Sādhanā (2020)45:185
https://doi.org/10.1007/s12046-020-01422-1

Solution and stability analysis of non-homogeneous difference equation followed by real life application in fuzzy environment

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MS received 6 November 2019; revised 1 June 2020; accepted 23 June 2020

Abstract. The study fuzzy difference equation becomes very important as huge numbers of real-life problems in the field of engineering; ecology social science, etc. can be mathematically represented in the form of difference equation where impreciseness is inherently involved. In this paper, we have focused on the solution techniques of non-homogeneous fuzzy linear difference equation with different cases involving fuzzy initial conditions, fuzzy forcing function and fuzzy coefficient. The idea of fuzzy equilibrium point is introduced and its stability analysis has been performed. The whole theoretical work is followed by real-life applications which show the impact of fuzzy concepts in mathematical modelling for better understanding the behaviour of the system in an elegant manner.

Keywords. Fuzzy set theory; difference equation; stability criteria for difference equation; mathematical modelling.

1. Introduction

In the year 1965, the innovative philosophy of Fuzzy Sets theory was instigated by Zadeh [1] which gradually became very popular in the measure of uncertainty and added a new dimension in the field of mathematical modelling under various uncertain environments. Chang and Zadeh [2] presented the perception of fuzzy numbers. In numerous cases in our daily life, especially those cases where human thinking and judgements are involved, uncertainty is inherently involved and perception of fuzzy plays a crucial role. Thus, the role of fuzzy set theory cannot be ignored in the modelling of various real-life discrete systems where uncertainty is inherently embedded. Many socio-economic, ecological and engineering problems have been represented in the form of difference equation where uncertainty is very common. Hence, the study of various difference equations in the fuzzy environment is very much essential. Several research works have already been initiated in this regard; for example Lakshmikantham and Vatsala [3] illustrated basic perception of fuzzy difference equations; Papaschinopoulos *et al* [4, 5], Schinas [6] have focused on various structure fuzzy difference equations. Stefanidou *et al* [7] have analyzed exponential type fuzzy difference equations and Din [8] has carried out the asymptotic behaviour of fuzzy difference equation. Zhang *et al* [9] have addressed non-linear fuzzy difference equations whereas volterra type fuzzy difference equations are addressed by Memarbashi and Ghasemabadi [10]. A fuzzy difference equation of rational type with asymptotic behaviour is explained by Stefanidou and Papaschinopoulos [11, 12]. Several contributions in the fuzzy difference equation are also made [13–16].

In light of the above mentioned works, we have made an attempt to do the following advancement in the field of fuzzy difference equation as follows.

(1) The solution technique of non-homogeneous linear difference equation with fuzzy coefficients has been discussed with fuzzy forcing function and initial conditions as fuzzy numbers which was not done earlier.
(2) The idea of fuzzy equilibrium point is introduced and a smooth relationship has been identified between fuzzy equilibrium points and crisp equilibrium points

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Published online: 21 July 2020

(3) The real life examples in investment in annuity and prescription for digoxin are given in fuzzy environment as an application of proposed study.

2. Preliminaries idea

Definition 2.1 (*Fuzzy Set*) A fuzzy set $\tilde{A}$ is defined as a set of ordered pair $\left(X, \mu_{\tilde{A}}(x)\right)$, where X is nonempty universal set and $x \in X, A$ is the classical set. $\mu_{\tilde{A}}(x)\colon X \to [0,1]$, membership function and $\mu_{\tilde{A}}(x)$ is the grade of membership of $x \in X$ in $\tilde{A}$.

Definition 2.2 (*Triangular fuzzy number*) A triangular fuzzy number (TFN) is defined as an ordered triplet $\tilde{A} = (a_1, a_2, a_3)$ and it's membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}; & a_1 \le x \le a_2 \\ 1; & x = a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}; & a_2 \le x \le a_3 \\ 0; & otherwise \end{cases}.$$

Definition 2.3 (*Fuzzy function*) Let the set of all real numbers and real valued fuzzy numbers are denoted by $\mathcal{R}$ and $\mathcal{R}_{\mathcal{F}}$ respectively. The function $W\colon \mathcal{R} \to [0,1]$ is called a fuzzy number valued function if w satisfies the following properties.

(1) W is upper semi continuous
(2) W is fuzzy convex
i.e., $W(\lambda s_1 + (1-\lambda)s_2) \ge min\{W(s_1), W(s_2)\}$, for all $s_1, s_2 \in \mathcal{R}$ and $\lambda \in [0,1]$
(3) W is normal i.e., $\exists$ a $s_0 \in \mathcal{R}$ such that $W(s_0) = 1$
(4) Closure of $supp(W)$ is compact, where $supp(W) = \{s \in \mathcal{R} |\ W(s) > 0\}$.

Definition 2.4 (*Hukuhara-difference on fuzzy function*) Let E^* be the set of all fuzzy function and $\tilde{s}, \tilde{t} \in E^*$. If $\exists$ a fuzzy number $\tilde{w} \in E^*$ and $\tilde{w}$ satisfy the relation $\tilde{s} = \tilde{w} + \tilde{t}$, then $\tilde{w}$ is said to be the Hukuhara-difference of $\tilde{s}$ and $\tilde{t}$, denoted by $\tilde{w} = \tilde{s} \ominus \tilde{t}$.

Theorem 2.1 [17, 18] *If $F\colon R^m \to R^n$ is continuous, then the Zadeh's extension $\tilde{F}\colon \left(R_f\right)^m - \left(R_f\right)^n$ is well defined, is continuous and $\left[\tilde{F}(u)\right]_\alpha = F\left([\tilde{u}]_\alpha\right)$ $\forall \alpha \in [0,1]$.*

The above is still valid if $F\colon U \to R^n$, where U is an open subset in R^n.

Theorem 2.2 (Characterization theorem) *Let us consider the fuzzy difference equation problem*

$$\tilde{x}_{n+1} = \tilde{f}(x_n, n), \tag{2.1}$$

$$\text{With initial value } \tilde{x}_{n=0} = \tilde{x}_0 \tag{2.2}$$

Where $f\colon E^* \times \mathbb{Z}_{\ge 0} \to E^*$ such that

(1) The parametric form of the function is

$$[f((x_n, n))]_\alpha = \left[\underline{f}(\underline{x}_n(\alpha), \bar{x}_n(\alpha), n, \alpha), \bar{f}(\underline{x}_n(\alpha), \bar{x}_n(\alpha), n, \alpha)\right]$$

(2) *The functions $\underline{f}(\underline{x}_n(\alpha), \bar{x}_n(\alpha), n, \alpha)$ and $\bar{f}(\underline{x}_n(\alpha), \bar{x}_n(\alpha), n, \alpha)$ are taken as continuous functions if for any $\in_1 > 0 \,\exists$ a $\delta_1 > 0$ such that*

$$\left|\underline{f}(\underline{x}_n(\alpha), \bar{x}_n(\alpha), n) - \underline{f}\left(\underline{x}_{n_1}(\alpha), \bar{x}_{n_1}(\alpha), n_1\right)\right| < \in_1 \quad \text{for all } \alpha \in [0,1]$$

With $\left\|(\underline{x}_n(\alpha), \bar{x}_n(\alpha), n) - \left(\underline{x}_{n_1}(\alpha), \bar{x}_{n_1}(\alpha), n_1\right)\right\| < \delta_1$, where and $\in_2 > 0 \,\exists$ an $\delta_2 > 0$ such that

$$\left|\bar{f}(\underline{x}_n(\alpha), \bar{x}_n(\alpha), n, \alpha) - \bar{f}\left(\underline{x}_{n_2}(\alpha), \bar{x}_{n_2}(\alpha), n_2\right)\right| < \in_2 \quad \text{for all } \alpha \in [0,1]$$

with $\left\|(\underline{x}_n(\alpha), \bar{x}_n(\alpha), n) - \left(\underline{x}_{n_2}(\alpha), \bar{x}_{n_2}(\alpha), n_2\right)\right\| < \delta_2$.

Then the difference equation (2.1) reduces to system of two difference equation as

$$\underline{x}_{n+1}(\alpha) = \underline{f}(\underline{x}_n(\alpha), \bar{x}_n(\alpha), n, \alpha)$$
$$\bar{x}_{n+1}(\alpha) = \bar{f}(\underline{x}_n(\alpha), \bar{x}_n(\alpha), n, \alpha)$$

With initial conditions

$$\underline{x}_{n=0}(\alpha) = \underline{x}_0(\alpha)$$
$$\bar{x}_{n=0}(\alpha) = \bar{x}_0(\alpha).$$

Note 2.1 By characterisation theorem every single fuzzy difference equation is converted into a system of two crisp difference equations. In this paper we have taken only single fuzzy difference equation in fuzzy environment. Hence, the difference equation converted into a pair of crisp difference equation.

Definition 2.5 (*Strong and weak solution of fuzzy difference equation*) The solutions of difference equation (2.1) with initial condition (2.2) to be regarded as

(1) A strong solution if $\underline{x}_n(\alpha) \le \bar{x}_n(\alpha)$ for every $\alpha \in [0,1]$ and $\frac{\partial}{\partial \alpha}[\underline{x}_n(\alpha)] > 0$, $\frac{\partial}{\partial \alpha}[\bar{x}_n(\alpha)] < 0$ for every $\alpha \in [0,1]$
(2) A weak solution if $\underline{x}_n(\alpha) \ge \bar{x}_n(\alpha)$ for every $\alpha \in [0,1]$ and $\frac{\partial}{\partial \alpha}[\underline{x}_n(\alpha)] < 0$, $\frac{\partial}{\partial \alpha}[\bar{x}_n(\alpha)] > 0$ for every $\alpha \in [0,1]$.

Definition 2.6 [19] Let p, q are fuzzy numbers and $[p]_\alpha = \left[\underline{p}(\alpha), \bar{p}(\alpha)\right]$, $[q]_\alpha = \left[\underline{q}(\alpha), \bar{q}(\alpha)\right]$ for all $\alpha \in [0,1]$. Then metric on fuzzy number space is defined as

$$d(p,q) = sup \max_{\alpha \in [0,1]} \left\{\left|\underline{p}(\alpha) - \underline{q}(\alpha)\right|, |\bar{p}(\alpha) - \bar{q}(\alpha)|\right\}.$$

Note 2.2 May be somewhere strictly the strong and weak solution strategy may not occur. In this case for particular time interval or particular interval of α the strong and weak solution both exists. The scenario where the strong or weak both solutions not occur, then we will call them non recommended fuzzy solution. We recommended for taking strong solution cases.

Theorem 2.2 (Equilibrium points of a fuzzy difference equation in terms of converted crisp difference equation) *Let us consider the crisp system $x_{n+1} = f(x_n)$. Now let the equilibrium point of the system be $x = x^*$ then we can say that at point x^*, $x_{n+1} = x_n$. That means if we interested to find the equilibrium point then we have to solve the equation $f(x) = x$. From this concept we can find the crisp equilibrium points of a crisp system. Now we are looking for fuzzy system.*

Let us take the fuzzy difference equation $\tilde{x}_{n+1} = \tilde{f}(x_n)$. By characterisation theorem the difference equation can be converted to two system of crisp system as

$$\underline{x}_{n+1}(\alpha) = F(\underline{x}_n(\alpha), \bar{x}_n(\alpha))$$
$$\bar{x}_{n+1}(\alpha) = G(\underline{x}_n(\alpha), \bar{x}_n(\alpha))$$

Where $[F(\underline{x}_n(\alpha), \bar{x}_n(\alpha)), G(\underline{x}_n(\alpha), \bar{x}_n(\alpha))]$ is the parametric form of the fuzzy function $\tilde{f}(x_n)$ and $[\underline{x}_n(\alpha), \bar{x}_n(\alpha)]$ are the parametric form of fuzzy parameter x_n.

Now the stability of the above converted system can be found easily. Let us consider the fuzzy equilibrium point is $[\underline{x}^*(\alpha), \bar{x}^*(\alpha)]$, that means there is no change of the above equation at this point. In other words, we can say that if we are interested to find the fuzzy equilibrium point then we have for solve the equation.

$$F(\underline{x}_n(\alpha), \bar{x}_n(\alpha)) = 0 \text{ and } G(\underline{x}_n(\alpha), \bar{x}_n(\alpha)) = 0.$$

Note 2.3 Equilibrium point for fuzzy difference equation has great importance to determine the stability criteria of the system. To find out the equilibrium point of fuzzy system the following approaches are followed:

(1) Convert the fuzzy system into corresponding crisp system
(2) Finding the equilibrium point from the crisp system.

3. Difference equation with fuzzy variable

Definition 3.1 A difference equation (sometime called a recurrence relation) is an equation that relates consecutive terms of a sequence of numbers.

A qth order linear difference equation can be expressed in the form

$$x_{n+q} = d_1 x_{n+q-1} + d_2 x_{n+q-2} + \cdots + d_q x_n + b_n \tag{3.1}$$

Where $d_1, d_2, \ldots, d_q$ and b_n are known constant.

If $b_n = 0$, for all n, the equation (3.1) is homogeneous difference equation and non-homogeneous difference equation if $b_n \neq 0$.

b_n is called the forcing factor.

We consider an autonomous linear difference equation of the form

$$x_{n+1} = \beta x_n + \gamma (\beta \neq 0) \tag{3.2}$$

If x_e be the equilibrium point of the equation (3.2), then

$$x_{n+1} = x_n = x_e (\textit{if there are no change from n ton} + 1\ \textit{steps})$$
$$\Rightarrow \beta x_e + \gamma = x_e$$

$x_e = \frac{\gamma}{1-\beta}$, exist if $\beta \neq 1$

The equilibrium point x_e is said to be stable if all the solution of equation (3.2) converges to x_e as n becomes large $(n \to \infty)$. The equilibrium point is said to be unstable if all solution of equation (3.2) diverges from x_e to $\pm\infty$.

Theorem 3.1 [20] *Let $m \in \mathbb{N}$, $m \geq 2$. A linear inhomogeneous system of m first order difference equations are given by in matrix form as*

$$X_{n+1} = AX_n + B \tag{3.3}$$

Where, $X_n = \left(X_n^1, X_n^2, \ldots, X_n^m\right)^T$, $A = \left(a_{ij}\right)_{m \times m}, i,j = 1, 2, \ldots, m$ and $B = (b_1, b_2, \ldots, b_m)^T$

Then the solution of equation (3.3) can be written as

$$X_n = A^n X_0 + \sum_{j=0}^{n-1} A^j B, \quad n \in \mathbb{N} \tag{3.4}$$

The above difference equation (3.1) is called fuzzy difference equation if

(i) *The initial condition or conditions are fuzzy number*
(ii) *The coefficient or coefficients are fuzzy number*
(iii) *The initial conditions and coefficient or coefficients are fuzzy numbers.*

4. Solution and stability analysis of non-homogeneous difference equation

Considering linear inhomogeneous difference equations

$$u_{n+1} = au_n + \tilde{b} \quad (\text{Type I}) \tag{4.1}$$

Where $b \neq 0$. In fuzzy sense, another in-equivalent form of (4.1) is taken as

$$u_{n+1} - au_n = \tilde{b} \quad (\text{Type II}) \tag{4.2}$$

Remarks 4.1 The equation (4.1) and (4.2) are equivalent in crisp sense but in fuzzy sense it is not equivalent.

Proof If we take the fuzzy difference equation (4.1) then it becomes using Theorem 2.1

$$[u_{n+1}]_\alpha = \left[au_n + \tilde{b}\right]_\alpha = [au_n]_\alpha + \left[\tilde{b}\right]_\alpha$$
$$= a[u_n]_\alpha + \left[\tilde{b}\right]_\alpha$$
$$\text{or, } \left[\underline{u}_{n+1}(\alpha), \bar{u}_{n+1}(\alpha)\right] = a[\underline{u}_n(\alpha), \bar{u}_n(\alpha)] + [\underline{b}(\alpha), \bar{b}(\alpha)]$$
$$\text{i.e., } \begin{cases} \underline{u}_{n+1}(\alpha) = a\underline{u}_n(\alpha) + \underline{b}(\alpha) \\ \bar{u}_{n+1}(\alpha) = a\bar{u}_n(\alpha) + \bar{b}(\alpha) \end{cases} \tag{4.3}$$

but when we take (4.2) then it becomes using Theorem 2.1

$$[u_{n+1}]_\alpha - [au_n]_\alpha = \left[\tilde{b}\right]_\alpha$$
$$\text{or, } \left[\underline{u}_{n+1}(\alpha), \bar{u}_{n+1}(\alpha)\right] - a[\underline{u}_n(\alpha), \bar{u}_n(\alpha)] + [\underline{b}(\alpha), \bar{b}(\alpha)]$$
$$\text{i.e., } \begin{cases} \underline{u}_{n+1}(\alpha) - a\bar{u}_n(\alpha) = \underline{b}(\alpha) \\ \bar{u}_{n+1}(\alpha) - a\underline{u}_n(\alpha) = \bar{b}(\alpha) \end{cases}$$
$$\text{or, } \begin{cases} \underline{u}_{n+1}(\alpha) = a\bar{u}_n(\alpha) + \underline{b}(\alpha) \\ \bar{u}_{n+1}(\alpha) = a\underline{u}_n(\alpha) + \bar{b}(\alpha) \end{cases} \tag{4.4}$$

Clearly from (4.3) and (4.4) we conclude that they are different.

