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On the distribution-free continuous-review production-inventory model with service level constraint

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Abstract. In this article, we study a continuous-review production-inventory model that assembles lost sales and backorders with service level constraint. The study under consideration assumes that the distribution of demand during the lead-time is known partially. The objective of this paper is twofold. Firstly, the distribution-free procedure is applied to obtain a closed-form solution of optimal production quantity, re-order level and lead-time in the random framework. Secondly, considering demand as a fuzzy random variable, the procedure is extended to the fuzzy random framework in which an algorithm is proposed to find the optimal global solution. Two numerical examples are provided to illustrate the methods. Furthermore, sensitivity analysis is performed to present some managerial inferences.

Keywords. Inventory; production quantity; continuous review; fuzzy random variable; min–max distribution-free procedure.

1. Introduction

The EPQ model is one of the more commonly used inventory systems in the industry and business organization. The classical EPQ models were formulated in the deterministic framework in which the demand is considered as a constant variable. However, over the last few years, researchers such as Coates *et al* [1], Sarker and Coates [2], Sarkar and Moon [3], Pal *et al* [4] and many others have studied EPQ models in the random environment. Sarkar and Moon [3] and Pal et al. [4] established the single-period production-inventory model under a random variable demand rate and constant production rate. On the other hand, a random variable with finite rate is incorporated for the manufacturing lead-time by Coates *et al* [1] and Sarker and Coates [2].

The lead-time associated with inventory system was treated as a constant or random variable in most of the earlier literatures [5–7]. However, research on the inventory system has been changed dynamically over the last few decades. Research works have pointed out that one can make the lead-time as control variable, including crashing costs. Consequently, the organizations are being benefited due to having a low amount of safety stock.

Regarding *lead-time reduction*, Liao and Shyu [8] were the inventors to propose an inventory model considering variable lead-time and predetermined order quantity. Afterwards, the problem of lead-time reduction in the inventory system has been extensively studied by several researchers (see [9–21] among others).

Besides the problem of lead-time reduction, distribution of the lead-time demand is one of the most key aspects of the inventory system. It is challenging, and limited to obtaining the exact information of the lead-time demand distribution. In this context, a well-known technique, namely min–max distribution-free method, has been applied to get the organizational decision for the inventory system in which only the mean and variance of the lead-time demand distribution are known to us. This procedure was introduced by Scarf [22] and enhanced by Gallego and Moon [23]. Afterwards, numerous researchers (see [11, 15, 19, 24–28] among others) have considered the min–max distribution-free technique to get managerial decisions for different kinds of inventory system.

Furthermore, in many realistic situations, the stock-out cost contains several intangible integrants, such as a deficit of goodwill and hidden detention of the supplementary parts in the inventory system. Hence, it is not always attainable to determine the exact amount of stock-out cost [29]. A service level has been measured by a fraction of demands satisfied directly from stock. The replacement of stock-out cost by a service level constraint is a common practice in the theory of inventory system. Many authors thus attempted to substitute the cost during the stock-out period by a service level constraint. Moon and Choi [24] were the pioneer inventors to consider service level constraint in an inventory model system. After this, several

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related studies considered the service level constraint (see [27] and the references therein).

In the middle of the 1980s, researchers noticed that fuzziness is an intrinsic property of the key parameters for an inventory system (see [30–32] and the reference therein). In addition to randomness, randomness and vagueness frequently arise together in many realistic situations.

Kwakernaak [33] introduced the fuzzy random variable to characterize the fuzziness and randomness of an event concurrently. Afterwards, several researchers (see [34–36] among others) enhanced the concept of the fuzzy random variable and applied it into many mathematical fields. Dutta *et al* [37] first applied it to the field of inventory control and developed a single-period inventory model under discrete fuzzy random variable demand. The continuous-review inventory model is analysed in fuzzy random circumstances by Chang *et al* [38] and Dutta *et al* [39]. Dey and Chakraborty [40] develop a periodic-review inventory model in which the annual demand is considered as a discrete fuzzy random variable. Bag *et al* [41] consider the demand as the discrete fuzzy random variable to develop a production-inventory model with flexibility and reliability consideration in the production process. Dey and Chakraborty [42] enhanced the model of [40] considering variable lead-time; Bhuiya and Chakraborty [43] further extended the model of [42] by incorporating controllable backorder rate. A fuzzy random framework is constructed for the manufacturer and the buyer into a two-echelon integrated inventory system by Kumar *et al* [44]. Kumar and Goswami [45] develop the min–max distribution-free procedure to obtain the optimal solutions of the inventory system in fuzzy random environments. Kumar and Goswami [46, 47] analysed the fuzzy random EPQ models incorporating learning effect, possibility and necessity constraints. Bhuiya and Chakraborty [48] propose a fuzzy random EPQ model in which the production and inspection process is imperfect. Adhikary *et al* [49] adopted the procedure of [45] and developed a distribution-free newsvendor inventory system by considering demand as a fuzzy random variable. Khan and Dey [50, 51] reformulate the fuzzy random continuous- and periodic-review inventory model in which the demand rate is considered as a continuous fuzzy random variable. Recently, Bhuiya *et al* [52] developed the fuzzy random distribution-free continuous-review (Q, r, L) inventory model in which the backorder rate is dependent on lead-time, and purchasing cost is dependent on the order quantity. Thus, they extended the continuous-review (Q, r, L) inventory model by considering some more realistic features.

In the earlier existing literature on EPQ models, there have been many studies on obtaining the optimal production quantity and runtime. Kumar and Goswami [45] reported that the existing models had been developed by considering various realistic situations such as imperfect production process, inspection errors, learning effect and many others in the deterministic, probabilistic, fuzzy and fuzzy probabilistic framework. However, no production-inventory models have formulated with re-ordering strategy over the infinite time horizon. Moreover, in any manufacturing system, it is not always possible to start the production process instantaneously as and when required due to the set-up of the machine, delay in arrival of raw materials, unavailability of human resources, etc. Consequently, the lead-time is necessary for dealing with such a situation. Considering these facts in mind, Kumar and Goswami [45] developed a continuous-review production-inventory model. However, the lead-time is constant in their model, and the shortage during the production downtime is fully backlogged. As mentioned earlier, the production organizations and customers are both benefited by controlling the lead-time. Moreover, the inventory system that assembles backorders and lost sales cases is more practical than the ones founded on single cases. On the other hand, the service level constraint is not yet considered in the continuous-review production-inventory model. This article proposes an inventory model to address these gaps for the distribution-free continuous-review production-inventory model. For this purpose, we developed a distribution-free continuous-review production-inventory model with variable lead-time under a mixture of backorders and lost sales in the random framework. A service level constraint is imposed on the model instead of the stock-out cost. A methodology is proposed for obtaining closed-form optimal solutions. We justify rigorously that the developed method yields optimal global solutions. Furthermore, the random model is extended into the fuzzy random environment employing the fuzzy random variable as a demand rate. An algorithm is proposed to find the optimal global solution in this framework. Numerical examples are provided to illustrate the proposed methodologies. The rest of the paper is organized as follows.

