-y|} \\ &\quad + \frac{|(\varphi-\phi)(\sigma_1(y)) - (\varphi-\phi)(\sigma_1(x))|}{|\sigma_1(y)-\sigma_1(x)|}|\sigma_1(y)-\sigma_1(x)| \\ &\quad + \frac{|(\varphi-\phi)(\sigma_1(x)) - (\varphi-\phi)(\sigma_2(y))|}{|\sigma_1(x)-\sigma_2(y)|}|\sigma_1(x)-\sigma_2(y)|.\end{aligned}$$

This yields

$$\begin{aligned}&|\Omega_{x,y}| \\ &\leq \left[\frac{x}{|x-y|}|\sigma_1(x)-\sigma_1(y)| + \frac{1-x}{|x-y|}|\sigma_2(x)-\sigma_2(y)| \right. \\ &\quad \left. + |\sigma_1(y)-\sigma_1(x)| + |\sigma_1(x)-\sigma_2(y)|\right]\|\varphi-\phi\| \\ &\leq (ax + b(1-x) + a|y-x| + c|x-y|)\|\varphi-\phi\| \\ &\leq (2a+b+c)\|\varphi-\phi\|\end{aligned}$$

and thus

$$\begin{aligned}d(A\varphi, A\phi) &= \|A\varphi - A\phi\| \\ &\leq (2a+b+c)\|\varphi-\phi\| \\ &= (2a+b+c)d(\varphi,\phi).\end{aligned}$$

As $2a+b+c<1$, by Theorem 2.4, we get the unique solution of (3.3). □

Remark 3.2 Sometimes it is very difficult to obtain the condition (3.4); therefore we try to use another condition to seek the solution of the functional equation (3.3).

Theorem 3.3 *Consider the functional equation* (3.3). *Suppose that* $\sigma_1, \sigma_2 : [0,1] \to [0,1]$ *are contraction mappings with contractive coefficients* a *and* b, *respectively, such that* $\sigma_2(0) = 0$,

$$2a + 2b < 1$$

and there exists a point $\bar{x} \in [0,1]$ *such that* $\sigma_1(\bar{x}) = \sigma_2(\bar{x})$. *Then* (3.3) *has a unique solution. Moreover, the sequence* $\{\varphi_n\}$ *in* $\mathcal{X}$ *defined by*

$$\varphi_n(x) = x\varphi_{n-1}(\sigma_1(x)) + (1-x)\varphi_{n-1}(\sigma_2(x))$$

for all $n \in \mathbb{N}$, *where* φ_0 *is given in* $\mathcal{X}$, *converges to a unique solution of* (3.3).

Proof We consider the metric d and the operator $A : \mathcal{X} \to \mathcal{X}$ as in Theorem 3.1. For each $x, y \in [0,1]$ with $x \neq y$, we set the following notation:

$$\Omega_{x,y} := \frac{A(\varphi-\phi)(x) - A(\varphi-\phi)(y)}{x-y}.$$

For each $\varphi, \phi \in \mathcal{X}$ and $x, y \in [0,1]$ with $x \neq y$, we get

$$\begin{aligned}|\Omega_{x,y}| &= \left|\frac{1}{x-y}\Big[x(\varphi-\phi)(\sigma_1(x)) + (1-x)(\varphi-\phi)(\sigma_2(x)) \right. \\ &\quad \left. - y(\varphi-\phi)(\sigma_1(y)) - (1-y)(\varphi-\phi)(\sigma_2(y))\Big]\right| \\ &\leq \left|\frac{1}{x-y}\Big[x(\varphi-\phi)(\sigma_1(x)) - x(\varphi-\phi)(\sigma_1(y))\Big]\right| \\ &\quad + \left|\frac{1}{x-y}\Big[(1-x)(\varphi-\phi)(\sigma_2(x)) \right. \\ &\quad \left. - (1-x)(\varphi-\phi)(\sigma_2(y))\Big]\right| \\ &\quad + \left|\frac{1}{x-y}\Big[(x-y)(\varphi-\phi)(\sigma_1(y)) \right. \\ &\quad \left. - (x-y)(\varphi-\phi)(\sigma_1(\bar{x}))\Big]\right| \\ &\quad + \left|\frac{1}{x-y}\Big[(x-y)(\varphi-\phi)(\sigma_2(y)) \right. \\ &\quad \left. - (x-y)(\varphi-\phi)(\sigma_2(\bar{x}))\Big]\right|.\end{aligned} \tag{3.5}$$

Now, we will discuss the following two cases.