So, in crisp sense (4.1) and (4.2) are same but not in fuzzy sense.

4.1 *Solution and stability analysis of non-homogeneous difference equation of Type I*

Considering equation (4.1) with fuzzy initial condition $[u_{n=0}]_\alpha = [\underline{u}_0(\alpha), \bar{u}_0(\alpha)]\ \forall\alpha \in [0,1]$.

Taking α-levels of equation (4.1) we get,

$$\left[\underline{u}_{n+1}(\alpha), \bar{u}_{n+1}(\alpha)\right] = a[\underline{u}_n(\alpha), \bar{u}_n(\alpha)] + [\underline{b}(\alpha), \bar{b}(\alpha)] \tag{4.1.1}$$

4.1a *Study when $a > 0$, a crisp number and u_0 be a fuzzy number*:

Then from equation (4.1.1), we get the coupled equations,

$$\underline{u}_{n+1}(\alpha) = a\underline{u}_n(\alpha) + \underline{b}(\alpha) \tag{4.1.2}$$

$$\bar{u}_{n+1}(\alpha) = a\bar{u}_n(\alpha) + \bar{b}(\alpha) \tag{4.1.3}$$

Solutions of the above equations are

$$\underline{u}_n(\alpha) = a^n\underline{u}_0(\alpha) + \underline{b}(\alpha)\left(\frac{1-a^n}{1-a}\right) \tag{4.1.4}$$

$$\bar{u}_n(\alpha) = a^n\bar{u}_0(\alpha) + \bar{b}(\alpha)\left(\frac{1-a^n}{1-a}\right) \tag{4.1.5}$$

And fuzzy equilibrium point is $E_1^1\left[\frac{\underline{b}(\alpha)}{1-a}, \frac{\bar{b}(\alpha)}{1-a}\right]$.

Remarks 4.1 Both the solutions and equilibrium points exist if $a \neq 1$.

Corollary 4.1 *If $\tilde{b}$ is positive and $a < 1$ then the equilibrium point E_1 is positive equilibrium point.*

Corollary 4.2 *If $\tilde{b}$ is positive and $a > 1$ then the equilibrium point is E_1^1 negative equilibrium point and the point is $\left[\frac{\bar{b}(\alpha)}{1-a}, \frac{\underline{b}(\alpha)}{1-a}\right]$.*

Corollary 4.3 *If $\tilde{b}$ is negative and $a < 1$ then the equilibrium point E_1^1 is negative equilibrium point.*

Corollary 4.4 *If $\tilde{b}$ is negative and $a > 1$ equilibrium point E_1^1 is positive equilibrium point and the equilibrium point is $\left[\frac{\bar{b}(\alpha)}{1-a}, \frac{\underline{b}(\alpha)}{1-a}\right]$.*

Lemma 4.1 *Consider the in-homogeneous difference equation (4.1), where* a > 0 *a crisp number and* u_0 *be a fuzzy number. Then the positive fuzzy equilibrium point* $E_1^1\left[\frac{\underline{b}(\alpha)}{1-a}, \frac{\bar{b}(\alpha)}{1-a}\right]$ *is stable if* $0 < a < 1$.

Proof Clearly, from equation (4.1.4) and (4.1.5), $\underline{u}_n(\alpha) \to \frac{\underline{b}(\alpha)}{1-a}$ and $\bar{u}_n(\alpha) \to \frac{\bar{b}(\alpha)}{1-a}$ as $n \to \infty$ if $0 < a < 1$.

Let u_n be a sequence of solutions of equation (1) under Case 4.1 a such that $[u_n]_\alpha = [\underline{u}_n(\alpha), \bar{u}_n(\alpha)]$ and u be fuzzy equilibrium point with α-cut $[u]_\alpha = [\underline{u}(\alpha), \bar{u}(\alpha)]$

For equilibrium point E_1^1, $\underline{u}(\alpha) = \frac{\underline{b}(\alpha)}{1-a}, \bar{u}(\alpha) = \frac{\bar{b}(\alpha)}{1-a}$. So we have, $\lim_{n\to\infty} D(u_n, u) = \lim_{n\to\infty} \sup max\{|\underline{u}_n(\alpha) - \underline{u}(\alpha)|, |\bar{u}_n(\alpha) - \bar{u}(\alpha)|\} = 0$ for all $\alpha \in [0,1]$, if $0 < a < 1$.

Therefore, fuzzy equilibrium point E_1^1 is stable if $0 < a < 1$.

Note 4.1 The equilibrium point $\left[\frac{\bar{b}(\alpha)}{1-a}, \frac{\underline{b}(\alpha)}{1-a}\right]$ is unstable positive equilibrium point if $a > 1$.

In this case, the equilibrium point $[\underline{u}(\alpha), \bar{u}(\alpha)] = \left[\frac{\bar{b}(\alpha)}{1-a}, \frac{\underline{b}(\alpha)}{1-a}\right]$, $\underline{u}_n(\alpha) \to \infty$ and $\bar{u}_n(\alpha) \to \infty$ as $n \to \infty$ if $a > 1$.

Therefore, $\lim_{n\to\infty} D(u_n, u) = \lim_{n\to\infty} \sup max\{|\underline{u}_n(\alpha) - \underline{u}(\alpha)|, |\bar{u}_n(\alpha) - \bar{u}(\alpha)|\} \neq 0$ for all $\alpha \in [0,1]$, if a>1.

4.1b *Study when $a = 1$ and initial value u_0 be a fuzzy number*:

Since $\tilde{b} \neq 0$ equations (4.1a) has no equilibrium point. The sequence of solutions are $\underline{u}_n(\alpha) = \underline{u}_0(\alpha) + n\underline{b}(\alpha)$

$\bar{u}_n(\alpha) = \bar{u}_0(\alpha) + n\bar{b}(\alpha)$, which leads a divergent solutions.

4.1c *Study when $a < 0$ and initial value u_0 be a fuzzy number*:

Let $a = -\mu$, $\mu > 0$.

Then from equation (4.1a) we have,

$$\left[\underline{u}_{n+1}(\alpha), \bar{u}_{n+1}(\alpha)\right] = -\mu[\underline{u}_n(\alpha), \bar{u}_n(\alpha)] + [\underline{b}(\alpha), \bar{b}(\alpha)] \tag{4.1.6}$$

$$\text{Therefore, } \underline{u}_{n+1}(\alpha) = -\mu\bar{u}_n(\alpha) + \underline{b}(\alpha) \tag{4.1.7}$$

$$\bar{u}_{n+1}(\alpha) = -\mu \underline{u}_n(\alpha) + \bar{b}(\alpha) \tag{4.1.8}$$

Pairs of equations (4.1.7) and (4.1.8) can be written in the matrix form as

$$\begin{pmatrix} \underline{u}_{n+1}(\alpha) \\ \bar{u}_{n+1}(\alpha) \end{pmatrix} = \begin{pmatrix} 0 & -\mu \\ -\mu & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_n(\alpha) \\ \bar{u}_n(\alpha) \end{pmatrix} + \begin{pmatrix} \underline{b}(\alpha) \\ \bar{b}(\alpha) \end{pmatrix} \tag{4.1.9}$$

Since $\tilde{b}$ is a non-zero fuzzy number, equation (4.1.9) has no trivial equilibrium point.

Let $E_2^1[\underline{x}(\alpha), \bar{x}(\alpha)]$ be the nontrivial equilibrium point of (4.1.9), then $\underline{x}(\alpha) = \frac{\underline{b}(\alpha)-\mu\bar{b}(\alpha)}{1-\mu^2}$ and $\bar{x}(\alpha) = \frac{\bar{b}(\alpha)-\mu\underline{b}(\alpha)}{1-\mu^2}$. This equilibrium point exists if $\mu \neq 1$.

Corollary 4.5 *If $\underline{b}(\alpha) - \mu\bar{b}(\alpha) > 0, \bar{b}(\alpha) - \mu\underline{b}(\alpha) > 0$ and $0 < \mu < 1$ then the equilibrium point E_2^1 is positive.*

Corollary 4.6 *If $\underline{b}(\alpha) - \mu\bar{b}(\alpha) < 0, \bar{b}(\alpha) - \mu\underline{b}(\alpha) < 0$ and $\mu > 1$ then the equilibrium point is E_2^1 positive.*

From the equation (4.1.9), let the co-efficient matrix $A_1 = \begin{pmatrix} 0 & -\mu \\ -\mu & 0 \end{pmatrix}$

Therefore,

$$A_1^n = \begin{cases} \begin{pmatrix} \mu^n & 0 \\ 0 & \mu^n \end{pmatrix}; & n \text{ is even natural number} \\ \begin{pmatrix} 0 & -\mu^n \\ -\mu^n & 0 \end{pmatrix}; & n \text{ is odd natural number} \end{cases}.$$

Therefore, solution of (4.1.6), using theorem (3.1), is given by

$$\begin{pmatrix} \underline{u}_n(\alpha) \\ \bar{u}_n(\alpha) \end{pmatrix} = A_1^n \begin{pmatrix} \underline{u}_0(\alpha) \\ \bar{u}_0(\alpha) \end{pmatrix} + \sum_{j=0}^{n-1} A_1^j \begin{pmatrix} \underline{b}(\alpha) \\ \bar{b}(\alpha) \end{pmatrix} \tag{4.1.10}$$

When n is even natural number, then (4.1.10) gives

$$\underline{u}_n(\alpha) = \mu^n \underline{u}_0(\alpha) + \underline{b}(\alpha)\left(1 + \mu^2 + \mu^4 + \cdots + \mu^{n-2}\right) - \bar{b}(\alpha)\left(\mu + \mu^3 + \cdots + \mu^{n-1}\right)$$

$$\bar{u}_n(\alpha) = \mu^n \bar{u}_0(\alpha) + \bar{b}(\alpha)\left(1 + \mu^2 + \mu^4 + \cdots + \mu^{n-2}\right) - \underline{b}(\alpha)\left(\mu + \mu^3 + \cdots + \mu^{n-1}\right)$$

Which implies,
$$\underline{u}_n(\alpha) = \mu^n \underline{u}_0(\alpha) + \frac{\underline{b}(\alpha) - \mu\bar{b}(\alpha)}{1-\mu^2}(1-\mu^n) \tag{4.1.11}$$

$$\bar{u}_n(\alpha) = \mu^n \bar{u}_0(\alpha) + \frac{\bar{b}(\alpha) - \mu\underline{b}(\alpha)}{1-\mu^2}(1-\mu^n) \tag{4.1.12}$$

When n is odd natural number, then from (4.1.10) we have

$$\underline{u}_n(\alpha) = -\mu^n \bar{u}_0(\alpha) + \underline{b}(\alpha)\left(1 + \mu^2 + \mu^4 + \cdots + \mu^{n-1}\right) - \bar{b}(\alpha)\left(\mu + \mu^3 + \cdots + \mu^{n-2}\right)$$

$$\bar{u}_n(\alpha) = -\mu^n \underline{u}_0(\alpha) + \bar{b}(\alpha)\left(1 + \mu^2 + \mu^4 + \cdots + \mu^{n-1}\right) - \underline{b}(\alpha)\left(\mu + \mu^3 + \cdots + \mu^{n-2}\right)$$

That implies,

$$\underline{u}_n(\alpha) = -\mu^n \bar{u}_0(\alpha) + \frac{\underline{b}(\alpha)}{1-\mu^2}\left(1-\mu^{n+1}\right) - \frac{\mu\bar{b}(\alpha)}{1-\mu^2}\left(1-\mu^{n-1}\right) \tag{4.1.13}$$

$$\bar{u}_n(\alpha) = -\mu^n \underline{u}_0(\alpha) + \frac{\bar{b}(\alpha)}{1-\mu^2}\left(1-\mu^{n+1}\right) - \frac{\mu\underline{b}(\alpha)}{1-\mu^2}\left(1-\mu^{n-1}\right) \tag{4.1.14}$$

Lemma 4.2 *Consider the in-homogeneous difference equation (4.1), where $a < 0$ and initial value u_0 be a fuzzy number. Suppose $a = -\mu, \mu > 0$ and $\tilde{b}$ is a non-zero fuzzy numbers. Then the nontrivial equilibrium point $E_2^1[\underline{x}(\alpha), \bar{x}(\alpha)]$, where $\underline{x}(\alpha) = \frac{\underline{b}(\alpha)-\mu\bar{b}(\alpha)}{1-\mu^2}$ and $\bar{x}(\alpha) = \frac{\bar{b}(\alpha)-\mu\underline{b}(\alpha)}{1-\mu^2}$, exists if $\mu \neq 1$ and stable positive equilibrium point if $0 < \mu < 1$.*

Proof Taking equations (4.1.10) and (4.1.11), for n is even and equations (4.1.12) and (4.1.13) for n is odd, we have, $\underline{u}_n(\alpha) \to \frac{\underline{b}(\alpha)-\mu\bar{b}(\alpha)}{1-\mu^2}$ and $\bar{u}_n(\alpha) \to \frac{\bar{b}(\alpha)-\mu\underline{b}(\alpha)}{1-\mu^2}$ as $n \to \infty$ if $0 < \mu < 1$.

Let u_n be a sequence of solutions of equation (4.1) under sub-section 4.1c such that $[u_n]_\alpha = [\underline{u}_n(\alpha), \bar{u}_n(\alpha)]$ and u be fuzzy equilibrium point with it's α-cut $[u]_\alpha = [\underline{u}(\alpha), \bar{u}(\alpha)]$.

For equilibrium point E_2^1, $\underline{u}(\alpha) = \frac{\underline{b}(\alpha)-\mu\bar{b}(\alpha)}{1-\mu^2}$, $\bar{u}(\alpha) = \frac{\bar{b}(\alpha)-\mu\underline{b}(\alpha)}{1-\mu^2}$. So we have, $\lim_{n\to\infty} D(u_n, u) = \lim_{n\to\infty} \sup max\{|\underline{u}_n(\alpha) - \underline{u}(\alpha)|, |\bar{u}_n(\alpha) - \bar{u}(\alpha)|\} = 0$ for all $\alpha \in [0,1]$, when $0 < \mu < 1$.

Therefore, fuzzy equilibrium point E_2^1 is stable positive equilibrium point if $0 < \mu < 1$.

4.1d *Study when $a > 0$ be a fuzzy number and initial value u_0 is a crisp number*:

Let, $[a]_\alpha = [\underline{a}(\alpha), \bar{a}(\alpha)]$, $\forall \alpha \in [0,1]$. Then from equation (4.1.1)

$$\left[\underline{u}_{n+1}(\alpha), \bar{u}_{n+1}(\alpha)\right] = [\underline{a}(\alpha), \bar{a}(\alpha)][\underline{u}_n(\alpha), \bar{u}_n(\alpha)] + [\underline{b}(\alpha), \bar{b}(\alpha)] \tag{4.1.15}$$

$$\text{Therefore, } \underline{u}_{n+1}(\alpha) = \underline{a}(\alpha)\underline{u}_n(\alpha) + \underline{b}(\alpha) \tag{4.1.16}$$

$$\bar{u}_{n+1}(\alpha) = \bar{a}(\alpha)\bar{u}_n(\alpha) + \bar{b}(\alpha) \tag{4.1.17}$$

Now solution of (4.1.16) and (4.1.17) are given by

$$\underline{u}_n(\alpha) = (\underline{a}(\alpha))^n u_0 + \underline{b}(\alpha)\left(\frac{1-(\underline{a}(\alpha))^n}{1-\underline{a}(\alpha)}\right) \tag{4.1.18}$$

$$\bar{u}_n(\alpha) = (\bar{a}(\alpha))^n u_0 + \bar{b}(\alpha)\left(\frac{1-(\bar{a}(\alpha))^n}{1-\bar{a}(\alpha)}\right) \tag{4.1.19}$$

The nontrivial equilibrium point of (4.1.15) is $E_3^1\left[\frac{\underline{b}(\alpha)}{1-\underline{a}(\alpha)}, \frac{\bar{b}(\alpha)}{1-\bar{a}(\alpha)}\right]$. This equilibrium point exist if $1 \notin [\underline{a}(\alpha), \bar{a}(\alpha)]$ and valid α-cut representation.