In Sect. 2, we give some simple concepts and properties of fuzzy sets that will be required to formulate the fuzzy random model. In Sect. 3, the formulation of the random model is provided. An optimization procedure is designed to obtain the closed-form optimal global solution in the same section. The model is extended into the fuzzy random framework in Sect. 4. In this section, an algorithm is developed to find the global solutions of the fuzzy random model. In Sect. 5, two numerical examples are presented to demonstrate the proposed models. Finally, the conclusion and future direction of the work are provided in Sect. 6.

2. Preliminaries

This section presents a few basic definitions, which are applied to develop the model.

Definition 2.1 [33]. Let Ω be the event space of a crisp probability space and $\mathbb{F}(\mathbb{R})$ be the set of all fuzzy numbers.

A map $\tilde{Z}:\Omega \longrightarrow \mathbb{F}(\mathbb{R})$ is said to be a fuzzy random variable if the two real-valued function $\tilde{Z}_\alpha^- : \Omega \longrightarrow \mathbb{R}$ and $\tilde{Z}_\alpha^+ : \Omega \longrightarrow \mathbb{R}$ are real-valued random variables for all $\alpha \in [0,1]$, where $\tilde{Z}_\alpha^-$ and $\tilde{Z}_\alpha^+$ are the α-cut of $\tilde{Z}$.

Definition 2.2 [33]). The expected value of a fuzzy random variable $\tilde{Z}$ is denoted by $E(\tilde{Z})$, and defined as

$$E(\tilde{Z}) = \int_\Omega \tilde{Z} d\mathbb{P} = \bigcup_{\alpha\in[0,1]} \alpha \left[\int_\Omega \tilde{Z}_\alpha^- d\mathbb{P}, \int_\Omega \tilde{Z}_\alpha^+ d\mathbb{P} \right]. \quad (1)$$

Definition 2.3 [53]. Let $\tilde{B}$ be a fuzzy number; then the signed distance of $\tilde{B}$ is denoted by $S(\tilde{B},0)$, and defined as

$$S(\tilde{B},0) = \frac{1}{2}\int_0^1 (\tilde{B}_\alpha^- + \tilde{B}_\alpha^+) d\alpha. \quad (2)$$

3. Mathematical modelling and analysis in random framework

The following notations and assumptions are used to develop the inventory system under consideration, which are the same as in [45].

3.1 *Notations*

- D Mean random demand per unit time
- σ Standard deviation of the random demand per unit time
- A Set-up cost per set-up
- h Holding cost per product per unit time
- L Duration of the lead-time (decision variable)
- P Production rate per unit time
- $\hat{\alpha}$ Proportion of demands that are not met from stock; thus, $(1-\hat{\alpha})$ is service level
- β Backordered proportion rate; $\beta \in [0,1]$
- r Re-order level (decision variable)
- Q_p Production lot size (decision variable)
- Z Distribution of the lead-time demand
- z^+ Max$\{0, z\}$

3.2 *Assumptions*

1. Shortages are partially backlogged.
2. Let us suppose that Z_i denotes the random demand of the i^{th} unit period. We consider that Z_i, $i = 1,2,\ldots$ are i.i.d. random variables with mean D and standard deviation σ.
3. Applying the convolution of the demand rate and the length of the lead-time [45], the lead-time demand, Z, is given by

$$E(Z) = \sum_{i=1}^{L} E(Z_i) = DL \text{ and}$$

$$Var(Z) = \sum_{i=1}^{L} Var(Z_i) = \sigma^2 L.$$

4. The safety stock is defined as the difference between re-ordering point and demand during the lead-time. Let k be the safety factor; then the re-order point r can be defined as $r = DL + k\sigma\sqrt{L}$.
5. We assume that m mutually components characterize the lead-time L. Let a_i, b_i and c_i denote, respectively, the minimum duration, normal duration and unit crashing cost for the i^{th} component. Moreover, these c_i are considered to be arranged in ascending order.
6. The total normal lead-time is denoted by L_0, given by $L_0 = \sum_{j=1}^{m} b_j$. Let L_i be the length of lead-time corresponding to i^{th} component crashed to their minimum duration. For each $i = 1,2,\ldots,m$, L_i can be formulated as

$$L_i = \sum_{j=1}^{m} b_j - \sum_{j=1}^{i} (b_j - a_j).$$

Consequently, the crashing cost $C(L)$ associated with the lead-time in every inventory cycle for a given L in $[L_i, L_{i-1}]$ is represented as follows:

$$C(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j).$$

7. The customers' demand and production co-occur during the production runtime. We assume that $P > D$ to protect the stock-out during the production runtime [54], [55].
8. Using the approach of Hadley [56], the average inventory can be calculated as '$\frac{1}{2}$[minimum expected net inventory + maximum expected net inventory]'. The expected net inventory is minimum at the time of the beginning of the production process and is maximum at the time of the end of the production process.

3.3 *Mathematical formulation*

In this subsection, we shall formulate the continuous-review production-inventory system under a service level constraint and variable lead-time with a mixture of backorders and lost sales in the random framework. As mentioned in [56], an order of quantity Q is placed when the inventory position falls to re-order level r in a (Q, r) continuous-review inventory model. However, the replenishment of the whole order quantity Q in a single

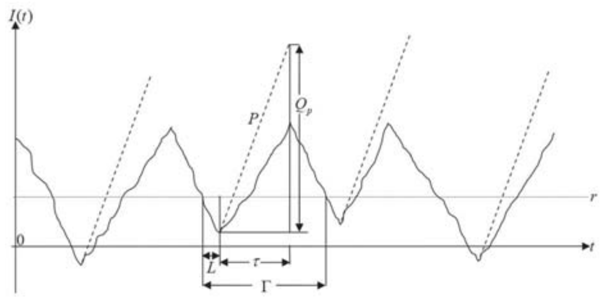

Figure 1. Graphical representation of the inventory system.

shipment at a time is not always practically possible. In this scenario, a distributor wishes to distribute the full order quantity in a time interval at some fixed rate. Moreover, sometimes, the production house manufactures the product internally, and concurrently satisfies the customer's demand. The graphical diagram of the behaviour of the inventory system in such circumstances is presented in figure 1, under the assumption that the products are internally manufactured [45].

As graphically illustrated, for the model in figure 1, the manufacturing company initiates to begin the production process to satisfy the demand of the succeeding cycle, when the inventory level falls down the re-order level r. However, the actual production process begins at the production rate P after an elapsed time L (called lead-time) due to the set-up time of the machine, deferment of receiving the raw materials, insufficiency of human resources, etc. The manufacturing process is carried on up to time τ. The minimum expected on-hand inventory is $(r - DL + (1-\beta) E(Z-r)^+)$ in the cycle, which will occur at the starting time of scheduling period. The maximum expected on-hand inventory is $r - DL + (P-D)\tau + (1-\beta)E(Z-r)^+ = (r - DL) + (P-D)\frac{Q_p}{P} + (1-\beta)E(Z-r)^+ = (r-DL) + Q_p\left(1 - \frac{D}{P}\right) + (1-\beta)E(Z-r)^+$ in the cycle, which will occur at the end of production process. Using assumption 3, the average on-hand inventory is $\frac{1}{2}[2\{r - DL + (1-\beta)E(Z-r)^+\} + (P-D)\tau]$. This problem aims at minimizing the expected cost per unit time, which is the sum of the set-up cost, production cost and holding cost subject to a constraint on the service level. The expected cost per cycle is given by

$$\begin{aligned} EC(Q_p, r) = & A + cQ_p \\ & + \frac{h}{2}[2\{r - DL + (1-\beta)E(Z-r)^+\} \\ & + (P-D)\tau]\Gamma. \end{aligned} \tag{3}$$

