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# Metric issues for systems in state space

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**Abstract.** A metric structure is imposed on state space such that solutions to 1st order state variable model of a linear system are geodesics subject to restrictions on the system and input matrices. For linear time invariant systems, a 2nd order state variable model of an extended system to a given 1st order model is derived. The metric is tailored to systems under consideration.

**Keywords.** Linear system; state variable; state space; metric; geodesic.

## 1. Introduction

The state variable models of linear systems consist of a set of first order coupled differential equations whose solutions as functions of time are curves in state space with the state variables as coordinates. There is no a priori assumed geometric structure to state space. Here, a metric structure is imposed and its consequences derived. The geodesic equation is expressed in a form in which the metric is expressed as a function of curve parameter (time). The relevant mathematical background can be found in [1].

## 2. Metric and geodesic equation

Consider a n dimensional state space, a coordinate system $x^\alpha$, vector field $\mathbf{V} = V^\alpha \mathbf{e}_\alpha$ with components $V^\alpha$ and basis vectors $\mathbf{e}_\alpha$, $\alpha = 1,\ldots,n$ in which summation is implied between the same index appearing as superscript and subscript. The coordinates and vector field components are expressed as column vectors x and V, respectively. The state space has a metric $\mathbf{g}$ whose components are expressed as a symmetric matrix g with corresponding connection coefficients Γ. Consider a curve in state space with parameter t whose tangent vector field is $\mathbf{V}$ ($V^\alpha = dx^\alpha/dt$) and the derivative of $\mathbf{V}$ along the curve such that $d\mathbf{V}/dt = \mathbf{0}$. This is parallel transport of $\mathbf{V}$ along the curve which is then a geodesic.

$$d\mathbf{V}/dt = dV^\alpha/dt\, \mathbf{e}_\alpha + V^\alpha d\mathbf{e}_\alpha/dt \tag{1}$$

From definition of connection coefficients

$$d\mathbf{e}_\alpha/dt = \partial \mathbf{e}_\alpha/\partial x^\beta\, dx^\beta/dt = \Gamma^\mu_{\alpha\beta}\mathbf{e}_\mu V^\beta \tag{2}$$

Substituting (2) in (1) and relabelling index

$$d\mathbf{V}/dt = dV^\alpha/dt\, \mathbf{e}_\alpha + V^\alpha \Gamma^\mu_{\alpha\beta} V^\beta \mathbf{e}_\mu$$

$$d\mathbf{V}/dt = \left(dV^\mu/dt + V^\alpha \Gamma^\mu_{\alpha\beta} V^\beta\right)\mathbf{e}_\mu \tag{3}$$

The components of g are $g_{\mu\eta}$ and $g^{-1}$ are $g^{\mu\eta}$. The connection coefficients in terms of the metric are

$$\Gamma^\mu_{\alpha\beta} = (1/2)g^{\mu\eta}(g_{\eta\alpha,\beta} + g_{\eta\beta,\alpha} - g_{\alpha\beta,\eta}) \tag{4}$$

$$\Gamma^\mu_{\alpha\beta}V^\alpha V^\beta = (1/2)g^{\mu\eta}(g_{\eta\alpha,\beta}V^\alpha V^\beta + g_{\eta\beta,\alpha}V^\alpha V^\beta - g_{\alpha\beta,\eta}V^\alpha V^\beta) \tag{5}$$

Notation $g_{\alpha\beta,\eta} = \partial g_{\alpha\beta}/\partial x^\eta$. Index lowering property $g_{\alpha\beta}V^\alpha = V_\beta$. Since $\mathbf{V}$ is parallel transported, $\mathbf{g}(\mathbf{V},\mathbf{V}) = g_{\alpha\beta}V^\alpha V^\beta = V^T gV$ = constant on the t curve or that $d(V^T gV)/dt = 0$; this does not imply that its coordinate derivative equals 0; its coordinate derivative will equal 0 if $V^T gV$ is a true constant (below). From (5)

$$g_{\eta\alpha,\beta}V^\alpha V^\beta = \partial g_{\eta\alpha}/\partial x^\beta\, dx^\beta/dt V^\alpha = dg_{\eta\alpha}/dt\, V^\alpha$$

$$g_{\eta\beta,\alpha}V^\alpha V^\beta = \partial g_{\eta\beta}/\partial x^\alpha\, dx^\alpha/dt V^\beta = dg_{\eta\beta}/dt\, V^\beta$$

$$\begin{aligned} g_{\alpha\beta,\eta}V^\alpha V^\beta &= (g_{\alpha\beta}V^\alpha V^\beta)_{,\eta} - V_\beta V^\beta_{,\eta} - V_\alpha V^\alpha_{,\eta} \\ &= -V_\beta V^\beta_{,\eta} - V_\alpha V^\alpha_{,\eta} \end{aligned} \tag{6}$$