Corollary 4.7 *The equilibrium point is E_3^1 positive if $\tilde{b}$ is positive fuzzy number and $0<\underline{a}(\alpha)\le\bar{a}(\alpha)<1$.*

Corollary 4.8 *The equilibrium point is E_3^1 positive if $\tilde{b}$ is negative fuzzy number and $\bar{a}(\alpha)\ge\underline{a}(\alpha)>1$ and the equilibrium point is $\left[\frac{\bar{b}(\alpha)}{1-\bar{a}(\alpha)}, \frac{\underline{b}(\alpha)}{1-\underline{a}(\alpha)}\right]$.*

Lemma 4.3 *Consider the in-homogeneous difference equation (4.1), where $a>0$ a fuzzy number and u_0 be a crisp number, $\tilde{b}$ is non-zero fuzzy number. Then the positive fuzzy equilibrium point $E_3^1\left[\frac{\underline{b}(\alpha)}{1-\underline{a}(\alpha)}, \frac{\bar{b}(\alpha)}{1-\bar{a}(\alpha)}\right]$ is stable if $0<\underline{a}(\alpha)<\bar{a}(\alpha)<1$.*

Proof It is evident from equation (4.1.18) and (4.1.19),

$$\underline{u}_n(\alpha) \to \frac{\underline{b}(\alpha)}{1-\underline{a}(\alpha)} \text{ as } n\to\infty \text{ if } |\underline{a}(\alpha)|<1$$

$$\bar{u}_n(\alpha) \to \frac{\bar{b}(\alpha)}{1-\bar{a}(\alpha)} \text{ as } n\to\infty \text{ if } |\bar{a}(\alpha)|<1$$

Therefore, both solutions are convergent if $0<\underline{a}(\alpha)<\bar{a}(\alpha)<1$.

Let u_n be a sequence of solution of equation (4.1.15) such that $[u_n]_\alpha = [\underline{u}_n(\alpha), \bar{u}_n(\alpha)]$ and u be fuzzy equilibrium point with it's α-cut $[u]_\alpha = [\underline{u}(\alpha), \bar{u}(\alpha)]$.

For equilibrium point E_3^1, $\underline{u}(\alpha) = \frac{\underline{b}(\alpha)}{1-\underline{a}(\alpha)}, \bar{u}(\alpha) = \frac{\bar{b}(\alpha)}{1-\bar{a}(\alpha)}$. So we have, $\lim_{n\to\infty} D(u_n, u) = \lim_{n\to\infty} \sup max\{|\underline{u}_n(\alpha) - \underline{u}(\alpha)|, |\bar{u}_n(\alpha) - \bar{u}(\alpha)|\} = 0$ for all $\alpha\in[0,1]$, if $0<\underline{a}(\alpha)<\bar{a}(\alpha)<1$.

Therefore, fuzzy equilibrium point E_3^1 is stable positive equilibrium point if

$$0<\underline{a}(\alpha)<\bar{a}(\alpha)<1.$$

4.1e *Study when $a<0$ be a fuzzy number and the initial value u_0 is a crisp number*:

Let $a=-\mu$, where μ is a positive fuzzy number. $[\mu]_\alpha = \left[\underline{\mu}(\alpha), \bar{\mu}(\alpha)\right]$.

Then the equation (4.1.1) reduces to the form

$$\left[\underline{u}_{n+1}(\alpha), \bar{u}_{n+1}(\alpha)\right] = -\left[\underline{\mu}(\alpha), \bar{\mu}(\alpha)\right]\left[\underline{u}_n(\alpha), \bar{u}_n(\alpha)\right] + \left[\underline{b}(\alpha), \bar{b}(\alpha)\right] \tag{4.1.20}$$

$$\text{This implies, } \quad \underline{u}_{n+1}(\alpha) = -\bar{\mu}(\alpha)\bar{u}_n(\alpha) + \underline{b}(\alpha) \tag{4.1.21}$$

$$\text{And } \bar{u}_{n+1}(\alpha) = -\underline{\mu}(\alpha)\underline{u}_n(\alpha) + \bar{b}(\alpha) \tag{4.1.22}$$

In matrix form, equations (4.1.21) and (4.1.22) can be written as

$$\begin{pmatrix}\underline{u}_{n+1}(\alpha)\\ \bar{u}_{n+1}(\alpha)\end{pmatrix} = \begin{pmatrix}0 & -\bar{\mu}(\alpha)\\ -\underline{\mu}(\alpha) & 0\end{pmatrix}\begin{pmatrix}\underline{u}_n(\alpha)\\ \bar{u}_n(\alpha)\end{pmatrix} + \begin{pmatrix}\underline{b}(\alpha)\\ \bar{b}(\alpha)\end{pmatrix} \tag{4.1.23}$$

The fuzzy equilibrium point of (4.1.21) and (4.1.22) is $E_4^1\left[\frac{\underline{b}(\alpha)-\bar{b}(\alpha)\bar{\mu}(\alpha)}{1-\underline{\mu}(\alpha)\bar{\mu}(\alpha)}, \frac{\bar{b}(\alpha)-\underline{b}(\alpha)\underline{\mu}(\alpha)}{1-\underline{\mu}(\alpha)\bar{\mu}(\alpha)}\right]$ and exist if $\underline{\mu}(\alpha)\bar{\mu}(\alpha)\neq 1$ and for a valid α-cut representation.

Corollary 4.9 *The equilibrium point is E_4^1 positive equilibrium point if*

$$\underline{b}(\alpha) - \bar{b}(\alpha)\bar{\mu}(\alpha) > 0, \bar{b}(\alpha) - \underline{b}(\alpha)\underline{\mu}(\alpha) > 0 \text{ and } \underline{\mu}(\alpha)\bar{\mu}(\alpha)<1.$$

i.e; $\frac{\underline{b}(\alpha)}{\bar{b}(\alpha)} > \bar{\mu}(\alpha)$, $\frac{\bar{b}(\alpha)}{\underline{b}(\alpha)} > \underline{\mu}(\alpha)$ and $\underline{\mu}(\alpha)\bar{\mu}(\alpha)<1$.

Corollary 4.10 *The equilibrium point is negative equilibrium point if*

$$\underline{b}(\alpha) - \bar{b}(\alpha)\bar{\mu}(\alpha)<0, \bar{b}(\alpha) - \underline{b}(\alpha)\underline{\mu}(\alpha)<0 \text{ and } \underline{\mu}(\alpha)\bar{\mu}(\alpha)>1$$

i.e., $\frac{\underline{b}(\alpha)}{\bar{b}(\alpha)}<\bar{\mu}(\alpha)$, $\frac{\bar{b}(\alpha)}{\underline{b}(\alpha)}<\underline{\mu}(\alpha)$ and $\underline{\mu}(\alpha)\bar{\mu}(\alpha)>1$ and the equilibrium point is $\left[\frac{\bar{b}(\alpha)-\underline{b}(\alpha)\underline{\mu}(\alpha)}{1-\underline{\mu}(\alpha)\bar{\mu}(\alpha)}, \frac{\underline{b}(\alpha)-\bar{b}(\alpha)\bar{\mu}(\alpha)}{1-\underline{\mu}(\alpha)\bar{\mu}(\alpha)}\right]$.

Solution of (4.1.20) is given by

$$\begin{pmatrix}\underline{u}_n(\alpha)\\ \bar{u}_n(\alpha)\end{pmatrix} = A_2^n\begin{pmatrix}u_0\\ u_0\end{pmatrix} + \sum_{j=0}^{n-1} A_2^j\begin{pmatrix}\underline{b}(\alpha)\\ \bar{b}(\alpha)\end{pmatrix} \tag{4.1.24}$$

Where, $A_2 = \begin{pmatrix}0 & -\bar{\mu}(\alpha)\\ -\underline{\mu}(\alpha) & 0\end{pmatrix}$ and

$$A_2^n = \begin{cases}\begin{pmatrix}\left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n}{2}} & 0\\ 0 & \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n}{2}}\end{pmatrix}; & when\, n\, is\, even\\ \begin{pmatrix}0 & -\left(\underline{\mu}(\alpha)\right)^{\frac{n-1}{2}}\left(\bar{\mu}(\alpha)\right)^{\frac{n+1}{2}}\\ -\left(\underline{\mu}(\alpha)\right)^{\frac{n+1}{2}}\left(\bar{\mu}(\alpha)\right)^{\frac{n-1}{2}} & 0\end{pmatrix}; & when\, n\, is\, odd\end{cases}$$

Now solution of equation (4.1.24), when n is even is

$$\underline{u}_n(\alpha) = \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n}{2}} u_0 + d_1(\alpha)\underline{b}(\alpha) - d_2(\alpha)\bar{b}(\alpha) \tag{4.1.25}$$

$$\bar{u}_n(\alpha) = \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n}{2}} u_0 + d_1(\alpha)\bar{b}(\alpha) - d_3(\alpha)\underline{b}(\alpha) \tag{4.1.26}$$

Where, $d_1(\alpha) = 1 + \underline{\mu}(\alpha)\bar{\mu}(\alpha) + \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^2 + \cdots + \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n-2}{2}}$

$$d_2(\alpha) = \bar{\mu}(\alpha) + (\bar{\mu}(\alpha))^2\underline{\mu}(\alpha) + \cdots + (\bar{\mu}(\alpha))^{\frac{n}{2}}\left(\underline{\mu}(\alpha)\right)^{\frac{n-2}{2}}$$

$$d_3(\alpha) = \underline{\mu}(\alpha) + \left(\underline{\mu}(\alpha)\right)^2\bar{\mu}(\alpha) + \cdots + (\bar{\mu}(\alpha))^{\frac{n-2}{2}}\left(\underline{\mu}(\alpha)\right)^{\frac{n}{2}}$$

This implies,

$$\underline{u}_n(\alpha) = \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n}{2}} u_0 + \frac{\underline{b}(\alpha) - \bar{b}(\alpha)\bar{\mu}(\alpha)}{1 - \underline{\mu}(\alpha)\bar{\mu}(\alpha)}\left(1 - \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n}{2}}\right) \tag{4.1.27}$$

$$\bar{u}_n(\alpha) = \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n}{2}} u_0 + \frac{\bar{b}(\alpha) - \underline{b}(\alpha)\underline{\mu}(\alpha)}{1 - \underline{\mu}(\alpha)\bar{\mu}(\alpha)}\left(1 - \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n}{2}}\right) \tag{4.1.28}$$

Solution of equation (4.1.24), when n is odd

$$\underline{u}_n(\alpha) = -\left(\underline{\mu}(\alpha)\right)^{\frac{n-1}{2}}(\bar{\mu}(\alpha))^{\frac{n+1}{2}} u_0 + d_4(\alpha)\underline{b}(\alpha) - d_5(\alpha)\bar{b}(\alpha) \tag{4.1.29}$$

$$\bar{u}_n(\alpha) = -\left(\underline{\mu}(\alpha))\right)^{\frac{n+1}{2}}(\bar{\mu}(\alpha))^{\frac{n-1}{2}} u_0 + d_4(\alpha)\bar{b}(\alpha) - d_6(\alpha)\underline{b}(\alpha) \tag{4.1.30}$$

Where, $d_4(\alpha) = 1 + \underline{\mu}(\alpha)\bar{\mu}(\alpha) + \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^2 + \cdots + \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n-1}{2}}$

$$d_5(\alpha) = \bar{\mu}(\alpha) + (\bar{\mu}(\alpha))^2\underline{\mu}(\alpha) + \cdots + (\bar{\mu}(\alpha))^{\frac{n-1}{2}}(\underline{\mu}(\alpha))^{\frac{n-3}{2}}$$

$$d_6(\alpha) = \underline{\mu}(\alpha) + \left(\underline{\mu}(\alpha)\right)^2\bar{\mu}(\alpha) + \cdots + (\bar{\mu}(\alpha))^{\frac{n-3}{2}}(\underline{\mu}(\alpha))^{\frac{n-1}{2}}$$

This implies,

$$\underline{u}_n(\alpha) = -\left(\underline{\mu}(\alpha)\right)^{\frac{n-1}{2}}(\bar{\mu}(\alpha))^{\frac{n+1}{2}} u_0 + \underline{b}(\alpha)\left(\frac{1 - \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n+1}{2}}}{1 - \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)}\right) - \bar{b}(\alpha)\bar{\mu}(\alpha)\left(\frac{1 - \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n-1}{2}}}{1 - \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)}\right) \tag{4.1.31}$$

$$\bar{u}_n(\alpha) = -\left(\underline{\mu}(\alpha))\right)^{\frac{n+1}{2}}(\bar{\mu}(\alpha))^{\frac{n-1}{2}} u_0 + \bar{b}(\alpha)\left(\frac{1 - \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n+1}{2}}}{1 - \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)}\right) - \underline{b}(\alpha)\underline{\mu}(\alpha)\left(\frac{1 - \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n-1}{2}}}{1 - \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)}\right). \tag{4.1.32}$$

Lemma 4.4 *Consider the in-homogeneous difference equation (4.1), where $a<0$ is a fuzzy number and initial value u_0 be a crisp number. Let $a = -\mu, \mu > 0$ and $\tilde{b}$ is a non-zero fuzzy numbers. Then the non-trivial equilibrium point $E_4^1\left[\frac{\underline{b}(\alpha)-\bar{b}(\alpha)\bar{\mu}(\alpha)}{1-\underline{\mu}(\alpha)\bar{\mu}(\alpha)}, \frac{\bar{b}(\alpha)-\underline{b}(\alpha)\underline{\mu}(\alpha)}{1-\underline{\mu}(\alpha)\bar{\mu}(\alpha)}\right]$ is stable positive equilibrium point if $0<\left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)<1$, for all $\alpha \in [0, 1]$.*

Proof Using equations (4.1.27) and (4.1.28) for n is even and equations (4.1.31), (4.1.32) for n is odd positive integer numbers,

$$\underline{u}_n(\alpha) \to \frac{\underline{b}(\alpha) - \bar{b}(\alpha)\bar{\mu}(\alpha)}{1 - \underline{\mu}(\alpha)\bar{\mu}(\alpha)}, \bar{u}_n(\alpha) \to \frac{\bar{b}(\alpha) - \underline{b}(\alpha)\underline{\mu}(\alpha)}{1 - \underline{\mu}(\alpha)\bar{\mu}(\alpha)} \text{ as } n \to \infty \text{ if } 0<\left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)<1.$$

Let u_n be a sequence of solution of equation (4.1.20) such that $[u_n]_\alpha = [\underline{u}_n(\alpha), \bar{u}_n(\alpha)]$ and u be a fuzzy equilibrium point with α-cut $[u]_\alpha = [\underline{u}(\alpha), \bar{u}(\alpha)]$, $\alpha \in [0, 1]$.

For equilibrium point E_4^1, $\underline{u}(\alpha) = \frac{\underline{b}(\alpha)-\bar{b}(\alpha)\bar{\mu}(\alpha)}{1-\underline{\mu}(\alpha)\bar{\mu}(\alpha)}$, $\bar{u}(\alpha) = \frac{\bar{b}(\alpha)-\underline{b}(\alpha)\underline{\mu}(\alpha)}{1-\underline{\mu}(\alpha)\bar{\mu}(\alpha)}$. So we have, $\lim_{n\to\infty} D(u_n, u) = \lim_{n\to\infty} \sup max\{|\underline{u}_n(\alpha) - \underline{u}(\alpha)|, |\bar{u}_n(\alpha) - \bar{u}(\alpha)|\} = 0$ for all $\alpha \in [0, 1]$, if $0<\left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)<1$.

Therefore, fuzzy equilibrium point E_4^1 is stable positive equilibrium point if

$$0<\left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)<1, \quad \text{for all } \alpha \in [0, 1].$$

Note 4.2 The equilibrium point E_4^1 is unstable equilibrium point if $\left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right) > 1$.

4.1f *Study when $a > 0$ and u_0 are both fuzzy number*:

Let α-levels of a is $= [\underline{a}(\alpha), \bar{a}(\alpha)]$, $\forall \alpha \in [0, 1]$ and $[u_{n=0}]_\alpha = [\underline{u}_0(\alpha), \bar{u}_0(\alpha)]$.