The expected scheduling period is $\Gamma = \frac{Q_p}{D}$. Using the renewal reward theorem, the expected cost per unit time is calculated as follows:

$$\begin{aligned} EUC(Q_p, r, L) = & \frac{AD}{Q_p} + cD + \frac{h}{2}\left[2\{r - DL + (1-\beta)E(Z-r)^+\}\right. \\ & \left. + Q_p\left(1 - \frac{D}{P}\right)\right] + \frac{C(L)D}{Q_p}. \end{aligned} \tag{4}$$

Thus, our goal is to optimize the mean value of the cost function per unit time subject to a restriction on service level by applying the distribution-free method. In this context, let $\mathcal{F}$ denote the class of distribution functions whose mean and variance are DL and $\sigma^2 L$, respectively. Therefore, the objective is to find the most unfavorable pdf $f_z \in \mathcal{F}$ for every decision variable, and later to minimize the decision spaces. Mathematically, the problem can be written as the following optimization problem:

$$(\mathrm{P}_1) \quad \begin{cases} \min\limits_{Q_p, r, L} & \left[\max\limits_{f_x \in \mathcal{F}} EUC(Q_p, r, L)\right] \\ \text{such that} & E(Z-r)^+ \leq \hat{\alpha} Q_p \\ & Q_p, r, L \geq 0. \end{cases} \tag{5}$$

Proposition 1 [23]. *For any* $f_z \in \mathcal{F}$

$$\begin{aligned} E(Z-r)^+ & \leq \frac{1}{2}\left[\sqrt{\sigma^2 L + (r - DL)^2} - (r - DL)\right] \\ & = \frac{1}{2}\sigma\sqrt{L}\left[\sqrt{1+k^2} - k\right]. \end{aligned} \tag{6}$$

Let us denote $EUCW(Q_p, k|L) = \max\limits_{f_x \in \mathcal{F}} EUC(Q_p, r, L)$. Using proposition 1 and assuming k to be a decision variable, instead of r, the problem (P_1) is reduced to

$$(\mathrm{P}_2) \quad \begin{cases} \min & EUCW(Q_p, k|L) \\ \text{such that} & \frac{1}{2}\sigma\sqrt{L}\left[\sqrt{1+k^2} - k\right] \leq \hat{\alpha} Q_p \\ & Q_p, k \geq 0, \end{cases} \tag{7}$$

where

$$\begin{aligned} EUCW(Q_p, k|L) = & \frac{[A + C(L)]D}{Q_p} + cD \\ & + h\left[\frac{Q_p(1 - \frac{D}{P})}{2} + k\sigma\sqrt{L}\right] \\ & + \frac{1}{2}\sigma\sqrt{L}\left(\sqrt{1+k^2} - k\right)(1-\beta)h. \end{aligned} \tag{8}$$

Adding a slack variable s^2, the constraint of the problem (P_2) is transformed into an equality constraint. Therefore, the Lagrangian for the optimization problem is given by

$$\mathfrak{L}(Q_p, k, \lambda, s|L) = \frac{[A + C(L)]D}{Q_p} + cD + h\left[\frac{Q_p\left(1 - \frac{D}{P}\right)}{2} + k\sigma\sqrt{L}\right] + \frac{1}{2}\sigma\sqrt{L}\left(\sqrt{1+k^2} - k\right)(1-\beta)h + \lambda\left[\sigma\sqrt{L}(\sqrt{1+k^2} - k) - 2\hat{\alpha}Q_p + s^2\right]. \quad (9)$$

Proposition 2 *The minimum value of the Lagrangian function $\mathfrak{L}(Q_p, k, \lambda, s|L)$ will occur at end point of $[L_i, L_{i-1}]$ for fixed Q_p and k.*

Proof The Lagrangian $\mathfrak{L}(Q_p, k, \lambda, s|L)$ is a concave function of $L \in [L_i, L_{i-1}]$ for fixed Q_p and k since $\frac{\partial^2 \mathfrak{L}(Q_p,k,\lambda,s|L)}{\partial L^2} < 0$. Hence the proof. □

Thus, for any fixed $L \in [L_i, L_{i-1}]$, Karush–Kuhn–Tucker (KKT) necessary conditions for a stationary point can be written as follows:

$$\frac{\partial \mathfrak{L}}{\partial Q_p} = -\frac{[A + C(L_i)]D}{Q_p^2} + \frac{h\left(1 - \frac{D}{P}\right)}{2} - 2\lambda\hat{\alpha} = 0, \quad (10)$$

$$\frac{\partial \mathfrak{L}}{\partial k} = h\sigma\sqrt{L_i} + \frac{1}{2}\sigma\sqrt{L_i}\left(\frac{k}{\sqrt{1+k^2}} - 1\right)[(1-\beta)h + 2\lambda] = 0, \quad (11)$$

$$\sigma\sqrt{L_i}\left[\sqrt{1+k^2} - k\right] - 2\hat{\alpha}Q_p \le 0, \quad (12)$$

$$\lambda\left[\sigma\sqrt{L_i}\left\{\sqrt{1+k^2} - k\right\} - 2\hat{\alpha}Q_p\right] = 0, \quad (13)$$

$$\lambda \ge 0. \quad (14)$$

From optimality conditions (10), (11) and complementary slackness condition (13), we get

$$Q_p = \sqrt{\frac{2D[A + C(L_i)]}{h\left(1 - \frac{D}{P}\right) - 4\lambda\hat{\alpha}}}, \quad (15)$$

$$\lambda = h\left[\frac{\sqrt{(1+k^2)}}{\sqrt{(1+k^2)} - k} - \frac{1}{2}(1-\beta)\right], = h\left[\left\{1 - \frac{1}{2}(1-\beta)\right\} + \frac{k}{\sqrt{(1+k^2)} - k}\right] \quad (16)$$

and

$$\sqrt{1+k^2} - k = \frac{2\hat{\alpha}Q_p}{\sigma\sqrt{L_i}}. \quad (17)$$

Manipulating Eqs. (15)–(17), we get the closed-form solutions of KKT point as follows:

$$Q_p = \sqrt{\frac{4\hat{\alpha}D[A + C(L_i)] + h\sigma^2 L_i}{2\hat{\alpha}h\left[\left(1 - \frac{D}{P}\right) - 2\hat{\alpha}\beta\right]}}, \quad (18)$$

$$k = \frac{\sigma^2 L_i - 4\hat{\alpha}^2 Q_p^2}{4\hat{\alpha}\sigma Q_p\sqrt{L_i}}, \quad (19)$$

and

$$\lambda = h\left[\frac{\beta}{2} + \frac{\sigma^2 L_i}{8\hat{\alpha}^2 Q_p^2}\right]. \quad (20)$$