Case 1: If $\bar{x} = y$, then by (3.5) we have

$$\begin{aligned}|\Omega_{x,y}| \leq & x\frac{|(\varphi-\phi)(\sigma_1(x)) - (\varphi-\phi)(\sigma_1(y))|}{|\sigma_1(x)-\sigma_1(y)|}\frac{|\sigma_1(x)-\sigma_1(y)|}{|x-y|} \\ & + (1-x)\frac{|(\varphi-\phi)(\sigma_2(x)) - (\varphi_1-\phi)(\sigma_2(y))|}{|\sigma_2(x)-\sigma_2(y)|} \\ & \frac{|\sigma_2(x)-\sigma_2(y)|}{|x-y|}.\end{aligned}$$

This yields

$$\begin{aligned}|\Omega_{x,y}| &\leq \left[\frac{x}{|x-y|}|\sigma_1(x)-\sigma_1(y)| + \frac{1-x}{|x-y|}|\sigma_2(x)-\sigma_2(y)|\right]\|\varphi-\phi\| \\ &\leq [ax + b(1-x)]\|\varphi-\phi\| \\ &\leq (a+b)\|\varphi-\phi\| \\ &\leq 2(a+b)\|\varphi-\phi\|.\end{aligned} \tag{3.6}$$

Case 2: If $\bar{x} \neq y$, then by (3.5) we have

$$\begin{aligned}|\Omega_{x,y}| \leq & x\frac{|(\varphi-\phi)(\sigma_1(x)) - (\varphi-\phi)(\sigma_1(y))|}{|\sigma_1(x)-\sigma_1(y)|} \\ & \frac{|\sigma_1(x)-\sigma_1(y)|}{|x-y|} \\ & + (1-x)\frac{|(\varphi-\phi)(\sigma_2(x)) - (\varphi-\phi)(\sigma_2(y))|}{|\sigma_2(x)-\sigma_2(y)|} \\ & \frac{|\sigma_2(x)-\sigma_2(y)|}{|x-y|} \\ & + \frac{|(\varphi-\phi)(\sigma_1(y)) - (\varphi-\phi)(\sigma_1(\bar{x}))|}{|\sigma_1(y)-\sigma_1(\bar{x})|}|\sigma_1(y)-\sigma_1(\bar{x})| \\ & + \frac{|(\varphi-\phi)(\sigma_2(\bar{x})) - (\varphi-\phi)(\sigma_2(y))|}{|\sigma_2(\bar{x})-\sigma_2(y)|}|\sigma_2(\bar{x})-\sigma_2(y)|.\end{aligned}$$

This yields

$$\begin{aligned}|\Omega_{x,y}| &\leq \left[\frac{x}{|x-y|}|\sigma_1(x)-\sigma_1(y)| + \frac{1-x}{|x-y|}|\sigma_2(x)-\sigma_2(y)| + |\sigma_1(\bar{x})-\sigma_1(y)| + |\sigma_2(\bar{x})-\sigma_2(y)|\right]\|\varphi-\phi\| \\ &\leq (ax + b(1-x) + a|\bar{x}-y| + b|\bar{x}-y|)\|\varphi-\phi\| \\ &\leq (2a+2b)\|\varphi-\phi\|.\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$\begin{aligned}d(A\varphi, A\phi) = \|A\varphi - A\phi\| &\leq (2a+2b)\|\varphi-\phi\| \\ &= (2a+2b)d(\varphi,\phi).\end{aligned} \tag{3.8}$$

From the fact that $2a+2b<1$, we get A as a Banach contraction mapping. By Theorem 2.4, we get the unique solution of (3.3). □

4. An illustrative example

Next, an illustrative example is presented that demonstrates the validity of the hypotheses and degree of utility of our results.

Example 4.1 Consider the following functional equation:

$$\varphi(x) = x\varphi\left(\tfrac{x+1}{7}\right) + (1-x)\varphi\left(\tfrac{2x}{7}\right) \tag{4.1}$$

for all $x \in [0,1]$, where $\varphi : [0,1] \to \mathbb{R}$ is an unknown function. If we set the mappings $\sigma_1, \sigma_2 : [0,1] \to [0,1]$ by

$$\sigma_1(x) = \tfrac{x+1}{7} \text{ and } \sigma_2(x) = \tfrac{2x}{7}$$

for all $x \in [0,1]$, then the functional equation (4.1) reduces to the functional equation (3.3).