Then the solution of equation (4.1.1), which are followed from equation (4.1.18) and (4.1.19), are given by

$$\underline{u}_n(\alpha) = (\underline{a}(\alpha))^n \underline{u}_0(\alpha) + \underline{b}(\alpha)\left(\frac{1-(\underline{a}(\alpha))^n}{1-\underline{a}(\alpha)}\right) \quad (4.1.33)$$

$$\bar{u}_n(\alpha) = (\bar{a}(\alpha))^n \bar{u}_0(\alpha) + \bar{b}(\alpha)\left(\frac{1-(\bar{a}(\alpha))^n}{1-\bar{a}(\alpha)}\right). \quad (4.1.34)$$

Remarks 4.2 The convergence condition, equilibrium points and positivity conditions and stability condition are same as equation (4.1.15).

4.1g *Study when $a<0$ and u_0 are both fuzzy numbers*:

Let $a=-\mu, \mu>0$. The α-level cut of μ is $[\mu]_\alpha = \left[\underline{\mu}(\alpha), \bar{\mu}(\alpha)\right]$, $[u_{n=0}]_\alpha = [\underline{u}_0(\alpha), \bar{u}_0(\alpha)]$.

Then the solution of equation (4.1.1) which are followed from equations (4.1.27) and (4.1.28), are given by

$$\underline{u}_n(\alpha) = \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n}{2}} \underline{u}_0(\alpha) + \frac{\underline{b}(\alpha) - \bar{b}(\alpha)\bar{\mu}(\alpha)}{1-\underline{\mu}(\alpha)\bar{\mu}(\alpha)}\left(1-\left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n}{2}}\right) \quad (4.1.35)$$

$$\bar{u}_n(\alpha) = \left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n}{2}} \bar{u}_0(\alpha)] + \frac{\bar{b}(\alpha) - \underline{b}(\alpha)\underline{\mu}(\alpha)}{1-\underline{\mu}(\alpha)\bar{\mu}(\alpha)}\left(1-\left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n}{2}}\right) \quad (4.1.36)$$

The above two equations show the solution for n is even only. When n is odd the solutions which are followed from equations (4.1.31) and (4.1.32),

This implies,

$$\underline{u}_n(\alpha) = -\left(\underline{\mu}(\alpha)\right)^{\frac{n-1}{2}} (\bar{\mu}(\alpha))^{\frac{n+1}{2}} \underline{u}_0(\alpha) + \underline{b}(\alpha)\left(\frac{1-\left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n+1}{2}}}{1-\left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)}\right) - \bar{b}(\alpha)\bar{\mu}(\alpha)\left(\frac{1-\left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n-1}{2}}}{1-\left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)}\right) \quad (4.1.37)$$

$$\bar{u}_n(\alpha) = -\left(\underline{\mu}(\alpha))\right)^{\frac{n+1}{2}} (\bar{\mu}(\alpha))^{\frac{n-1}{2}} \bar{u}_0(\alpha) + \bar{b}(\alpha)\left(\frac{1-\left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n+1}{2}}}{1-\left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)}\right) - \underline{b}(\alpha)\underline{\mu}(\alpha)\left(\frac{1-\left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)^{\frac{n-1}{2}}}{1-\left(\underline{\mu}(\alpha)\bar{\mu}(\alpha)\right)}\right). \quad (4.1.38)$$

Remarks 4.3 In this case equilibrium points, positivity of equilibrium points and stability condition are same as equation (4.1.20).

4.2 *Solution and stability analysis of non-homogeneous difference equation of Type II*

Taking α-levels of equation (4.2), reduces to the form

$$\left[\underline{u}_{n+1}(\alpha), \bar{u}_{n+1}(\alpha)\right] - a[\underline{u}_n(\alpha), \bar{u}_n(\alpha)] = [\underline{b}(\alpha), \bar{b}(\alpha)]. \quad (4.2.1)$$

4.2a *Study when $a=1$ and initial condition u_0 be a fuzzy number*:

Then from equation (4.2.1) we get

$$\underline{u}_{n+1}(\alpha) = \bar{u}_n(\alpha) + \underline{b}\,(\alpha) \quad (4.2.2)$$

$$\bar{u}_{n+1}(\alpha) = \underline{u}_n(\alpha) + \bar{b}(\alpha) \quad (4.2.3)$$

In matrix form, the above two equations can be written as

$$\begin{pmatrix} \underline{u}_{n+1}(\alpha) \\ \bar{u}_{n+1}(\alpha) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \underline{u}_n(\alpha) \\ \bar{u}_n(\alpha) \end{pmatrix} + \begin{pmatrix} \underline{b}(\alpha) \\ \bar{b}(\alpha) \end{pmatrix} \quad (4.2.4)$$

Since, $\tilde{b} \neq \tilde{0}$ fuzzy number, so the equation (4.2.4) has no equilibrium point.

The solution of equations (4.2.2) and (4.2.3) are, when n is even,

$$\underline{u}_n(\alpha) = \underline{u}_0(\alpha) + \frac{n}{2}(\underline{b}(\alpha) + \bar{b}(\alpha)) \quad (4.2.5)$$

$$\bar{u}_n(\alpha) = \bar{u}_0(\alpha) + \frac{n}{2}(\underline{b}(\alpha) + \bar{b}(\alpha)) \quad (4.2.6)$$

When n is odd the solutions are

$$\underline{u}_n(\alpha) = \bar{u}_0(\alpha) + \frac{n+1}{2}\underline{b}(\alpha) + \frac{n-1}{2}\bar{b}(\alpha) \quad (4.2.7)$$

$$\bar{u}_n(\alpha) = \underline{u}_0(\alpha) + \frac{n-1}{2}\underline{b}(\alpha) + \frac{n+1}{2}\bar{b}(\alpha) \quad (4.2.8)$$

For both cases n is even and odd, $\underline{u}_n(\alpha)$ and $\bar{u}_n(\alpha)$ leads a divergent solution.

4.2c *Study when $a>0$, a real valued number and initial condition u_0 be a fuzzy number*:

Then, from equation (4.2.1), we get the pair of equations,

$$\underline{u}_{n+1}(\alpha) - a\bar{u}_n(\alpha) = \underline{b}(\alpha) \quad (4.2.9)$$

$$\bar{u}_{n+1}(\alpha) - a\underline{u}_n(\alpha) = \bar{b}(\alpha) \quad (4.2.10)$$

In matrix form (4.2.9) and (4.2.10) can be written as

$$\begin{pmatrix} \underline{u}_{n+1}(\alpha) \\ \bar{u}_{n+1}(\alpha) \end{pmatrix} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\begin{pmatrix} \underline{u}_n(\alpha) \\ \bar{u}_n(\alpha) \end{pmatrix} + \begin{pmatrix} \underline{b}(\alpha) \\ \bar{b}(\alpha) \end{pmatrix} \quad (4.2.11)$$

Equilibrium point of (4.2.11) is $E_1^2\left[\frac{\underline{b}(\alpha)}{1-a}, \frac{\bar{b}(\alpha)}{1-a}\right]$, exist if $a \neq 1$.

Corollary 4.11 *The equilibrium point E_1^2 is positive equilibrium point if $0<a<1$ and $\tilde{b}$ is positive fuzzy number.*

Corollary 4.12 *The equilibrium point is positive equilibrium point if $a > 1$ and $\tilde{b}$ is negative fuzzy number and the point is* $\left[\frac{\bar{b}(\alpha)}{1-a}, \frac{\underline{b}(\alpha)}{1-a}\right]$.

Let the matrix $A_4 = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$, then

$$A_4^n = \begin{cases} \begin{pmatrix} a^n & 0 \\ 0 & a^n \end{pmatrix}; & \text{if } n \text{ is even} \\ \begin{pmatrix} 0 & a^n \\ a^n & 0 \end{pmatrix}; & \text{if } n \text{ is odd} \end{cases}$$

The solutions of (4.2.9) and (4.2.10) are, when n is even,

$$\begin{aligned} \underline{u}_n(\alpha) &= a^n \underline{u}_0(\alpha) + \left(1 + a^2 + a^4 + \cdots + a^{n-2}\right)\underline{b}(\alpha) \\ &\quad + \left(a + a^3 + \cdots + a^{n-1}\right)\bar{b}(\alpha) \\ \bar{u}_n(\alpha) &= a^n \bar{u}_0(\alpha) + \left(a + a^3 + \cdots + a^{n-1}\right)\underline{b}(\alpha) \\ &\quad + \left(1 + a^2 + a^4 + \cdots + a^{n-2}\right)\bar{b}(\alpha) \end{aligned}$$

That implies, $$\underline{u}_n(\alpha) = a^n \underline{u}_0(\alpha) + \frac{\underline{b}(\alpha) + a\bar{b}(\alpha)}{1 - a^2}(1 - a^n) \tag{4.2.12}$$

$$\bar{u}_n(\alpha) = a^n \bar{u}_0(\alpha) + \frac{\bar{b}(\alpha) + a\underline{b}(\alpha)}{1 - a^2}(1 - a^n) \tag{4.2.13}$$

The solutions of (4.2.9) and (4.2.10) are, when$n$ is odd,

$$\begin{aligned} \underline{u}_n(\alpha) &= a^n \bar{u}_0(\alpha) + \left(1 + a^2 + a^4 + \cdots + a^{n-1}\right)\underline{b}(\alpha) \\ &\quad + \left(a + a^3 + \cdots + a^{n-2}\right)\bar{b}(\alpha) \\ \bar{u}_n(\alpha) &= a^n \underline{u}_0(\alpha) + \left(a + a^3 + \cdots + a^{n-2}\right)\underline{b}(\alpha) \\ &\quad + \left(1 + a^2 + a^4 + \cdots + a^{n-1}\right)\bar{b}(\alpha) \end{aligned}$$

That implies,

$$\underline{u}_n(\alpha) = a^n \bar{u}_0(\alpha) + \frac{\underline{b}(\alpha)}{1 - a^2}\left(1 - a^{n+1}\right) + \frac{a\bar{b}(\alpha)}{1 - a^2}\left(1 - a^{n-1}\right) \tag{4.2.14}$$

$$\bar{u}_n(\alpha) = a^n \underline{u}_0(\alpha) + \frac{\bar{b}(\alpha)}{1 - a^2}\left(1 - a^{n+1}\right) + \frac{a\underline{b}(\alpha)}{1 - a^2}\left(1 - a^{n-1}\right). \tag{4.2.15}$$

Lemma 4.5 *Consider the in-homogeneous difference equation* (4.2)*, where* $a > 0$*, a real valued number and initial condition* u_0 *be a fuzzy number. Then the positive fuzzy equilibrium point* $E_1^2\left[\frac{\underline{b}(\alpha)}{1-a}, \frac{\bar{b}(\alpha)}{1-a}\right]$*, exists if* $a \neq 1$ *and unstable positive equilibrium point if* $0 < a < 1$.

Proof Clearly, from equations (4.2.12) and (4.2.13), for n is even and equations (4.2.14) and (4.2.15), for n is odd, we have, $\underline{u}_n(\alpha) \to \frac{\underline{b}(\alpha)+a\bar{b}(\alpha)}{1-a^2}$ and $\bar{u}_n(\alpha) \to \frac{\bar{b}(\alpha)+a\underline{b}(\alpha)}{1-a^2}$ as $n \to \infty$ if $0 < a < 1$, for all $\alpha \in [0, 1]$.

Let, u_n be a sequence of solution of equation (4.2) under sub section 4.2b such that $[u_n]_\alpha = [\underline{u}_n(\alpha), \bar{u}_n(\alpha)]$ and u be a fuzzy equilibrium point with it's α-cut $[u]_\alpha = [\underline{u}(\alpha), \bar{u}(\alpha)]$.

For equilibrium point E_1^2, $\underline{u}(\alpha) = \frac{\underline{b}(\alpha)}{1-a}, \bar{u}(\alpha) = \frac{\bar{b}(\alpha)}{1-a}$. So we have, $\lim_{n\to\infty} D(u_n, u) = \lim_{n\to\infty} \sup max\{|\underline{u}_n(\alpha) - \underline{u}(\alpha)|, |\bar{u}_n(\alpha) - \bar{u}(\alpha)|\} \neq 0$, for all $\alpha \in [0, 1]$, if $0 < a < 1$. Therefore, fuzzy equilibrium point E_1^2 is unstable positive equilibrium point if $0 < a < 1$, for all $\alpha \in [0, 1]$.

Remarks 4.4 If $a > 1$, for both n is even and odd, $\underline{u}_n(\alpha)$ and $\bar{u}_n(\alpha)$, gives a divergent solution. Similarly, we can show that equilibrium point $\left[\frac{\bar{b}(\alpha)}{1-a}, \frac{\underline{b}(\alpha)}{1-a}\right]$ is unstable positive equilibrium point.

4.2c *Study when* $a < 0$ *and initial condition* u_0 *is a fuzzy number*:

Let $a = -m, m > 0$

Then from the equation (4.2.1), we have the pair of equations

$$\underline{u}_{n+1}(\alpha) + m\underline{u}_n(\alpha) = \underline{b}(\alpha) \tag{4.2.16}$$

$$\bar{u}_{n+1}(\alpha) + m\bar{u}_n(\alpha) = \bar{b}(\alpha) \tag{4.2.17}$$

Equilibrium point of (4.2.16) and (4.2.17) is $E_2^2\left[\frac{\underline{b}(\alpha)}{1+m}, \frac{\bar{b}(\alpha)}{1+m}\right]$. Since, $m > 0$, the equilibrium point always exists.

Corollary 4.1.3 *The equilibrium point is positive if $\tilde{b}$ is positive fuzzy number and negative equilibrium point if $\tilde{b}$ is negative fuzzy number.*

Solving (4.2.16) *and* (4.2.17)*, we get,*

$$\underline{u}_n(\alpha) = (-m)^n \underline{u}_0(\alpha) + \underline{b}(\alpha)\left(\frac{1 - (-m)^n}{1 + m}\right) \tag{4.2.18}$$

$$\bar{u}_n(\alpha) = (-m)^n \bar{u}_0(\alpha) + \bar{b}(\alpha)\left(\frac{1 - (-m)^n}{1 + m}\right). \tag{4.2.19}$$

Lemma 4.6 *Consider the in-homogeneous difference equation* (4.2)*, where* $a < 0$*, a real valued number and initial condition* u_0 *be a fuzzy number. Then the positive fuzzy equilibrium point* $E_2^2\left[\frac{\underline{b}(\alpha)}{1+m}, \frac{\bar{b}(\alpha)}{1+m}\right]$ *is stable positive equilibrium point if* $0 < m < 1$*, for all* $\alpha \in [0, 1]$.

Proof The proof of this lemma is obvious.

4.2d *Study when* $a > 0$ *be a positive fuzzy number and initial condition* u_0 *not be a fuzzy number*:

Let, the α-levels of a is given by $[a]_\alpha = [\underline{a}(\alpha), \bar{a}(\alpha)]$, $\forall \alpha \in [0, 1]$. Then the equation (4.2.1) becomes,

$$\begin{aligned} &\left[\underline{u}_{n+1}(\alpha), \bar{u}_{n+1}(\alpha)\right] - [\underline{a}(\alpha), \bar{a}(\alpha)][\underline{u}_n(\alpha), \bar{u}_n(\alpha)] \\ &\quad = [\underline{b}(\alpha), \bar{b}(\alpha)] \end{aligned} \tag{4.2.20}$$

Which implies, $$\underline{u}_{n+1}(\alpha) = \bar{a}(\alpha)\bar{u}_n(\alpha) + \underline{b}(\alpha) \tag{4.2.21}$$

$$\bar{u}_{n+1}(\alpha) = \underline{a}(\alpha)\underline{u}_n(\alpha) + \bar{b}(\alpha) \tag{4.2.22}$$

In matrix form, equations (4.2.21) and (4.2.22) can be written as

$$\begin{pmatrix} \underline{u}_{n+1}(\alpha) \\ \bar{u}_{n+1}(\alpha) \end{pmatrix} = \begin{pmatrix} 0 & \bar{a}(\alpha) \\ \underline{a}(\alpha) & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_n(\alpha) \\ \bar{u}_n(\alpha) \end{pmatrix} + \begin{pmatrix} \underline{b}(\alpha) \\ \bar{b}(\alpha) \end{pmatrix} \tag{4.2.23}$$

Now, equilibrium point of equation (4.2.23) is $E_3^2\left[\frac{\underline{b}(\alpha)+\bar{b}(\alpha)\bar{a}(\alpha)}{1-\underline{a}(\alpha)\bar{a}(\alpha)}, \frac{\bar{b}(\alpha)+\underline{b}(\alpha)\underline{a}(\alpha)}{1-\underline{a}(\alpha)\bar{a}(\alpha)}\right]$.