Proposition 3 *The KKT point (Q_p, k, λ) is the global optimum point of the optimization problem* (P_2) *for any fixed $L \in [L_i, L_{i-1}]$.*

Proof Partial differentiations of $EUCW(Q_p, k|L)$ with respect to Q_p and k yield

$$\frac{\partial EUCW(Q_p, k|L)}{\partial Q_p} = -\frac{[A + C(L)]D}{Q_p^2} + \frac{h}{2}\left(1 - \frac{D}{P}\right), \quad (21)$$

and

$$\frac{EUCW(Q_p, k|L)}{\partial k} = h\sigma\sqrt{L} + \frac{1}{2}\sigma\sqrt{L}(1-\beta)h\sqrt{L}\left(\frac{k}{\sqrt{1+k^2}} - 1\right). \quad (22)$$

In addition

$$\frac{\partial^2 EUCW(Q_p, k|L)}{\partial Q_p^2} = \frac{[A + C(L)]2D}{Q_p^3} > 0, \quad (23)$$

$$\frac{\partial^2 EUCW(Q_p, k|L)}{\partial k^2} = \frac{h}{2}\sigma\sqrt{L}(1-\beta)\frac{1}{(1+k^2)^{\frac{3}{2}}} > 0 \quad (24)$$

and

$$\frac{\partial^2 EUCW(Q_p, k|L)}{\partial Q_p \partial k} = 0. \quad (25)$$

Thus, the objective function $EUCW(Q_p, k|L)$ of the optimization problem (P_2) is a convex function of Q_p and k, since

$$\frac{\partial^2 EUCW}{\partial Q_p^2} \times \frac{\partial^2 EUCW}{\partial k^2} - \left[\frac{\partial^2 EUCW}{\partial Q_p \partial k}\right]^2 = \frac{[A + C(L)]2D}{Q_p^3}\frac{h}{2}\sigma\sqrt{L}(1-\beta)\frac{1}{(1+k^2)^{\frac{3}{2}}} > 0. \tag{26}$$

Similarly, it can also be shown that the constraint $\frac{1}{2}\sigma\sqrt{L}[\sqrt{1+k^2}-k] - \hat{\alpha}Q_p$ of the problem (P_2) is also a convex function of Q_p and k. Thus, the problem (P_2) is a convex optimization problem. Hence, the proof is complete. □

Remark 1 It can be observed that our model has extended several ground-breaking models of this field. For instance, if we consider $P \to \infty$, $\beta = 1$ and L is constant then the present model is reduced to the model of [24]. This model is transformed to the model of [11] if we take $P \to \infty$.

Remark 2 It can also be noticed that the continuous-review production-inventory model is an extension of the primary continuous-review inventory model. Thus, the fundamental distinction between the present model and the model of Bhuiya *et al* [52] is the structural difference. Bhuiya *et al* [52] studied a more realistic continuous-review inventory model without service level constraint. On the other hand, this study incorporated the service level constraint into a continuous-review production-inventory model.

It is worthwhile to note that the model mentioned earlier describes the uncertainty of the demand parameter by representing the corresponding variable as a random variable. However, fuzziness and randomness often appear *combined* in many real-life situations [33]. For instance, the different analyses may have recommended distinctive estimations for forecasting the market demand; one expert may suggest that 'the demand is around D_1', while another expert may indicate that 'the demand is approximately D_2'. Due to the globalization of the business domain, consideration of both the fuzziness and randomness in the demand parameter of an inventory system is a more appropriate approach to tackle the present competitive and realistic situations. In this regard, we develop the fuzzy random model in the subsequent section.

4. Mathematical modelling and analysis in fuzzy random framework

In this section, the random model has been extended to the hybrid framework employing fuzziness and randomness in the demand parameter. In this context, we use the fuzzy random variable for capturing the randomness and vagueness of the demand parameter. Let $\tilde{Z}_i(\omega)$ denote the customer demand during i^{th} period, given by

$$\tilde{Z}_i(\omega) = (Z_i(\omega) - \Delta, Z_i(\omega), Z_i(\omega) + \Delta') = (z_i - \Delta, z_i, z_i + \Delta')$$

and $\tilde{D}$ denote the expectation of $\tilde{Z}_i(\omega)$, which is given by $\tilde{D} = (D - \Delta, D, D + \Delta')$, where z_i is the realization corresponding to each classical random variable Z_i. The expectation and standard deviation of Z_i are D and σ, respectively, subject to the restriction $D > \max\{\Delta, \Delta'\}$. The fuzzy random variable can be represented by a collection of random intervals $[\tilde{Z}_\alpha^-, \tilde{Z}_\alpha^+]$ in which $\tilde{Z}_\alpha^-$ and $\tilde{Z}_\alpha^+$ are real-valued random variables for $\alpha \in [0, 1]$ [52]. Moreover, the mean of $\tilde{Z}$ is a unique fuzzy number that is supported by finite intervals. Hence, the decision-maker is much more comfortable with calculating the demand per unit time by a range with membership value corresponding to every point instead of a single value [38, 45]. Even though the mean of a fuzzy random variable is a fuzzy number, a crisp measurement of inflation or diffusion of the fuzzy random variable about its expectation has been used to calculate the variance. It should have no fuzziness [35, 45]. Thus, the variance of the fuzzy random variable $\tilde{Z}_i$ is $\tilde{\sigma}^2$, and its membership function can be represented as

$$\mu_{\tilde{\sigma}^2}(t) = \begin{cases} 1 & \text{if } t = \sigma^2 \\ 0 & \text{if } t \neq \sigma^2. \end{cases} \tag{27}$$

Now, applying the convolution method, we calculate the fuzzy random lead-time demand rate. If the time gap between initiate the production process and start the actual production process is L units, then the lead-time demand can be estimated by L-fold convolution of the fuzzy random demand distribution. To calculate the fuzzy random demand during lead-time, we assume that the demands of i^{th} and $(i+1)^{\text{th}}$ unit time are independent as in [45]. Thus, for any $\omega \in \Omega, (\tilde{Z}_i + \tilde{Z}_{i+1})(\omega) = (z_i - \Delta, z_i, z_i + \Delta') + (z_{i+1} - \Delta, z_{i+1}, z_{i+1} + \Delta')$ [45]. Thus, if we apply the extension principle [57], then $(\tilde{Z}_i + \tilde{Z}_{i+1})(\omega) = (z_i + z_{i+1} - 2\Delta, z_i + z_{i+1}, z_i + z_{i+1} + 2\Delta')$ is a triangle fuzzy number. The fuzzy random demand during the lead-time length L is calculated by applying the same process as [45]

$$\tilde{Z} = \sum_{i=1}^{L} \tilde{Z}_i = \tilde{Z}_1 + \tilde{Z}_2 + \cdots + \tilde{Z}_L, \tag{28}$$

where $\tilde{Z}_i$, $i = 1, 2, \ldots, L$, represents the fuzzy random demand of i^{th} unit time with expectation $\tilde{D}$ and standard deviation $\tilde{\sigma}$. Employing the operation of α-cut in (28), we get

$$\tilde{Z}_\alpha^- = (\tilde{Z}_1)_\alpha^- + (\tilde{Z}_2)_\alpha^- + \cdots + (\tilde{Z}_L)_\alpha^- \tag{29}$$

and

$$\tilde{Z}_\alpha^+ = (\tilde{Z}_1)_\alpha^+ + (\tilde{Z}_2)_\alpha^+ + \cdots + (\tilde{Z}_L)_\alpha^+. \tag{30}$$