Next, we try to apply Theorem 3.3 for solving this problem. Here, σ_1 and σ_2 are contraction mappings with coefficient $a = \frac{1}{7}$ and $b = \frac{2}{7}$, respectively, and thus $2a+2b<1$. Also, there exists a point $\bar{x} \in [0,1]$ such that $\sigma_1(\bar{x}) = \sigma_2(\bar{x})$ (see Fig. 1).

Now, all the assumptions of Theorem 3.3 hold. Therefore, the functional equation (4.1) has a unique solution. Moreover, if we take an initial approximation $\varphi_0(x) = x$ for all $x \in [0,1]$, then the following iteration converges to a unique solution of (4.1):

$$\begin{aligned}\varphi_1(x) &= \frac{-x^2+3x}{7}, \\ \varphi_2(x) &= \frac{3x^3 - 27x^2 + 62x}{343}, \\ &\vdots \\ \varphi_n(x) &= x\varphi_{n-1}\left(\tfrac{x+1}{7}\right) + (1-x)\varphi_{n-1}\left(\tfrac{2x}{7}\right),\end{aligned}$$

for all $n \in \mathbb{N}$. Hence, by the error estimate, we can get a priori to know how many iterations are required to achieve absolute precision in approximating the solution φ using the iterates φ_n.

Consider, for example, the particular case $a = \frac{1}{7}$, $b = \frac{2}{7}$, and fix the desired computational precision to $\varepsilon = 10^{-p}$. By the inequality

$$\|\varphi_n - \varphi\| \leq \frac{\alpha^n}{1-\alpha}\|\varphi_1 - \varphi_0\|, n = 1, 2, 3, \ldots,$$

where $\alpha := 2a+2b$, we obtain that n must be the smallest number such that

$$\left(\frac{6}{7}\right)^n \leq \frac{1}{7}\varepsilon.$$

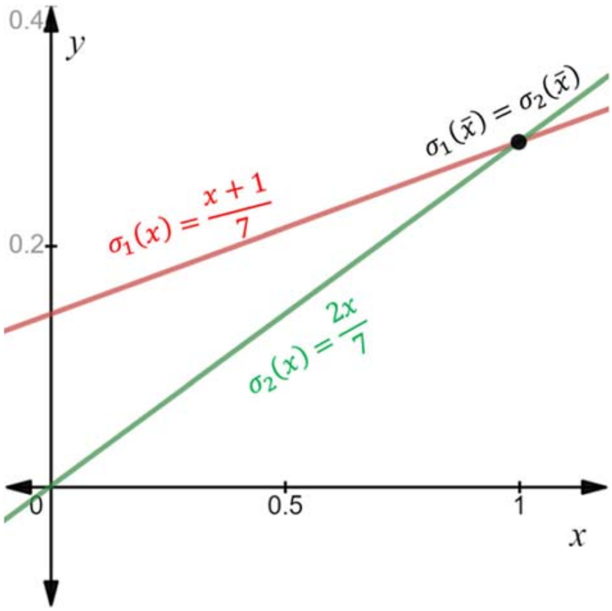

Figure 1. Graphs of σ_1 and σ_2 in Example 4.1.

In case $p = 3$, for example, this shows that we must compute $n \geq 58$ iterations or more (which requires a substantial computational skill) to calculate the solution with 3 exact digits.

Remark 4.2 Due to the linear convergence rate of the Picard iteration, we cannot expect fast convergence from the sequence of iterations. To tackle this problem, we can use an appropriate accelerative method (see [9–13]).

5. Conclusion

The proposed functional equation (1.6) connects two classes of functional equations arising in learning theory and psychology discussed in [4, 5]. It has great importance, especially in a two-choice situation, i.e., such a type of functional equation describes the interrelation between the predator animals and their two choices of prey with their corresponding probabilities. In [4, 5, 7] the authors used the boundary conditions in the proof of their main results, but in their comparison, we do not use such a type of conditions to find the existence and uniqueness results of the functional equation (1.6), which proves that our result covers more problems than the previous ones existing in the particular literature.

Acknowledgements

The second author would like to thank the Thailand Research Fund and Office of the Higher Education Commission under Grant No. MRG6180283 for financial support during the preparation of this manuscript.

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