Corollary 4.14 *The equilibrium point is positive if* $\underline{b}(\alpha) + \bar{b}(\alpha)\bar{a}(\alpha) > 0, \bar{b}(\alpha) + \underline{b}(\alpha)\underline{a}(\alpha) > 0$ *and* $\underline{a}(\alpha)\bar{a}(\alpha) < 1$.

Corollary 4.15 *Again the equilibrium point is positive if* $\underline{b}(\alpha) + \bar{b}(\alpha)\bar{a}(\alpha) < 0, \underline{b}(\alpha) + \bar{b}(\alpha)\bar{a}(\alpha) < 0$ *and* $\underline{a}(\alpha)\bar{a}(\alpha) > 1$.

Solution of equation (4.2.23), solving in the similar way as equation (4.1.24)

When n is even:

$$\underline{u}_n(\alpha) = (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}u_0 + k_1(\alpha)\underline{b}(\alpha) + k_2(\alpha)\bar{b}(\alpha)$$
$$\bar{u}_n(\alpha) = (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}u_0 + k_1(\alpha)\bar{b}(\alpha) + k_3(\alpha)\underline{b}(\alpha)$$

Where, $k_1(\alpha) = 1 + \underline{a}(\alpha)\bar{a}(\alpha) + (\underline{a}(\alpha)\bar{a}(\alpha))^2 + \cdots + (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n-2}{2}}$

$$k_2(\alpha) = \bar{a}(\alpha) + (\bar{a}(\alpha))^2\underline{a}(\alpha) + \cdots + (\bar{a}(\alpha))^{\frac{n}{2}}(\underline{a}(\alpha))^{\frac{n-2}{2}}$$
$$k_3(\alpha) = \underline{a}(\alpha) + (\underline{a}(\alpha))^2\bar{a}(\alpha) + \cdots + (\bar{a}(\alpha))^{\frac{n-2}{2}}(\underline{a}(\alpha))^{\frac{n}{2}}$$

This implies,

$$\underline{u}_n(\alpha) = (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}u_0 + \frac{\underline{b}(\alpha) + \bar{b}(\alpha)\bar{a}(\alpha)}{1 - \underline{a}(\alpha)\bar{a}(\alpha)}\left(1 - (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}\right) \tag{4.2.24}$$

$$\bar{u}_n(\alpha) = (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}u_0 + \frac{\bar{b}(\alpha) + \underline{b}(\alpha)\underline{a}(\alpha)}{1 - \underline{a}(\alpha)\bar{a}(\alpha)}\left(1 - (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}\right) \tag{4.2.25}$$

Solution of equation (4.2.23), when n is odd,

$$\underline{u}_n(\alpha) = (\underline{a}(\alpha))^{\frac{n-1}{2}}(\bar{a}(\alpha))^{\frac{n+1}{2}}u_0 + k_4(\alpha)\underline{b}(\alpha) + k_5(\alpha)\bar{b}(\alpha) \tag{4.2.26}$$

$$\bar{u}_n(\alpha) = (\underline{a}(\alpha)))^{\frac{n+1}{2}}(\bar{a}(\alpha))^{\frac{n-1}{2}}u_0 + k_4(\alpha)\bar{b}(\alpha) + k_6(\alpha)\underline{b}(\alpha) \tag{4.2.27}$$

Where, $k_4(\alpha) = 1 + \underline{a}(\alpha)\bar{a}(\alpha) + (\underline{a}(\alpha)\bar{a}(\alpha))^2 + \cdots + (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n-1}{2}}$

$$k_5(\alpha) = \bar{a}(\alpha) + (\bar{a}(\alpha))^2\underline{a}(\alpha) + \cdots + (\bar{a}(\alpha))^{\frac{n-1}{2}}(\underline{a}(\alpha))^{\frac{n-3}{2}}$$
$$k_6(\alpha) = \underline{a}(\alpha) + (\underline{a}(\alpha))^2\bar{a}(\alpha) + \cdots + (\bar{a}(\alpha))^{\frac{n-3}{2}}(\underline{a}(\alpha))^{\frac{n-1}{2}}$$

That implies,

$$\underline{u}_n(\alpha) = (\underline{a}(\alpha))^{\frac{n-1}{2}}(\bar{a}(\alpha))^{\frac{n+1}{2}}u_0 + \underline{b}(\alpha)\left(\frac{1 - (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}}{1 - \underline{a}(\alpha)\bar{a}(\alpha)}\right) + \bar{b}(\alpha)\bar{a}\left(\frac{1 - (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n-1}{2}}}{1 - \underline{a}(\alpha)\bar{a}(\alpha)}\right) \tag{4.2.28}$$

$$\bar{u}_n(\alpha) = (\underline{a}(\alpha)))^{\frac{n+1}{2}}(\bar{a}(\alpha))^{\frac{n-1}{2}}u_0 + \bar{b}(\alpha)\left(\frac{1 - (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}}{1 - \underline{a}(\alpha)\bar{a}(\alpha)}\right) + \underline{b}(\alpha)\underline{a}\left(\frac{1 - (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n-1}{2}}}{1 - \underline{a}(\alpha)\bar{a}(\alpha)}\right). \tag{4.2.29}$$

Lemma 4.7 *Consider the in-homogeneous difference equation* (4.2), *where* a > 0, *be a positive fuzzy number and initial condition* u_0 *not be a fuzzy number. Then the positive fuzzy equilibrium point* $E_3^2\left[\frac{\underline{b}(\alpha)+\bar{b}(\alpha)\bar{a}(\alpha)}{1-\underline{a}(\alpha)\bar{a}(\alpha)}, \frac{\bar{b}(\alpha)+\underline{b}(\alpha)\underline{a}(\alpha)}{1-\underline{a}(\alpha)\bar{a}(\alpha)}\right]$ *is stable positive equilibrium point if* $0 < \underline{a}(\alpha)\bar{a}(\alpha) < 1$, *for all* $\alpha \in [0, 1]$ *and if it's* α*-cut representation exist.*

Proof The proof of this lemma is obvious.

4.2e *Study when* $a < 0$ *be a fuzzy number and initial condition* u_0 *not be a fuzzy number*:

Also let $a = -m, m$ be a positive fuzzy number. The α-levels of m is $[m]_\alpha = [\underline{m}(\alpha), \bar{m}(\alpha)]$.

Then the equation (4.2.1) takes the form

$$[\underline{u}_{n+1}(\alpha), \bar{u}_{n+1}(\alpha)] + [\underline{m}(\alpha), \bar{m}(\alpha)][\underline{u}_n(\alpha), \bar{u}_n(\alpha)] = [\underline{b}(\alpha), \bar{b}(\alpha)] \tag{4.2.30}$$

$$\underline{u}_{n+1}(\alpha) = -\underline{m}(\alpha)\underline{u}_n(\alpha) + \underline{b}(\alpha) \tag{4.2.31}$$

$$\bar{u}_{n+1}(\alpha) = -\bar{m}(\alpha)\bar{u}_n(\alpha) + \bar{b}(\alpha) \tag{4.2.32}$$

Equilibrium point of (4.2.31) and (4.2.32) is $E_4^2\left[\frac{\underline{b}(\alpha)}{1+\underline{m}(\alpha)}, \frac{\bar{b}(\alpha)}{1+\bar{m}(\alpha)}\right]$, which exist for all fuzzy number $m > 0$ and $\alpha \in [0, 1]$.

Corollary 4.16 *The equilibrium point* E_4^2 *is positive if* $\tilde{b}$ *is a positive fuzzy number.*

Solving (4.2.31) and (4.2.32), we get

$$\underline{u}_n(\alpha) = (-\underline{m}(\alpha))^n u_0 + \underline{b}(\alpha)\left(\frac{1 - (-\underline{m}(\alpha))^n}{1 + \underline{m}(\alpha)}\right) \tag{4.2.33}$$

$$\bar{u}_n(\alpha) = (-\bar{m}(\alpha))^n u_0 + \bar{b}(\alpha)\left(\frac{1 - (-\bar{m}(\alpha)^n}{1 + \bar{m}(\alpha)}\right) \tag{4.2.34}$$

The above solutions converges to equilibrium point if $0 < \underline{m}(\alpha) < \bar{m}(\alpha) < 1$.

Remarks 4.5 The equilibrium point E_4^2 is stable if $0 < \underline{m}(\alpha) < \bar{m}(\alpha) < 1$, $\alpha \in [0, 1]$.

4.2f *Study when the initial condition u_0 and $a > 0$ are both fuzzy numbers*:

Let, the α-levels of u_0 and a are, $[u_{n=0}]_\alpha = [\underline{u}_0(\alpha), \bar{u}_0(\alpha)]$, $[a]_\alpha = [\underline{a}(\alpha), \bar{a}(\alpha)], \forall\ \alpha \in [0, 1]$, respectively.

In this case the solutions are given by, followed from equation (4.2.23),

When n is even,

$$\underline{u}_n(\alpha) = (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}\underline{u}_0(\alpha) + k_1(\alpha)\underline{b}(\alpha) + k_2(\alpha)\bar{b}(\alpha) \tag{4.2.35}$$

$$\bar{u}_n(\alpha) = (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}\bar{u}_0(\alpha) + k_1(\alpha)\bar{b}(\alpha) + k_3(\alpha)\underline{b}(\alpha) \tag{4.2.36}$$

Where, $k_1(\alpha) = 1 + \underline{a}(\alpha)\bar{a}(\alpha) + (\underline{a}(\alpha)\bar{a}(\alpha))^2 + \cdots + (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n-2}{2}}$

$$k_2(\alpha) = \bar{a}(\alpha) + (\bar{a}(\alpha))^2\underline{a}(\alpha) + \cdots + (\bar{a}(\alpha))^{\frac{n}{2}}(\underline{a}(\alpha))^{\frac{n-2}{2}}$$
$$k_3(\alpha) = \underline{a}(\alpha) + (\underline{a}(\alpha))^2\bar{a}(\alpha) + \cdots + (\bar{a}(\alpha))^{\frac{n-2}{2}}(\underline{a}(\alpha))^{\frac{n}{2}}$$

When, n is odd,

$$\underline{u}_n(\alpha) = (\underline{a}(\alpha))^{\frac{n-1}{2}}(\bar{a}(\alpha))^{\frac{n+1}{2}}\bar{u}_0 + k_4(\alpha)\underline{b}(\alpha) + k_5(\alpha)\bar{b}(\alpha) \tag{4.2.37}$$

$$\bar{u}_n(\alpha) = (\underline{a}(\alpha)))^{\frac{n+1}{2}}(\bar{a}(\alpha))^{\frac{n-1}{2}}\underline{u}_0 + k_4(\alpha)\bar{b}(\alpha) + k_6(\alpha)\underline{b}(\alpha) \tag{4.2.38}$$

Where, $k_4(\alpha) = 1 + \underline{a}(\alpha)\bar{a}(\alpha) + (\underline{a}(\alpha)\bar{a}(\alpha))^2 + \cdots + (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n-1}{2}}$

$$k_5(\alpha) = \bar{a}(\alpha) + (\bar{a}(\alpha))^2\underline{a}(\alpha) + \cdots + (\bar{a}(\alpha))^{\frac{n-1}{2}}(\underline{a}(\alpha))^{\frac{n-3}{2}}$$
$$k_6(\alpha) = \underline{a}(\alpha) + (\underline{a}(\alpha))^2\bar{a}(\alpha) + \cdots + (\bar{a}(\alpha))^{\frac{n-3}{2}}(\underline{a}(\alpha))^{\frac{n-1}{2}}.$$

Lemma 4.8 *Consider the in-homogeneous difference equation (4.2), where* $a > 0$ *and initial condition u_0 are both fuzzy numbers. Then the positive fuzzy equilibrium point* $\left[\frac{\underline{b}(\alpha)+\bar{b}(\alpha)\bar{a}(\alpha)}{1-\underline{a}(\alpha)\bar{a}(\alpha)}, \frac{\bar{b}(\alpha)+\underline{b}(\alpha)\underline{a}(\alpha)}{1-\underline{a}(\alpha)\bar{a}(\alpha)}\right]$ *is stable positive equilibrium point if* $0 < \underline{a}(\alpha)\bar{a}(\alpha) < 1$, *for all* $\alpha \in [0, 1]$.

Proof The proof of this lemma is immediate.

Remarks 4.6 In this case, the equilibrium point, positivity of equilibrium, existence criteria of equilibrium point and stability conditions are same as sub case 4.2.4, which implies the independent of u_0.

4.2g *Study when the initial condition u_0 and $a < 0$ are both fuzzy numbers*:

Let the α-levels of u_0, $[u_{n=0}]_\alpha = [\underline{u}_0(\alpha), \bar{u}_0(\alpha)]$. Also let $a = -m, m$ be a positive fuzzy number. The α-levels of m is $[m]_\alpha = [\underline{m}(\alpha), \bar{m}(\alpha)]\forall\ \alpha \in [0, 1]$.

In the similar way, as equation (4.2.30), we have the following solutions

$$\underline{u}_n(\alpha) = (-\underline{m}(\alpha))^n\underline{u}_0(\alpha) + \underline{b}(\alpha)\left(\frac{1-(-\underline{m}(\alpha))^n}{1+\underline{m}(\alpha)}\right) \tag{4.2.39}$$

$$\bar{u}_n(\alpha) = (-\bar{m}(\alpha))^n\bar{u}_0(\alpha) + \bar{b}(\alpha)\left(\frac{1-(-\bar{m}(\alpha))^n}{1+\bar{m}(\alpha)}\right) \tag{4.2.40}$$

The above solutions converges to equilibrium point if $0 < \underline{m}(\alpha) < \bar{m}(\alpha) < 1$.

Remarks 4.7 In this case equilibrium point, positivity of equilibrium point and stability condition is same as case 4.2.5. So, independent of u_0, whether it is a fuzzy or not.

Proposition 4.1 [21] *Let $a, b, u_0 \in \mathbb{R}_{\mathcal{F}}^+$. If $1 \notin supp(a)$, then $\exists$ a unique positive solution for equation (4.1). Also the solution is bounded and persistence.*

Proposition 4.2 *Let $a, b, u_0 \in \mathbb{R}_{\mathcal{F}}^+$ and $\underline{a}(\alpha)\bar{a}(\alpha) < 1$. Then $\exists$ a unique positive and bounded solution for equation (4.2) is given by $[\underline{u}_n(\alpha), \bar{u}_n(\alpha)]$ in it's α-cuts form, provided it exist and the solutions are,*

When, n is even,

$$\underline{u}_n(\alpha) = (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}\underline{u}_0 + \frac{\underline{b}(\alpha)+\bar{b}(\alpha)\bar{a}(\alpha)}{1-\underline{a}(\alpha)\bar{a}(\alpha)}\left(1-(\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}\right) \tag{4.2.a}$$

$$\bar{u}_n(\alpha) = (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}\bar{u}_0 + \frac{\bar{b}(\alpha)+\underline{b}(\alpha)\underline{a}(\alpha)}{1-\underline{a}(\alpha)\bar{a}(\alpha)}\left(1-(\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}\right) \tag{4.2.b}$$

And, for n is odd,

$$\underline{u}_n(\alpha) = (\underline{a}(\alpha))^{\frac{n-1}{2}}(\bar{a}(\alpha))^{\frac{n+1}{2}}\bar{u}_0 + \underline{b}(\alpha)\left(\frac{1-(\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}}{1-\underline{a}(\alpha)\bar{a}(\alpha)}\right) + \bar{b}(\alpha)\bar{a}\left(\frac{1-(\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n-1}{2}}}{1-\underline{a}(\alpha)\bar{a}(\alpha)}\right) \tag{4.2.c}$$

$$\bar{u}_n(\alpha) = (\underline{a}(\alpha)))^{\frac{n+1}{2}}(\bar{a}(\alpha))^{\frac{n-1}{2}}\underline{u}_0 + \bar{b}(\alpha)\left(\frac{1-(\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}}{1-\underline{a}(\alpha)\bar{a}(\alpha)}\right) + \underline{b}(\alpha)\underline{a}\left(\frac{1-(\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n-1}{2}}}{1-\underline{a}(\alpha)\bar{a}(\alpha)}\right) \tag{4.2.d}$$

$\forall\alpha \in [0, 1]$.