Here, $\tilde{Z}_\alpha^-$ and $\tilde{Z}_\alpha^+$ are also real-valued random variables because the sum of a finite number of real random variables is also a random variable. Therefore

$$E(\tilde{Z}_\alpha^-) = \sum_{i=1}^{L} E(\tilde{Z}_i)_\alpha^- = L\tilde{D}_\alpha^- = L(D - (1-\alpha)\Delta), \quad (31)$$

$$E(\tilde{Z}_\alpha^+) = \sum_{i=1}^{L} E(\tilde{Z}_i)_\alpha^+ = L\tilde{D}_\alpha^+ = L(D + (1-\alpha)\Delta'), \quad (32)$$

$$Var(\tilde{Z}_\alpha^-) = \sum_{i=1}^{L} Var(\tilde{Z}_i)_\alpha^- = \sigma^2 L \text{ and} \quad (33)$$

$$Var(\tilde{Z}_\alpha^+) = \sum_{i=1}^{L} Var(\tilde{Z}_i)_\alpha^+ = \sigma^2 L. \quad (34)$$

The subtraction of the re-order point and demand during the lead-time is generally considered as a safety stock of the inventory system. Thus, the safety stock is denoted by $\widetilde{SS}$, and given by $\widetilde{SS} = r - L\tilde{D}$. The $\widetilde{SS}$ is a fuzzy number due to mean customer demand during the lead-time $L\tilde{D} = \bigcup_{\alpha\in[0,1]} \alpha\left[L\tilde{D}_\alpha^-, L\tilde{D}_\alpha^+\right]$, which is a fuzzy number. We can re-write the safety stock $\widetilde{SS}$ as follows using the decomposition principle:

$$\begin{aligned}\widetilde{SS} &= \bigcup_{\alpha\in[0,1]} \alpha\left[r - L\tilde{D}_\alpha^+, r - L\tilde{D}_\alpha^-\right] \\ &= \bigcup_{\alpha\in[0,1]} \alpha[r - L(D + \Delta'(1-\alpha)), r - L(D - \Delta(1-\alpha))] \\ &= \bigcup_{\alpha\in[0,1]} \alpha\left[k\sigma\sqrt{L} - L\Delta'(1-\alpha), k\sigma\sqrt{L} + L\Delta(1-\alpha)\right].\end{aligned} \quad (35)$$

Now, we can find the fuzzy expected total cost function per unit time. The cost function of the fuzzy random (Q_p, r, L) inventory model has the following cost components:

1. mean set-up cost per unit of time = $\frac{A\tilde{D}}{Q_p}$,
2. mean production cost per unit of time = $c\tilde{D}$,
3. mean carrying cost per unit of time = $\frac{h}{2}\left[2\left\{r - \tilde{D}L + (1-\beta)E(\tilde{Z} - r)^+\right\} + Q_p(1 - \frac{\tilde{D}}{P})\right]$,
4. mean value of lead-time crashing cost per unit of time = $\frac{C(L)\tilde{D}}{Q_p}$.

Therefore, the goal of the decision-maker is to optimize the fuzzy mean value of the total cost function per unit of time, subject to a restriction on the service level. Mathematically, the model is formulated by the following optimization problem:

$$(\mathrm{P}_3)\quad \begin{cases} \min & \widetilde{EUC}(Q_p, r|L) \\ \text{such that} & E(\tilde{Z} - r)^+ \le \hat{\alpha} Q_p \\ & Q_p, r \ge 0, \end{cases} \quad (36)$$

where

$$\widetilde{EUC}(Q_p, r|L) = \frac{h}{2}\left[2\left\{r - \tilde{D}L + (1-\beta)E(\tilde{Z} - r)^+\right\} + Q_p\left(1 - \frac{\tilde{D}}{P}\right)\right] + \frac{\tilde{D}[A + C(L)]}{Q_p} + c\tilde{D}. \quad (37)$$

Applying the principle of decomposition, we modify the objective function of the optimization problem (P_3).

$$\widetilde{EUC}(Q_p, r|L) = \bigcup_{\alpha\in[0,1]} \alpha\left[\widetilde{EUC}_\alpha^-, \widetilde{EUC}_\alpha^+\right], \quad (38)$$

where

$$\begin{aligned}\widetilde{EUC}_\alpha^- = &\frac{h}{2}\left[2\left\{r - \tilde{D}_\alpha^+ L + (1-\beta)E(\tilde{Z}_\alpha^- - r)^+\right\} \right.\\ &\left. + Q_p\left(1 - \frac{\tilde{D}_\alpha^+}{P}\right)\right] \\ &+ \frac{\tilde{D}_\alpha^-[A + C(L)]}{Q_p} + c\tilde{D}_\alpha^-\end{aligned} \quad (39)$$

and

$$\begin{aligned}\widetilde{EUC}_\alpha^+ = &\frac{h}{2}\left[2\left\{r - \tilde{D}_\alpha^- L \right.\right.\\ &\left.\left. + (1-\beta)E(\tilde{Z}_\alpha^+ - r)^+\right\} + Q_p\left(1 - \frac{\tilde{D}_\alpha^-}{P}\right)\right] \\ &+ \frac{\tilde{D}_\alpha^+[A + C(L)]}{Q_p} + c\tilde{D}_\alpha^+.\end{aligned} \quad (40)$$

We de-fuzzify the objective cost function $\widetilde{EUC}(Q, r|L)$ by employing the signed distance method. Utilizing Eqs. (2) and (35), we calculate the de-fuzzify the objective cost function as follows:

$$\begin{aligned}\overline{EUC}(Q_p, k|L) &= \frac{1}{2}\int_0^1 \left[\widetilde{EUC}_\alpha^- + \widetilde{EUC}_\alpha^+\right] d\alpha \\ &= \left[\frac{(A + C(L) + cQ_p)(4D + \Delta' - \Delta)}{4Q_p}\right] + h\left[k\sigma\sqrt{L} + \frac{L}{4}(\Delta - \Delta') \right.\\ &\left. + (1-\beta)G^*(k, L) + \frac{Q_p}{2}\left(1 - \frac{4D + \Delta' - \Delta}{4P}\right)\right],\end{aligned} \quad (41)$$

where

$$G^*(k,L) = \frac{1}{2}\int_0^1 \left[E(\tilde{Z}_\alpha^- - r)^+ + E(\tilde{Z}_\alpha^+ - r)^+\right] d\alpha. \tag{42}$$

Therefore, the mathematical programming problem (P_3) is converted into the following correspondent optimization problem:

$$(P_4)\quad \begin{cases} \min & \overline{EUC}(Q_p, k|L) \\ \text{such that} & G^*(k,L) \le \hat{\alpha} Q_p \\ & Q_p, k, L \ge 0. \end{cases} \tag{43}$$