Proof Considering equation (4.2) in fuzzy environment in which $a, b, u_0 \in \mathbb{R}_{\mathcal{F}}^+$ and $\underline{a}(\alpha)\bar{a}(\alpha) < 1$. Now by characterisation theorem 2.1, equation (4.2) can be converted into a pair of crisp difference equation, which can be expressed in matrix form as

$$\begin{pmatrix} \underline{u}_{n+1}(\alpha) \\ \bar{u}_{n+1}(\alpha) \end{pmatrix} = \begin{pmatrix} 0 & \bar{a}(\alpha) \\ \underline{a}(\alpha) & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_{n}(\alpha) \\ \bar{u}_{n}(\alpha) \end{pmatrix} + \begin{pmatrix} \underline{b}(\alpha) \\ \bar{b}(\alpha) \end{pmatrix} \tag{4.2.e}$$

Using theorem 3.1, the above system of difference equation can be solved.

The solution of equation (4.2.e), solving in the similar way as equation (4.1.24), are given below:

When n is even,

$$\underline{u}_n(\alpha) = (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}\underline{u}_0(\alpha) + k_1(\alpha)\underline{b}(\alpha) + k_2(\alpha)\bar{b}(\alpha) \tag{4.2.35}$$

$$\bar{u}_n(\alpha) = (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}\bar{u}_0(\alpha) + k_1(\alpha)\bar{b}(\alpha) + k_3(\alpha)\underline{b}(\alpha) \tag{4.2.36}$$

Where, $k_1(\alpha) = 1 + \underline{a}(\alpha)\bar{a}(\alpha) + (\underline{a}(\alpha)\bar{a}(\alpha))^2 + \cdots + (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n-2}{2}}$

$$k_2(\alpha) = \bar{a}(\alpha) + (\bar{a}(\alpha))^2\underline{a}(\alpha) + \cdots + (\bar{a}(\alpha))^{\frac{n}{2}}(\underline{a}(\alpha))^{\frac{n-2}{2}}$$
$$k_3(\alpha) = \underline{a}(\alpha) + (\underline{a}(\alpha))^2\bar{a}(\alpha) + \cdots + (\bar{a}(\alpha))^{\frac{n-2}{2}}(\underline{a}(\alpha))^{\frac{n}{2}}, \quad \forall\alpha \in [0,1]$$

Which implies,

$$\underline{u}_n(\alpha) = (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}\underline{u}_0 + \frac{\underline{b}(\alpha) + \bar{b}(\alpha)\bar{a}(\alpha)}{1 - \underline{a}(\alpha)\bar{a}(\alpha)}\left(1 - (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}\right) \tag{4.2.a}$$

$$\bar{u}_n(\alpha) = (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}\bar{u}_0 + \frac{\bar{b}(\alpha) + \underline{b}(\alpha)\underline{a}(\alpha)}{1 - \underline{a}(\alpha)\bar{a}(\alpha)}\left(1 - (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}\right) \tag{4.2.b}$$

When, n is odd,

$$\underline{u}_n(\alpha) = (\underline{a}(\alpha))^{\frac{n-1}{2}}(\bar{a}(\alpha))^{\frac{n+1}{2}}\bar{u}_0(\alpha) + k_4(\alpha)\underline{b}(\alpha) + k_5(\alpha)\bar{b}(\alpha) \tag{4.2.37}$$

$$\bar{u}_n(\alpha) = (\underline{a}(\alpha)))^{\frac{n+1}{2}}(\bar{a}(\alpha))^{\frac{n-1}{2}}\underline{u}_0(\alpha) + k_4(\alpha)\bar{b}(\alpha) + k_6(\alpha)\underline{b}(\alpha) \tag{4.2.38}$$

Where, $k_4(\alpha) = 1 + \underline{a}(\alpha)\bar{a}(\alpha) + (\underline{a}(\alpha)\bar{a}(\alpha))^2 + \cdots + (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n-1}{2}}$

$$k_5(\alpha) = \bar{a}(\alpha) + (\bar{a}(\alpha))^2\underline{a}(\alpha) + \cdots + (\bar{a}(\alpha))^{\frac{n-1}{2}}(\underline{a}(\alpha))^{\frac{n-3}{2}}$$
$$k_6(\alpha) = \underline{a}(\alpha) + (\underline{a}(\alpha))^2\bar{a}(\alpha) + \cdots + (\bar{a}(\alpha))^{\frac{n-3}{2}}(\underline{a}(\alpha))^{\frac{n-1}{2}}$$
$\forall\alpha \in [0,1]$

Which implies,

$$\underline{u}_n(\alpha) = (\underline{a}(\alpha))^{\frac{n-1}{2}}(\bar{a}(\alpha))^{\frac{n+1}{2}}\bar{u}_0(\alpha) + \underline{b}(\alpha)\left(\frac{1 - (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}}{1 - \underline{a}(\alpha)\bar{a}(\alpha)}\right) + \bar{b}(\alpha)\bar{a}\left(\frac{1 - (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n-1}{2}}}{1 - \underline{a}(\alpha)\bar{a}(\alpha)}\right) \tag{4.2.c}$$

$$\bar{u}_n(\alpha) = (\underline{a}(\alpha)))^{\frac{n+1}{2}}(\bar{a}(\alpha))^{\frac{n-1}{2}}\underline{u}_0(\alpha) + \bar{b}(\alpha)\left(\frac{1 - (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n}{2}}}{1 - \underline{a}(\alpha)\bar{a}(\alpha)}\right) + \underline{b}(\alpha)\underline{a}\left(\frac{1 - (\underline{a}(\alpha)\bar{a}(\alpha))^{\frac{n-1}{2}}}{1 - \underline{a}(\alpha)\bar{a}(\alpha)}\right) \tag{4.2.d}$$

Since, $a, b, u_0 \in \mathbb{R}^+_{\mathcal{F}}$, therefore, $supp(a) \subset (0,\infty), supp(b) \subset (0,\infty)$ and $supp(u_0) \subset (0,\infty)$. So, $\underline{a}(\alpha), \underline{b}(\alpha), \underline{u}_0(\alpha), \bar{a}(\alpha), \bar{b}(\alpha)$ and $\bar{u}_0(\alpha)$ are all positive real valued numbers $\forall\alpha \in [0,1]$.

Firstly, we consider the case for n is even only. $k_1(\alpha), k_2(\alpha), k_3(\alpha)$ are all positive real numbers and exist as $\underline{a}(\alpha)\bar{a}(\alpha) < 1$, for all $\alpha \in [0,1]$ and for even $n \in \mathbb{N}$. Therefore, from equations (4.2.35) and (4.2.36), $\underline{u}_n(\alpha) > 0, \bar{u}_n(\alpha) > 0, \ \forall\alpha \in [0,1]$ and even $n \in \mathbb{N}$ i.e., $supp(u_n) \subset (0,\infty)\forall\alpha \in [0,1]$ and even $n \in \mathbb{N}$.

Therefore, if $a, b, u_0 \in \mathbb{R}^+_{\mathcal{F}}$ and $\underline{a}(\alpha)\bar{a}(\alpha) < 1$, equation (4.2) has sequence of positive solutions u_n with it's α-cuts form $[\underline{u}_n(\alpha), \bar{u}_n(\alpha)]$, $\forall\alpha \in [0,1]$ and even $n \in \mathbb{N}$. In this case to prove the uniqueness of solutions, let $\exists$ a solution $\hat{u}_n$ of equation (4.2) with it's α-cuts $\left[\hat{\underline{u}}_n(\alpha), \bar{\hat{u}}_n(\alpha)\right]$, $\forall\alpha \in [0,1]$ and even $n \in \mathbb{N}$ under the same initial conditions $u_0 \in \mathbb{R}^+_{\mathcal{F}}$ and $a, b \in \mathbb{R}^+_{\mathcal{F}}$. Then, finally we shall get, $\left[\hat{\underline{u}}_n(\alpha), \bar{\hat{u}}_n(\alpha)\right] = [\underline{u}_n(\alpha), \bar{u}_n(\alpha)]$, $\forall\alpha \in [0,1]$ and even $n \in \mathbb{N}$. This proves the uniqueness of the solution of (4.2).

Similarly, unique positive solution of (4.2) can be shown for n is odd case also.

Since, $0 < \underline{a}(\alpha)\bar{a}(\alpha) < 1$, from equations (4.2.a) and (4.2.b) for even n and equations (4.2.c), (4.2.d) for n is odd, it is clear that $\underline{u}_n(\alpha) \to \frac{\underline{b}(\alpha)+\bar{b}(\alpha)\bar{a}(\alpha)}{1-\underline{a}(\alpha)\bar{a}(\alpha)}$ as $n \to \infty$ and $\bar{u}_n(\alpha) \to \frac{\bar{b}(\alpha)+\underline{b}(\alpha)\underline{a}(\alpha)}{1-\underline{a}(\alpha)\bar{a}(\alpha)}$ as $n \to \infty$. Since, $a, b, u_0 \in \mathbb{R}^+_{\mathcal{F}}$ and $\underline{a}(\alpha)\bar{a}(\alpha) < 1$, so $\exists$ two positive numbers $\underline{M} > 0, \bar{M} > 0$ such that $\underline{M} = \frac{\underline{b}(\alpha)+\bar{b}(\alpha)\bar{a}(\alpha)}{1-\underline{a}(\alpha)\bar{a}(\alpha)}$ and $\bar{M} = \frac{\bar{b}(\alpha)+\underline{b}(\alpha)\underline{a}(\alpha)}{1-\underline{a}(\alpha)\bar{a}(\alpha)}$. Therefore, $supp(u_n) \subset [\underline{M}, \bar{M}]$. So the solutions of equations (4.2) is bounded. This completes the proof.

5. Application in real life problem

5.1 *An investment annuity*

Return now to the savings account problem and consider an annuity. Annuity is often planed for retirement purposes. They are basically savings accounts that pay

Table 1. Values of $\underline{I}_n(\alpha)$ and $\bar{I}_n(\alpha)$ for different values of n and α.

	$n = 4$		$n = 8$		$n = 12$	
α	$\underline{I}_n(\alpha)$	$\bar{I}_n(\alpha)$	$\underline{I}_n(\alpha)$	$\bar{I}_n(\alpha)$	$\underline{I}_n(\alpha)$	$\bar{I}_n(\alpha)$
0	90000.0000	109395.5625	90000.0000	132970.9899	90000.0000	161627.0693
0.1	90396.9950	107657.2220	90795.5804	128810.1159	91195.7625	154147.7055
0.2	90869.8051	105950.0987	91746.5895	124778.9350	92630.4092	147001.6721
0.3	91421.1793	104271.6710	92859.4897	120870.4095	94315.1377	140170.9802
0.4	92053.8823	102619.9941	94140.8244	117078.6716	96261.3585	133639.6666
0.5	92770.6938	100993.5354	95597.2185	113398.7017	98480.6991	127393.3126
0.6	93574.4093	99391.0634	97235.3796	109826.1103	100985.0072	121418.7150
0.7	94467.8396	97811.5703	99062.0984	106356.9875	103786.3535	115703.6550
0.8	95453.8112	96254.2181	101084.2498	102987.7957	106897.0360	110236.7324
0.9	96535.1660	94718.2986	103308.7932	99715.2924	110329.5828	105007.2429
1	97714.7619	93203.2052	105742.7741	96536.4734	114096.7557	100005.0857

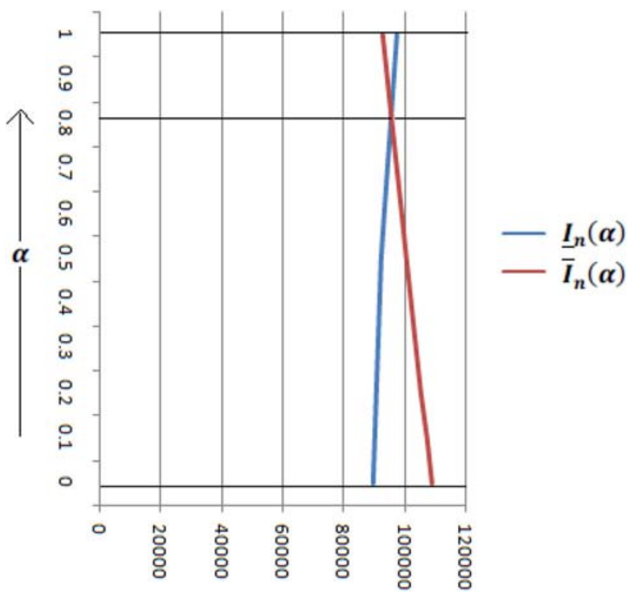

Figure 1. Fuzzy solution for $n = 4$.

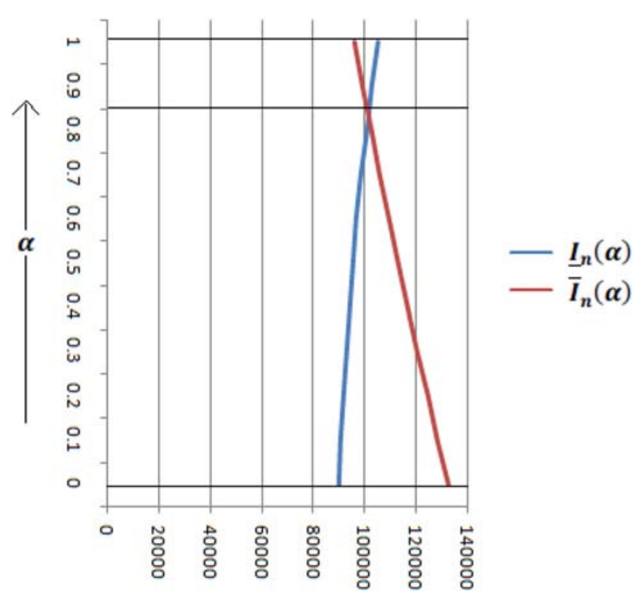

Figure 2. Fuzzy solution for $n = 8$.

interest on the amount present and allow the investor to withdraw a fixed amount each month until the account is depleted. An interesting issue (posed in the problems) is to determine the amount on must save monthly to build an annuity allowing for withdrawals, beginning at a certain age with a specified amount for a desired number of years, before the account's depletion. For now consider around and above 1% (i.e., fuzzy number $(1, 1.01, 1.05)$) as the monthly interest rate and a monthly withdrawals of near about \$1000 (i.e., fuzzy number $(900, 1000, 1100)$). Now suppose we made the following initial investment: $(1){:}\, I_0 = 90000 (2){:}\, I_0 = 100000 (3){:}\, I_0 = 110000$. Then solve the problem.

Solution This gives a fuzzy discrete dynamical system

$$I_{n+1} = (1, 1.01, 1.05)I_n - (900, 1000, 1100)$$

With crisp initial condition $(1){:}\, I_0 = 90000 (2){:}\, I_0 = 100000 (3){:}\, I_0 = 110000$

Case 5.1a *When the initial condition is* $I_0 = 90000$ (table 1):

Solution The solution can be written as

$$\underline{I}_n(\alpha) = 90000(1 + 0.01\alpha)^n - 100\alpha(900 + 100\alpha)(1 - (1 + 0.01\alpha)^n)$$

$$\bar{I}_n(\alpha) = 90000(1.05 - 0.04\alpha)^n + (1100 - 100\alpha)\left(\frac{1 - (1 + 0.01\alpha)^n}{0.05 + 0.04\alpha}\right)$$

Note 5.1 It is not always true that if we consider triangular fuzzy initially in a problem then the resultant is also

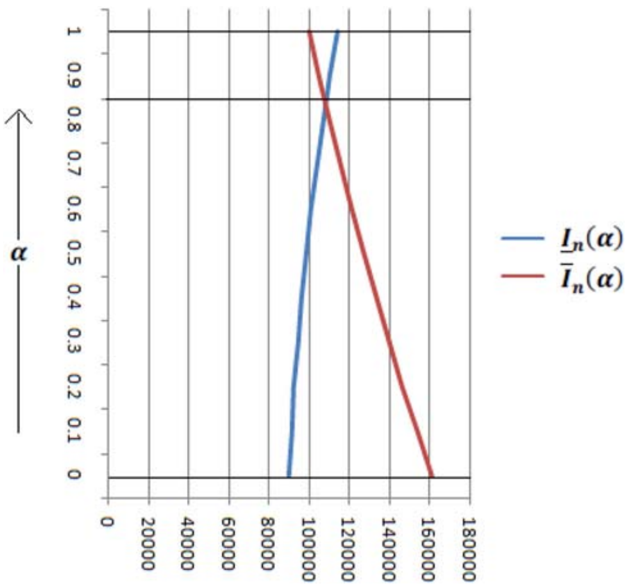

Figure 3. Fuzzy solution for $n = 12$.

in the form of triangular fuzzy number. It is a fuzzy interval but need not necessary a triangular fuzzy number. Many authors take initial conditions as triangular fuzzy number but in result time the write it as a approximated triangular fuzzy number (figures 1, 2, 3).