Proposition 4 *If the lead-time demand is distribution-free, then $G^*(k,L) \le M(k,L)$, where $M(k, L)$ is given by* [52]

$$M(k,L) = \frac{1}{4}\left[\int_0^1 \left\{\sqrt{\sigma^2 L + [k\sigma\sqrt{L} + L\Delta(1-\alpha)]^2} - 2k\sigma\sqrt{L} + \sqrt{\sigma^2 L + [k\sigma\sqrt{L} - L\Delta'(1-\alpha)]^2} + L(1-\alpha)(\Delta' - \Delta)\right\} d\alpha\right].$$

We now solve this problem using the distribution-free method. In this regard, we utilize proposition 4. Applying proposition 4 and the min–max distribution-free procedure, the optimization problem (P_4) is converted to

$$(P_5)\quad \begin{cases} \min & \overline{EUC}(Q_p, k|L) \\ \text{such that} & M(k,L) \le \hat{\alpha} Q_p \\ & Q_p, k, L \ge 0, \end{cases} \tag{44}$$

where

$$\overline{EUC}(Q,k|L) = \frac{[A + C(L) + cQ_p](4D + \Delta' - \Delta)}{4Q_p} + h\left[k\sigma\sqrt{L} + \frac{L}{4}(\Delta - \Delta') + (1-\beta)M(k,L) + \frac{Q_p}{2}\left(1 - \frac{4D + \Delta' - \Delta}{4P}\right)\right]. \tag{45}$$

We now introduce a slack variable $\hat{s}^2$ for converting the inequality constraint into an equality constraint. Therefore, the Lagrangian function is calculated by

$$\begin{aligned}\hat{\hat{\mathfrak{L}}}(Q_p, k, \lambda, \hat{s}|L) &= \frac{[A + C(L) + cQ_p](4D + \Delta' - \Delta)}{4Q_p} + h\left[k\sigma\sqrt{L} + \frac{L}{4}(\Delta - \Delta') + (1-\beta)M(k,L) + \frac{Q_p}{2}\left(1 - \frac{4D + \Delta' - \Delta}{4P}\right)\right] \\ &\quad + \lambda(2M(k,L) - 2\hat{\alpha}Q_p + \hat{s}^2).\end{aligned} \tag{46}$$

For fixed $L \in [L_i, L_{i-1}]$, the KKT necessary conditions for a critical point are given by

$$\frac{\partial \hat{\hat{\mathfrak{L}}}}{\partial Q_p} = -\frac{[A + C(L)](4D + \Delta' - \Delta)}{4Q_p^2} + \frac{h}{2}\left(1 - \frac{4D + \Delta' - \Delta}{4P}\right) - 2\lambda\hat{\alpha} = 0, \tag{47}$$

$$\frac{\partial \hat{\hat{\mathfrak{L}}}}{\partial k} = h\sigma\sqrt{L} - \frac{\sigma\sqrt{L}}{4}[(1-\beta)h + 2\lambda]\left[\int_0^1 \left(2 - \frac{k\sigma\sqrt{L} + L\Delta(1-\alpha)}{\sqrt{\sigma^2 L + [k\sigma\sqrt{L} + L\Delta(1-\alpha)]^2}} - \frac{k\sigma\sqrt{L} - L\Delta'(1-\alpha)}{\sqrt{\sigma^2 L + [k\sigma\sqrt{L} - L\Delta'(1-\alpha)]^2}}\right) d\alpha\right] = 0. \tag{48}$$

$$M(k,L) - \hat{\alpha}Q_p \le 0, \tag{49}$$

$$\lambda(M(k,L) - \hat{\alpha}Q_p) = 0, \tag{50}$$

$$\lambda \ge 0. \tag{51}$$

From Eqs. (47) and (48), we obtain

$$Q_p = \sqrt{\frac{(4D + \Delta' - \Delta)(A + C(L))}{\left[2h\left(1 - \frac{4D+\Delta'-\Delta}{4P}\right) - 8\lambda\hat{\alpha}\right]}}, \tag{52}$$

and

$$[(1-\beta)h+2\lambda]\left[\Delta'\left\{\sqrt{\sigma^2L+(k\sigma\sqrt{L}+L\Delta)^2}-\sqrt{\sigma^2L+k^2\sigma^2L}\right\}\right.$$
$$\left.+\Delta\left\{\sqrt{\sigma^2L+k^2\sigma^2L}-\sqrt{\sigma^2L+(k\sigma\sqrt{L}-L\Delta')^2}\right\}-2L\Delta\Delta'\right]$$
$$+4hL\Delta\Delta'=0. \tag{53}$$

Proposition 5 *The KKT point* (Q_p, k, λ) *is the global optimum point of the optimization problem* (P_5) *for any fixed* $L \in [L_i, L_{i-1}]$.

Proof Two partial differentiations of $\overline{EUC}_M(Q_p, k|L)$ with respect to Q_p and k yield

$$\frac{\partial^2 \overline{EUC}}{\partial Q_p^2}=\frac{(4D+\Delta'-\Delta)}{2Q^3}[(A+C(L))]>0, \tag{54}$$

$$\frac{\partial^2 \overline{EUC}}{\partial k^2}=\frac{1}{2}h(1-\beta)\frac{\partial^2 M(k,L)}{\partial k^2}>0, \tag{55}$$

and

$$\frac{\partial^2 \overline{EUC}}{\partial k \partial Q_p}=0, \tag{56}$$

where

$$\frac{\partial^2 M(k,L)}{\partial k^2}=\frac{1}{4}\int_0^1\left(\frac{\sigma^4L^2}{\sqrt[3]{\sigma^2L+[k\sigma\sqrt{L}+L\Delta(1-\alpha)]^2}}+\frac{\sigma^4L^2}{\sqrt[3]{\sigma^2L+[k\sigma\sqrt{L}-L\Delta'(1-\alpha)]^2}}\right)d\alpha. \tag{57}$$

Thus, the objective function of (P_5) is a convex function of (Q_p, k) for any fixed $L \in [L_i, L_{i-1}]$. In a similar manner, it can easily be proved that the constraint $M(k,L)-\hat{\alpha}Q_p$ of (P_5) is also a convex function of (Q_p, k). Hence, the problem (P_5) is a convex optimization problem. This completes the proof. □

We notice that it is not possible to get closed-form expression for Q_p and k by calculating Eqs. (52) and (53). Thus, we propose algorithm (1) for obtaining the optimal global solution of the optimization problem (P_5) using the following concept. From the complementary slackness condition (51), we can say that either $\lambda = 0$ or $\lambda > 0$. Now, if we consider $\lambda = 0$, we can find Q_p from (52). Consequently, we can get k from (53) using the value of Q_p. If this (Q_p, k) satisfies the feasibility conditions, then (Q_p, k) is the optimal solution of the optimization problem (P_5) for a fixed $L \in [L_i, L_{i-1}]$. Otherwise, consider the initial approximate Q_p^0 using (18). Then we can make an iterative loop to obtain the values of (Q_p, k, λ) employing Eqs. (50), (53) and (52).

Algorithm 1 For obtaining the global optimal solution of Problem (P_5)

Require: A, D, h, c, $\hat{\alpha}$, and σ.