Case 5.1b: *When the initial condition is* $I_0 = 100000$ (table 2):

Solution The solution can be written as (figures 4, 5, 6)

$$\underline{I}_n(\alpha) = 100000(1 + 0.01\alpha)^n - 100\alpha(900 + 100\alpha)(1 - (1 + 0.01\alpha)^n)$$

$$\bar{I}_n(\alpha) = 100000(1.05 - 0.04\alpha)^n + (1100 - 100\alpha)\left(\frac{1 - (1 + 0.01\alpha)^n}{0.05 + 0.04\alpha}\right)$$

Case 5.1c *When the initial condition is* $I_0 = 110000$ (table 3):

Solution The solution can be written as

$$\underline{I}_n(\alpha) = 110000(1 + 0.01\alpha)^n - 100\alpha(900 + 100\alpha)(1 - (1 + 0.01\alpha)^n)$$

$$\bar{I}_n(\alpha) = 110000(1.05 - 0.04\alpha)^n + (1100 - 100\alpha)\left(\frac{1 - (1 + 0.01\alpha)^n}{0.05 + 0.04\alpha}\right)$$

Remarks on this application and Stability situation The equilibrium point in this cases is $\left[\frac{-100\alpha-900}{-0.5+0.04\alpha}, \frac{-1100+100\alpha}{-0.01\alpha}\right]$. Clearly this equilibrium point is positive equilibrium point for all $\alpha \in (0, 1]$. The equilibrium point is unstable since $0 < 1 + 0.01\alpha < 1.05 - 0.04\alpha < 1$ does not hold for all $\alpha \in [0, 1]$. The stability criteria does not depends on the initial value or non- homogeneity portion. From the table and graph we see that somewhere the solution not follow the strong weak solution concept on solution after certain range of α. In our problem reader can understood the fact as the line mention in the figure and highlighted data in table. Somewhere we see that the graph are not exact triangular shaped. The cause of the fact is due to taken the coefficient and non-homogeneity portion both as fuzzy number. Therefore if the investment annuity problem is fuzzy valued and the rate of interest fluctuates, then a unstable situation arise (figures 7, 8, 9).

5.2 *Prescription for digoxin*

Consider again the digoxin problem. Recall that digoxin is used in the treatment of heart patients. The objective of the problem is to consider the decay of digoxin in the bloodstream to prescribe a dosage that keeps the concentration between acceptable levels(so that is both safe and effective). Suppose we prescribe a daily drug dosage of near about 0.1 mg (i.e., fuzzy number $(0.08, 0.1, 0.12)$) and know that half of the digoxin remains in the system at the

Table 2. Values of $\underline{I}_n(\alpha)$ and $\bar{I}_n(\alpha)$ for different values of n and α.

	$n = 4$		$n = 8$		$n = 12$	
α	$\underline{I}_n(\alpha)$	$\bar{I}_n(\alpha)$	$\underline{I}_n(\alpha)$	$\bar{I}_n(\alpha)$	$\underline{I}_n(\alpha)$	$\bar{I}_n(\alpha)$
0	100000.0000	121550.6250	100000.0000	147745.5444	100000.0000	179585.6326
0.1	100437.0550	119628.1202	100875.8609	143140.3563	101316.4247	171302.2904
0.2	100950.0454	117738.9334	101907.7140	138676.5972	102873.0669	163385.3963
0.3	101541.7204	115880.5267	103102.0249	134346.9626	104681.1375	155815.7163
0.4	102214.8449	114050.9396	104465.3404	130145.3232	106752.0606	148576.0848
0.5	102972.1988	112248.6235	106004.2890	126066.4025	109097.4773	141650.9215
0.6	103816.5780	110472.3310	107725.5815	122105.5595	111729.2488	135025.9012
0.7	104750.7934	108721.0386	109636.0122	118258.6373	114659.4601	128687.7220
0.8	105777.6717	106993.8924	111742.4594	114521.8562	117900.4230	122623.9377
0.9	106900.0553	105290.1687	114051.8861	110891.7363	121464.6795	116822.8342
1	108120.8020	103609.2453	116571.3411	107365.0405	125365.0060	111273.3360

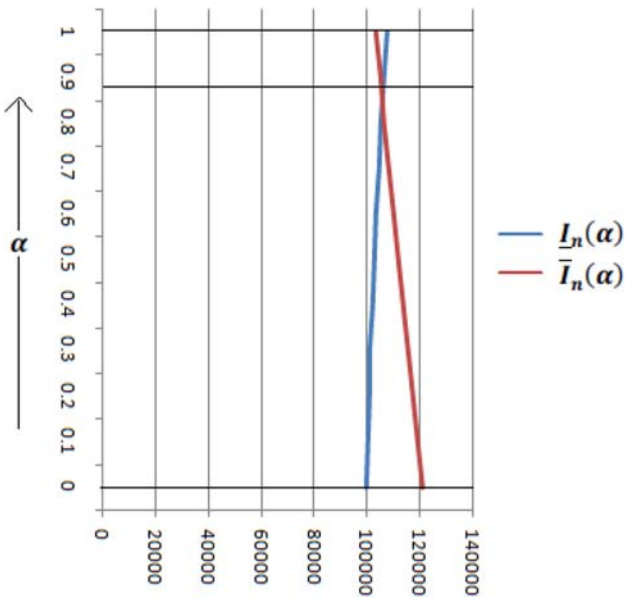

Figure 4. Fuzzy solution for $n = 4$.

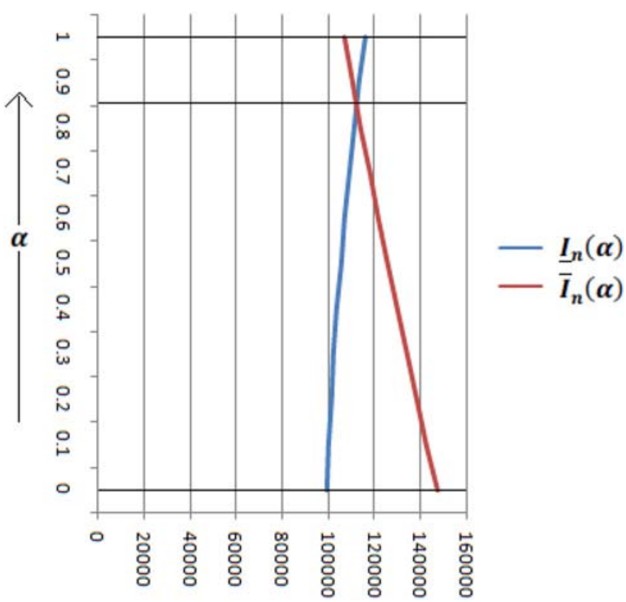

Figure 5. Fuzzy solution for $n = 8$.

end of each dosage period. Solve the problem with fuzzy initial data (1): $D_{n=0} = (0.08, 0.1, 0.12)$, (2): $D_{n=0} = (0.15, 0.2, 0.25)$, (3): $D_{n=0} = (0.25, 0.3, 0.40)$

Solution This resultant fuzzy dynamical system

$$D_{n+1} = 0.5D_n + (0.08, 0.1, 0.12)$$

With initial conditions $D_{n=0} = (0.08, 0.1, 0.12)$, $D_{n=0} = (0.15, 0.2, 0.25)$, $D_{n=0} = (0.25, 0.3, 0.40)$

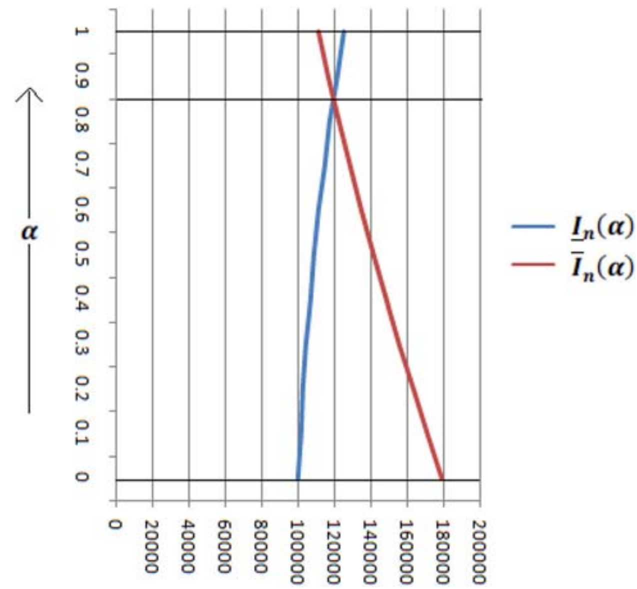

Figure 6. Fuzzy solution for $n = 12$.

Case 5.2a *When the initial condition is* $D_{n=0} = (0.08, 0.1, 0.12)$ (table 4):

Solution The solution can be written as

$$\underline{D}_n(\alpha) = (0.5)^n(0.08 + 0.02\alpha) + 2(0.08 + 0.02\alpha)(1 - (0.5)^n)$$
$$\bar{D}_n(\alpha) = (0.5)^n(0.12 - 0.02\alpha) + 2(0.12 - 0.02\alpha)(1 - (0.5)^n)$$

Case 5.2b: *When the initial condition is* $D_{n=0} = (0.15, 0.2, 0.25)$ (table 5):

Solution The solution can be written as (figures 10, 11, 12)

$$\underline{D}_n(\alpha) = (0.5)^n(0.15 + 0.05\alpha) + 2(0.08 + 0.02\alpha)(1 - (0.5)^n)$$
$$\bar{D}_n(\alpha) = (0.5)^n(0.25 - 0.05\alpha) + 2(0.12 - 0.02\alpha)(1 - (0.5)^n)$$

Case 5.2c *When the initial condition is* $D_{n=0} = (0.25, 0.3, 0.40)$ (table 6):

Solution The solution can be written as

$$\underline{u}_n(\alpha) = (0.5)^n(0.25 + 0.05\alpha) + 2(0.08 + 0.02\alpha)(1 - (0.5)^n)$$
$$\bar{u}_n(\alpha) = (0.5)^n(0.40 - 0.10\alpha) + 2(0.12 - 0.02\alpha)(1 - (0.5)^n)$$

Remarks on this application and Stability situation The equilibrium point in this case is $[0.16 + 0.04\alpha, 0.24 - 0.04\alpha]$. Clearly this equilibrium point is positive equilibrium point for all $\alpha \in [0, 1]$. The equilibrium point is stable since $0 < a < 1$ hold for all $\alpha \in [0, 1]$. The stability criteria does not depends on the

Table 3. Values of $\underline{I}_n(\alpha)$ and $\bar{I}_n(\alpha)$ for different values of n and α.

	$n = 4$		$n = 8$		$n = 12$	
α	$\underline{I}_n(\alpha)$	$\bar{I}_n(\alpha)$	$\underline{I}_n(\alpha)$	$\bar{I}_n(\alpha)$	$\underline{I}_n(\alpha)$	$\bar{I}_n(\alpha)$
0	110000.0000	133705.6875	110000.0000	162520.0988	110000.0000	197544.1959
0.1	110477.1151	131599.0184	110956.1415	157470.5967	111437.0869	188456.8753
0.2	111030.2857	129527.7680	112068.8385	152574.2594	113115.7246	179769.1205
0.3	111662.2615	127489.3825	113344.5601	147823.5158	115047.1373	171460.4524
0.4	112375.8074	125481.8851	114789.8565	143211.9747	117242.7626	163512.5030
0.5	113173.7038	123503.7116	116411.3594	138734.1033	119714.2554	155908.5304
0.6	114058.7466	121553.5986	118215.7834	134385.0087	122473.4905	148633.0875
0.7	115033.7471	119630.5068	120209.9260	130160.2871	125532.5668	141671.7891
0.8	116101.5322	117733.5667	122400.6690	126055.9166	128903.8099	135011.1430
0.9	117264.9445	115862.0389	124794.9790	122068.1801	132599.7763	128638.4254
1	118526.8421	114015.2854	127399.9082	118193.6076	136633.2563	122541.5863

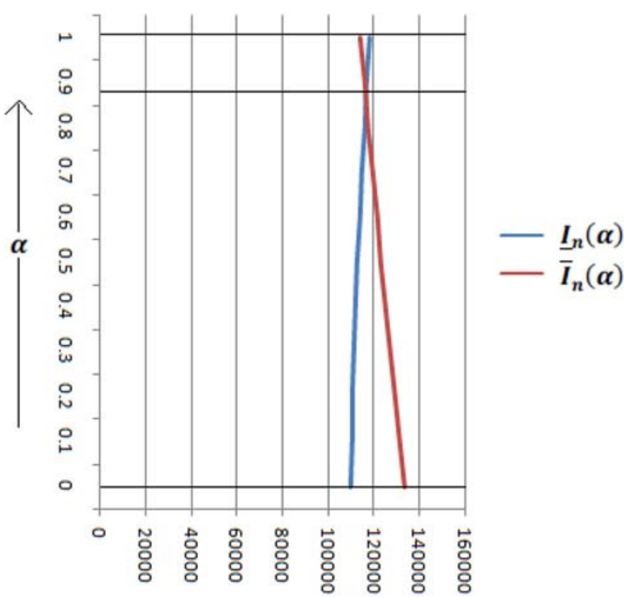

Figure 7. Fuzzy solution for $n = 4$.

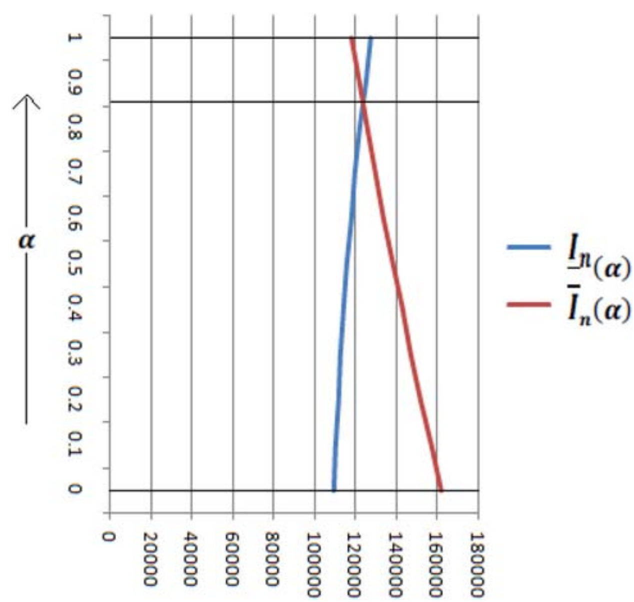

Figure 8. Fuzzy solution for $n = 8$.

initial value or non- homogeneity portion. From the table and graph we see that the solution almost follow the strong solution concept on solution $\alpha \in [0, 1]$. In our problem, readers can understand the fact as the line mention in figures and highlighted data in tables. We see that the graphs are almost exact triangular shaped. The cause of the fact is due to taking the coefficient and non-homogeneity portion both as fuzzy number. Therefore in prescription of digoxin problem not only half of the dosage but also dosage values strictly less than 1 remains in the system, then a stable situation arise (figures 13, 14, 15, 16, 17, 18).

6. Conclusion and future extension

Mathematical modelling is a very important tool to deal many real-life problems in the field of engineering, ecology, social science, etc. Maximum of these real-life problems can be represented in the form of differential equations when it obeys the rule of continuity; whereas discrete systems are described and presented through difference equations. Modelling with difference equation becomes more important now a days for discrete system modelling. On the other hand, uncertainty and impreciseness are inherently involved to maximum of these real-life

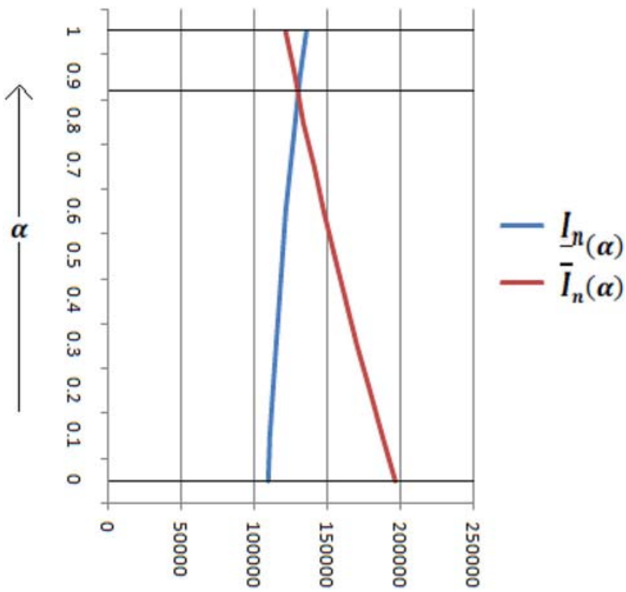

Figure 9. Fuzzy solution for $n = 12$.

Table 4. Values of $\underline{D}_n(\alpha)$ and $\bar{D}_n(\alpha)$ for different values of n and α.

	$n = 4$		$n = 8$		$n = 12$	
α	$\underline{D}_n(\alpha)$	$\bar{D}_n(\alpha)$	$\underline{D}_n(\alpha)$	$\bar{D}_n(\alpha)$	$\underline{D}_n(\alpha)$	$\bar{D}_n(\alpha)$
0	0.1550	0.2325	0.1597	0.2325	0.1600	0.2400
0.1	0.1589	0.2286	0.1637	0.2286	0.1640	0.2360
0.2	0.1628	0.2247	0.1677	0.2247	0.1680	0.2320
0.3	0.1666	0.2209	0.1717	0.2209	0.1720	0.2280
0.4	0.1705	0.2170	0.1757	0.2170	0.1760	0.2240
0.5	0.1744	0.2131	0.1796	0.2131	0.1800	0.2200
0.6	0.1783	0.2093	0.1836	0.2093	0.1840	0.2160
0.7	0.1821	0.2054	0.1876	0.2054	0.1880	0.2120
0.8	0.1860	0.2015	0.1916	0.2015	0.1920	0.2080
0.9	0.1899	0.1976	0.1956	0.1976	0.1960	0.2040
1	0.1938	0.1937	0.1996	0.1937	0.2000	0.2000

Table 5. Values of $\underline{D}_n(\alpha)$ and $\bar{D}_n(\alpha)$ for different values of n and α.

	$n = 4$		$n = 8$		$n = 12$	
α	$\underline{D}_n(\alpha)$	$\bar{D}_n(\alpha)$	$\underline{D}_n(\alpha)$	$\bar{D}_n(\alpha)$	$\underline{D}_n(\alpha)$	$\bar{D}_n(\alpha)$
0	0.1594	0.2406	0.1600	0.2400	0.1600	0.2400
0.1	0.1634	0.2366	0.1640	0.2360	0.1640	0.2360
0.2	0.1675	0.2325	0.1680	0.2320	0.1680	0.2320
0.3	0.1716	0.2284	0.1720	0.2280	0.1720	0.2280
0.4	0.1756	0.2244	0.1760	0.2240	0.1760	0.2240
0.5	0.1797	0.2203	0.1800	0.2200	0.1800	0.2200
0.6	0.1838	0.2163	0.1840	0.2160	0.1840	0.2160
0.7	0.1878	0.2122	0.1880	0.2120	0.1880	0.2120
0.8	0.1919	0.2081	0.1920	0.2080	0.1920	0.2080
0.9	0.1959	0.2041	0.1960	0.2040	0.1960	0.2040
1	0.2000	0.2000	0.2000	0.2000	0.2000	0.2000

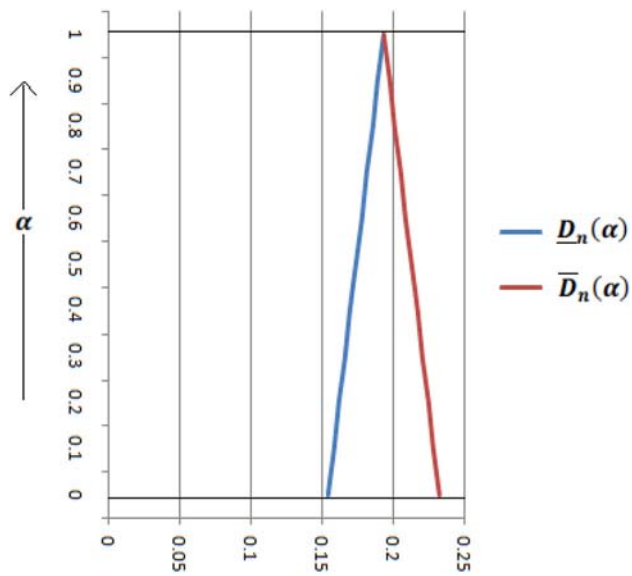

Figure 10. Fuzzy solution for $n = 4$.

problems and hence naturally fuzzy set theory comes to the picture in the area of difference equations. Several interesting works have already been done and many more is going in fuzzy difference equations. In this article, we discussed the solution technique of non-homogeneous linear fuzzy difference equation associated with fuzzy initial condition, forcing function and fuzzy coefficient which was not done earlier. Moreover, we address the issue of finding the equilibrium points and its stability analysis of a model system presented through fuzzy difference equation. Finally, the theoretical techniques and methodologies have been highlighted through its application in a real-life problem in the field of financial and biomathematics. The contributions and important outcomes of the article can be summarised as follows:

(a) The solution technique of non-homogeneous linear difference equation has been discussed with fuzzy initial conditions, fuzzy forcing function and fuzzy coefficients.
(b) The idea of fuzzy equilibrium point is introduced and a smooth relationship has been identified between fuzzy equilibrium points and crisp equilibrium points.
(c) The real-life examples in investment in annuity and prescription for digoxin are given in fuzzy environment.

The work done in this article can be extended into the nonlinear and non-homogeneous fuzzy difference equation

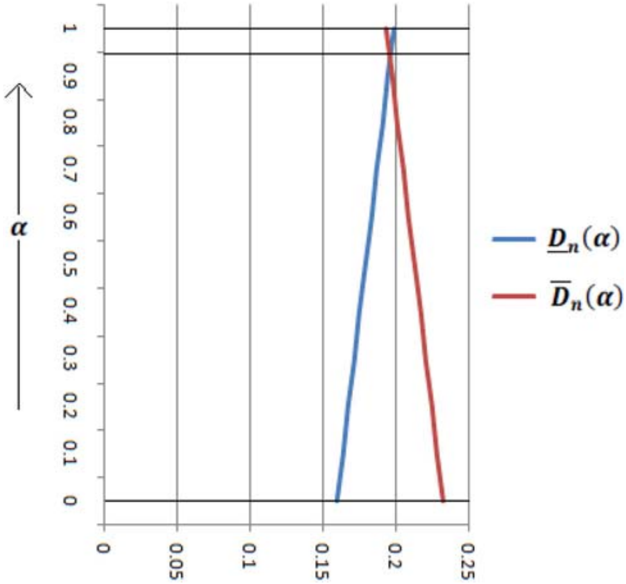

Figure 11. Fuzzy solution for $n = 8$.

Table 6. Values of $\underline{D}_n(\alpha)$ and $\bar{D}_n(\alpha)$ for different values of n and α.

	$n = 4$		$n = 8$		$n = 12$	
α	$\underline{D}_n(\alpha)$	$\bar{D}_n(\alpha)$	$\underline{D}_n(\alpha)$	$\bar{D}_n(\alpha)$	$\underline{D}_n(\alpha)$	$\bar{D}_n(\alpha)$
0	0.1656	0.2500	0.1604	0.2406	0.1600	0.2400
0.1	0.1697	0.2456	0.1644	0.2366	0.1640	0.2360
0.2	0.1738	0.2412	0.1684	0.2326	0.1680	0.2320
0.3	0.1778	0.2369	0.1724	0.2286	0.1720	0.2280
0.4	0.1819	0.2325	0.1764	0.2245	0.1760	0.2240
0.5	0.1859	0.2281	0.1804	0.2205	0.1800	0.2200
0.6	0.1900	0.2237	0.1844	0.2165	0.1840	0.2160
0.7	0.1941	0.2194	0.1884	0.2125	0.1880	0.2120
0.8	0.1981	0.2150	0.1924	0.2084	0.1920	0.2080
0.9	0.2022	0.2106	0.1964	0.2044	0.1960	0.2040
1	0.2063	0.2063	0.2004	0.2004	0.2000	0.2000

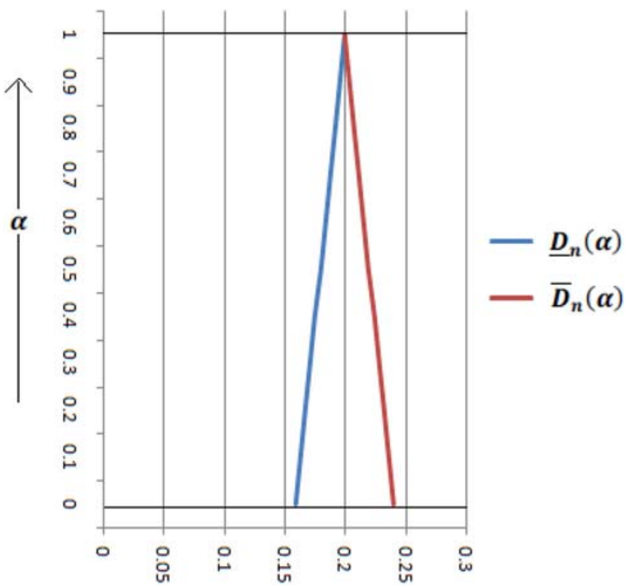

Figure 12. Fuzzy solution for $n = 12$.

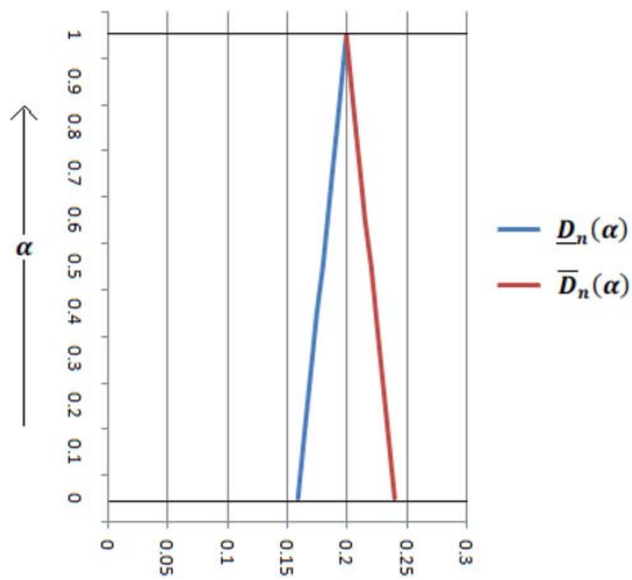

Figure 13. Fuzzy solution for $n = 4$.

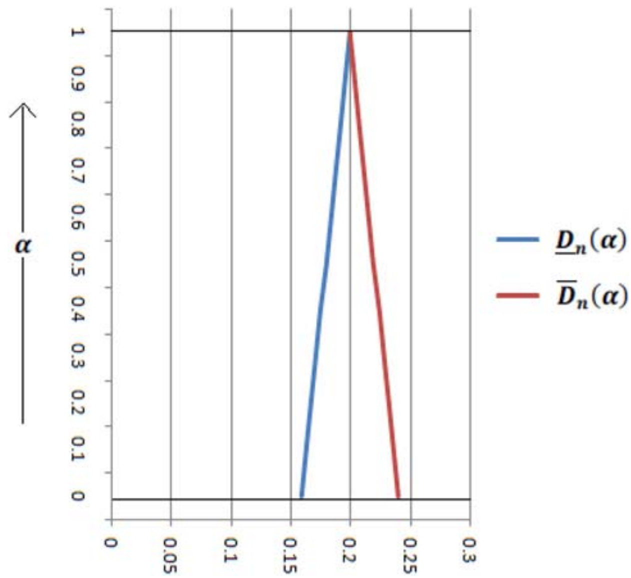

Figure 14. Fuzzy solution for $n = 8$.

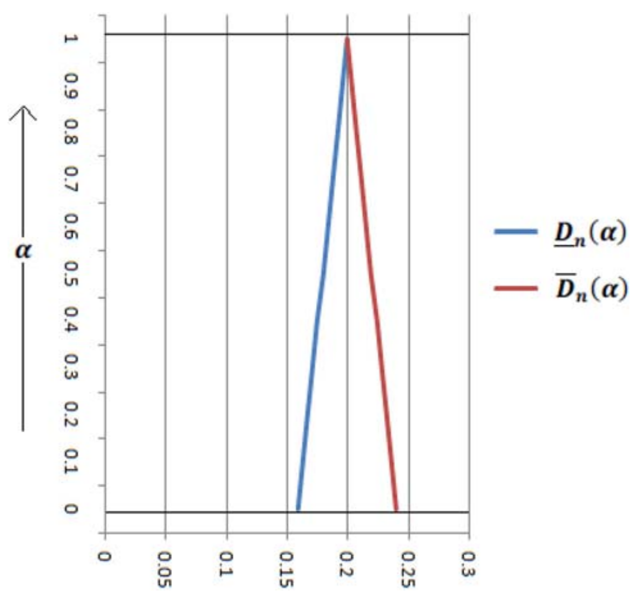

Figure 15. Fuzzy solution for $n = 12$.

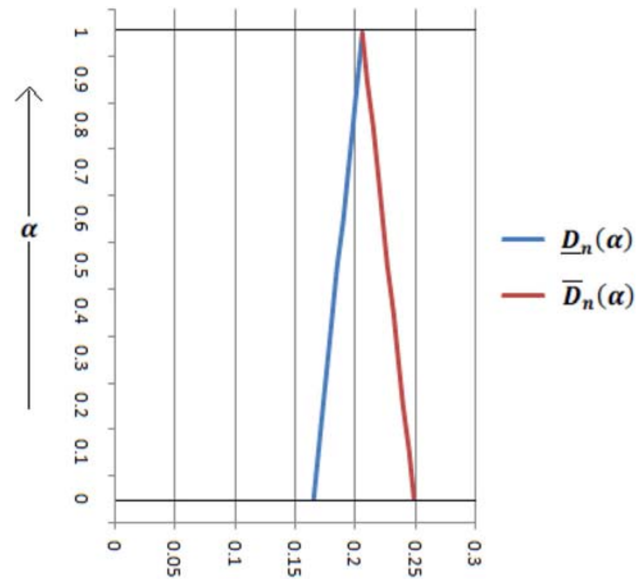

Figure 16. Fuzzy solution for $n = 4$.

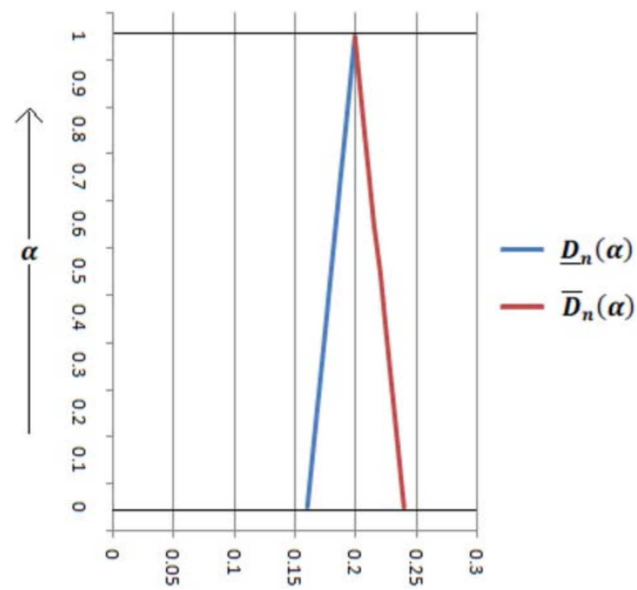

Figure 17. Fuzzy solution for $n = 8$.

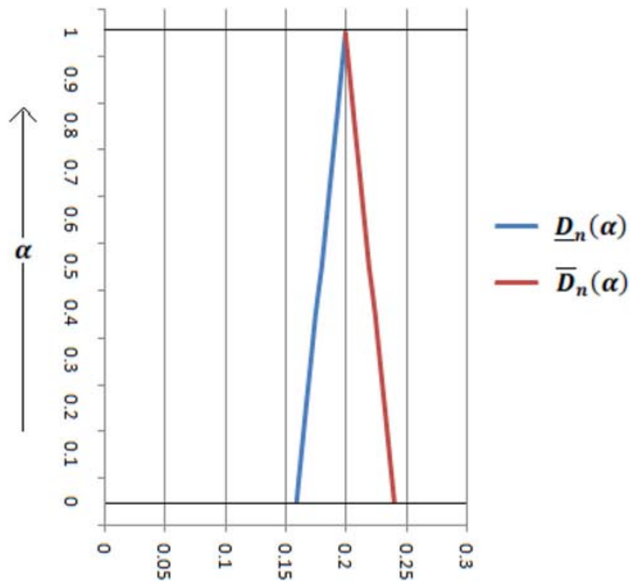

Figure 18. Fuzzy solution for $n = 12$.

followed by real-life applications. Furthermore, one can take different imprecise environment for better improvement of this study.

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