1: **for** $i = 0, 1, 2, \cdots, m$ **do**
2: Set $\lambda_i \leftarrow 0$.
3: Compute the production quantity ${Q_p}^i$ from equation (52).
4: Determine the safety factor k_i using equation (53).
5: **if** $M(k,L)-\hat{\alpha}{Q_p}^i \le 0$, **then**
6: $\left(\hat{Q_p}^i, \hat{k}_i, \hat{L}_i, \hat{\lambda}_i\right) \leftarrow \left(Q_p^i, k_i, L_i, \lambda_i\right)$.
7: Compute the total cost of the objective function $\overline{EUC}\left(\hat{Q_p}^i, \hat{k}_i \middle| \hat{L}_i\right)$ of Problem (P_5).
8: Go to Step 29.
9: **else**
10: Go to Step 12.
11: **end if**
12: Set $Q_p^i \leftarrow Q_p^0$ (Guess the value of Q_p^0 by using 18).
13: **repeat**
14: Use the value of Q_p^i in equation (50) to calculate the value of λ_i.
15: Put the valueλ_i in equation (53) and obtain the safety factor k_i.
16: Put the value of k_i in equation (52) and determine Q_p^i.
17: **until** no sufficient diversity arises in the values of Q_p^i, k_i, and λ_i
18: **if** $M(k,L)-\hat{\alpha}Q_p^i \le 0$, **then**
19: Set $\left(\hat{Q_p}^i, \hat{k}_i, \hat{L}_i, \hat{\lambda}_i\right) \leftarrow \left(Q_p^i, k_i, L_i, \lambda_i\right)$.
20: Compute the total cost of the objective function $\overline{EUC}\left(\hat{Q_p}^i, \hat{k}_i \middle| \hat{L}_i\right)$ of Problem (P_5).
21: **end if**
22: **if** $\left(\hat{Q_p}^i, \hat{k}_i, \hat{L}_i, \hat{\lambda}_i\right)$ is an optimal point **then**
23: Set $\overline{EUC}(Q_p^{i^*}, k_i^*|L_i^*) \leftarrow \overline{EUC}(\hat{Q_p}^i, \hat{k}_i|\hat{L}_i)$.
24: Go to Step 29.
25: **else**
26: The optimization problem (P_5) has no optimal point for i.
27: Go to Step 2 for next i.
28: **end if**
29: Denote the solution by $(Q_p^{i^*}, k_i^*, L_i^*, \lambda_i^*)$, and compute the corresponding total expected cost $\overline{EUC}(Q_p^{i^*}, k_i^*|L_i^*)$ of the problem (P_5).
30: Go to Step 2 for next i.
31: **end for**
32: **if** $\min_{0\le i\le m}\left\{\overline{EUC}(Q_p^{i^*}, k_i^*|L_i^*)\right\}$ has finite value **then**
33: Set $\overline{EUC}(Q_p^{**}, k^{**}|L^{**}) \leftarrow \min_{0\le i\le m}\left\{\overline{EUC}(Q_p^{i^*}, k_i^*|L_i^*)\right\}$.
34: The optimal solution of Problem (P_5) is $(Q_p^{**}, k^{**}, L^{**}, \lambda^{**})$.
35: **else**
36: The problem (P_5) has no optimal solution.
37: **end if**

5. Numerical examples

This section presents two numerical examples for exploring the methodologies of random and the fuzzy random models developed in Sects. 3 and 4, respectively. We illustrate the effectiveness of the proposed methodology and closed-form optimal solution of the random model by considering the following numerical example.

Example 1 We consider the inventory system with the data used in [11] and given by $D = 600$ units/year, $\sigma^2 = 49$ units/week, $A = \$200/$ set-up, $h = \$20/\text{unit}/\text{year}$, $\beta = 0.5$, service level $1 - \hat{\alpha} = 0.985$, i.e. the proportion of demand that is not met from stock is $\hat{\alpha} = 0.015$ and the lead-time has three components, which are given in table 1. Moreover, we take $P = 3000$ units/year.

Applying the accomplished closed-form solutions, we determine the optimum solutions for each lead-time component, which are summarized in table 2. From the table, we get the minimum cost $EUCW(Q_p^{**}, k^{**}|L^{**}) =$ \$393.1180, which is attained at optimal production quantity $Q_p^{**} = 159.1271$ units, re-order level $r^{**} = 64.169143$ units and lead-time $L^{**} = 4$ weeks. It is interesting to observe that the production quantity and safety factor decrease with the decrease in the length of lead-time, even though the crashing cost increases with decrease of lead-time. However, the minimum expected cost is attained at the lead-time $L^{**} = 4$ weeks.

In the next example, we apply the proposed methodology developed in Sect. 4.

Example 2 We consider the same data used in Example 1, excluding the demand of a product to be around 12 units per week, which is characterized as a triangular fuzzy number $\tilde{D} = (D - \Delta, D, D + \Delta')$, where $D = 12$, $\Delta = 2.3$ and $\Delta' = 2.7$.

We obtain the optimal global solution for different components of lead-time using algorithm 1. These solutions are tabulated in table 3. From table 3, we get the optimum values of the design variable of the inventory system in the fuzzy random environment as follows. The optimum order quantity $Q_p^{**} = 162.5952$ units, the optimum re-order point $r^{**} = 67.367515$ units, optimal lead-times $L^{**} = 4$ weeks and the minimum expected cost $\overline{EUC}(Q_p^{**}, k_i^{**}|L_i^{**})$ $= \$411.9181$. In table 2, it is observed that the re-order point and the value of the Lagrangian multiplier decrease with increasing crashing cost. A similar feature occurs in the random model. However, the production lot size and re-order level in the fuzzy random environment are bigger than in the random setting. Hence, in the mixed framework, the production house requires holding additional items as a safety stock to tackle the fuzziness and randomness of the demand parameter. Consequently, the expected cost of the fuzzy random inventory system is higher than that in the random inventory system. The human learning in the fuzziness of the parameters of the inventory system may be incorporated to reduce the effect of vagueness [58].

Table 1. Lead-time data.

i	Normal duration b_i (days)	Minimum duration a_i (days)	Unit crashing cost c_i (\$/day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

5.1 *Sensitivity*

In this subsection, we analyse and show different numerical results through the sensitivity analysis in both frameworks. Sensitivity analysis is conducted over the backorder rate β, service level parameter $\hat{\alpha}$, demand and cost parameters in both the random and fuzzy random frameworks. The results of sensitivity analysis are shown in tables 4, 5, 6 and 7. From tables 4 and 5, it is interestingly stated that the optimal production quantity Q_p increases; the optimal cost and re-order point r decrease with increasing backorder rate β in both the cases. Moreover, it is also observed that when the service level parameter $\hat{\alpha}$ increases, the decision-maker decreases the production quantity Q_p and re-order point r, to reduce the holding cost. Consequently, it decreases the total cost. From tables 6 and 7, it can be pointed out that the production quantity Q_p and total cost increase, and re-order point r decreases with the increase of set-up costs. Moreover, we observe that the production quantity Q_p and re-

Table 2. Solutions in the random framework (in weeks).

(i)	(L_i)	$C(L_i)$	$(Q_p^{i*}, k_i^*, \lambda_i^*)$	$EUCW(Q_p^{i*}, k_i^*\|L_i^*)$
0	8	0	(178.6954, 1.711241, 2.711794)	399.0099
1	6	5.6	(167.9248, 1.554893, 2.317565)	395.7669
2	4	22.4	(159.1271,1.295839, 1.745307)	393.1180
3	3	57.4	(160.9842, 1.056066, 1.304576)	393.6772

Table 3. Solutions in the fuzzy random environment (in weeks).

(i)	(L_i)	$C(L_i)$	$(Q_p^{i^*}, k_i^*, \lambda_i^*)$	$\overline{EUC}(Q_p^{i^*}, k_i^* \mid L_i^*)$
0	8	0	(179.7096, 1.900743, 2.530642)	418.3274
1	6	5.6	(169.9341, 1.695877, 2.161907)	414.7877
2	4	22.4	(162.5952,1.383394, 1.6218196)	411.9181
3	3	57.4	(164.5184, 1.125466, 1.225655)	412.4393

Table 4. Sensitivity analysis with respect to $\hat{\alpha}$ for different values of β in random framework.

β		Change in $\hat{\alpha}$				
		0.013	0.014	0.015	0.016	0.017
0.3	*EUCW*	394.5392	393.8805	393.3008	392.7861	392.3259
	Q_p^{**}	162.3579	160.3117	158.5225	156.9453	155.5454
	r^{**}	67.13231	65.61549	64.256524	63.02944	61.91375
	L^{**}	4	4	4	4	4
0.5	*EUCW*	394.3769	393.7080	393.1180	392.5931	392.1226
	Q_p^{**}	162.8934	160.8818	159.1271	157.5846	156.2193
	r^{**}	67.04903	65.53016	64.169143	62.9400	61.8223
	L^{**}	4	4	4	4	4
0.8	*EUCW*	394.1327	393.4482	392.8426	392.3021	391.8160
	Q_p^{**}	163.7067	161.7484	160.0472	158.5584	157.2469
	r^{**}	66.92351	65.40146	64.037328	62.80512	61.68431
	L^{**}	4	4	4	4	4

Table 5. Sensitivity analysis with respect to $\hat{\alpha}$ for different values of β in fuzzy random framework.

β		Change in $\hat{\alpha}$				
		0.013	0.014	0.015	0.016	0.017
0.3	$\overline{EUC}$	413.2823	412.6555	412.1039	411.6146	411.1772
	Q_p^{**}	165.5534	163.6368	161.2443	159.9773	157.8996
	r^{**}	70.24004	68.76925	67.55036	66.33147	65.34726
	L^{**}	4	4	4	4	4
0.5	$\overline{EUC}$	413.1173	412.4799	411.9181	411.4182	410.9695
	Q_p^{**}	166.0079	164.1973	162.5952	160.0729	159.5880
	r^{**}	70.17387	68.69088	67.36752	66.31889	65.13169
	L^{**}	4	4	4	4	4
0.8	$\overline{EUC}$	412.8691	412.2157	411.6375	411.1215	410.6571
	Q_p^{**}	167.1660	165.2858	163.4337	161.1006	159.8067
	r^{**}	70.00679	68.54005	67.25538	66.18449	65.10406
	L^{**}	4	4	4	4	4

order point r increase with the increase in demand rate for a fixed lead-time. On the other hand, production quantity Q_p decreases and the re-order point r increases with the increase in holding cost for a fixed lead-time. However, the overall cost has been increasing in both cases. It is also a notable observation that if the demand is low or the cost of carrying an item is high, then the lead-time is lower than the usual lead-time.

Table 6. Sensitivity analysis for the demand and cost parameters in random framework.

Parameter		Parameter change (%)						
		−50%	−40%	−20%	0%	+20%	+40%	+50%
A	$EUCW$	385.2389	386.9339	390.1325	393.1180	395.9281	398.5905	399.8730
	Q_p^{**}	132.9591	138.5886	149.2116	159.1271	168.4599	177.3023	181.5619
	r^{**}	68.60196	67.51953	65.68206	64.16914	62.89185	61.79214	61.29598
	L^{**}	4	4	4	4	4	4	4
D	$EUCW$	213.1415	249.5826	321.8663	393.1180	463.9572	534.4274	569.5340
	Q_p^{**}	119.4242	127.9739	144.4258	159.1271	173.9634	189.1186	196.8709
	r^{**}	35.98401	37.93723	57.27382	64.16914	71.40132	78.87468	82.68097
	L^{**}	3	3	4	4	4	4	4
h	$EUCW$	376.1760	379.7928	386.6272	393.1180	399.3887	405.2572	408.0439
	Q_p^{**}	205.7187	191.4522	171.9626	159.1271	149.9611	142.4599	139.1329
	r^{**}	58.82091	60.21819	62.44434	64.16914	65.56141	49.58148	50.04262
	L^{**}	4	4	4	4	4	3	3

Table 7. Sensitivity analysis for the demand and cost parameters in fuzzy random framework.

Parameter		Parameter change (%)						
		−50%	−40%	−20%	0%	+20%	+40%	+50%
A	$\overline{EUC}$	403.7606	405.5196	408.8323	411.9181	414.8175	417.5604	418.8804
	Q_p^{**}	134.3525	141.0290	151.2358	162.5952	171.3419	180.7148	185.6820
	r^{**}	71.87289	70.65926	68.99543	67.36752	66.24545	65.14862	64.60595
	L^{**}	4	4	4	4	4	4	4
D	$\overline{EUC}$	224.4401	262.3914	337.6916	411.9181	485.7046	559.0965	595.6544
	Q_p^{**}	121.3510	130.7281	146.4327	162.5952	177.2753	193.4325	202.1995
	r^{**}	35.98401	39.50268	60.15199	67.36752	75.13952	83.00627	86.96327
	L^{**}	3	3	4	4	4	4	4
h	$\overline{EUC}$	394.6766	398.3656	405.3234	411.9181	418.2787	424.1801	427.0012
	Q_p^{**}	210.6194	195.5383	175.4514	162.5952	152.3514	145.5569	141.8899
	r^{**}	62.20761	63.59812	65.75213	67.36752	68.82592	51.82099	52.29959
	L^{**}	4	4	4	4	4	3	3

6. Conclusion

In this paper, we have considered a continuous-review production-inventory model with a mixture of backorders and lost sales. The model has been developed under the assumption that the distribution is known partially, and the lead-time is a control variable. Moreover, a restriction on service level has been imposed on the model. Firstly, the model has been formulated considering demand as a random variable. In this framework, we have obtained a closed-form global optimal solution for the production quantity, re-order level and variable lead-time. Secondly, we have considered the demand as a fuzzy random variable for extending the model into the fuzzy random environment. In this framework, it is complicated to get closed-form solutions. In this context, an algorithm has been developed to find the optimal global solutions in the fuzzy random framework. Finally, we have demonstrated the proposed methodologies by two numerical examples.

There are several useful extensions of our model that can be done for future studies in this field. For example, this model can be extended by considering the imperfect production process and screening process. It would be interesting to study the effect of learning of inspection errors on the production quantity. Moreover, one could enhance the present model under various inventory frameworks such as integrated inventory model, two-warehouse inventory model, models with discount, etc.

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