From (3) and (6) (assuming $(g_{\alpha\beta}V^\alpha V^\beta)_{,\eta} = 0$)

$$d\mathbf{V}/dt = (dV^\mu/dt + g^{\mu\eta}\, d(g_{\eta\delta})/dt\, V^\delta + g^{\mu\eta}V_\delta V^\delta_{,\eta})\mathbf{e}_\mu = \mathbf{0} \tag{7}$$

$W = [\partial V/\partial x]$, $W_{ij} = \partial V^i/\partial x^j$, (7) is (T transpose)

$$dV/dt = -g^{-1}dg/dt\, V - g^{-1}W^T gV$$

$$d(gV)/dt + W^T gV = 0 \tag{8}$$

$dV(x,t)/dt = [\partial V/\partial x]V + \partial V/\partial t$; if constraint $dV/dt = WV$

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is imposed such that $V^T gV$ is a true constant, $(g_{\alpha\beta}V^\alpha V^\beta)_{,\eta} = 0$ and (8) is

$$\left(dg/dt + W^T g + gW\right)V = 0,\ \partial V/\partial t = 0 \tag{9}$$

The constraint on V and (9) implies

$$dg/dt = -\left(W^T g + gW\right),\ \partial V/\partial t = 0 \tag{10}$$

If $(g_{\alpha\beta}V^\alpha V^\beta)_{,\eta} \neq 0$, $d(gV)/dt = ½\,[V^T[\partial g/\partial x^i]V]$, $i = 1, \ldots, n$, a column vector.

## 3. Systems and geodesics

It is specifically assumed that V is of the form

$$V(t) = dx(t)/dt = A(t)x(t) + B(t)u(t) \tag{11}$$

This models a linear system with state variables x(t) and input u(t) [2]. This is a Linear Time Varying (LTV) system if A and B are functions of time, a Linear Time Invariant (LTI) system results if A and B are constant.

LTI System: With $u(t) = u$ a constant, $\partial V/\partial t = 0$ and (10) and its solution is

$$dg/dt = -\left(A^T g + gA\right)$$

$$g(t) = \exp(-tA^T)g(0)\exp(-tA) \tag{12}$$

Using (12), (8) is

$$gV = \exp(-tA^T)g(0)V(0)$$

$$V(t) = dx(t)/dt = \exp(tA)V(0)$$

$$V(0) = Ax(0) + Bu$$

$$x(t) = A^{-1}(\exp(tA) - I)V(0) + x(0) \tag{13}$$

In (13), x(t) is a geodesic curve and $V^T gV = V^T(0)g(0)V(0)$ a true constant.

LTV System: From (8), (10)

$$dg(t)/dt = -\left(A^T(t)g(t) + g(t)A(t)\right)$$

$$dV(t)/dt = A(t)V(t) \tag{14}$$

The general closed form solution of (14) can be expressed if A(t) commutes with $\int_t A(\tau)d\tau$,

$$V(t) = \exp\left(\int_t A(\tau)d\tau\right)V(0)$$

$$g(t) = \exp\left(-\int_t A^T(\tau)d\tau\right)[g(0)]\exp\left(-\int_t A(\tau)d\tau\right) \tag{15}$$

Then $V^T gV = V^T(0)g(0)V(0)$, a true constant. Using $\partial V/\partial t = 0$ in (11)

$$x(t) = -[dA(t)/dt]^{-1}d(B(t)u(t))/dt \tag{16}$$

This x(t) must satisfy (11) and one sub-optimum condition is for A(t), B(t)u(t) to satisfy $d^2A/dt^2 = -[dA/dt]\ A$, $d^2(Bu)/dt^2 = -[dA/dt]\ (Bu)$. This in addition to the commuting requirement.

Extended LTI System (2nd order state variable equation): For a LTI system, modify (8) to

$$d(gV)/dt + A^T gV = gABu(t) \tag{17}$$

In (17) if V = Ax is used, (10) for g implies (11). The LHS is "acceleration" and the RHS is "force". Consider a LTI system (system 0) $dx/dt = A\ x(t) + B\ u(t)$ whose solution ($x(0) = 0$) is [2]

$$x(t) = \exp(tA)\int_t \exp(-\tau A)Bu(\tau)d\tau \tag{18}$$

Consider (17) but with $V = dx/dt \neq Ax$. This system 1 with initial conditions 0 is

$$d(dx(t)/dt)/dt = A(dx(t)/dt) + ABu(t)$$

$$x(t) = \int_t \exp(tA)\left[\int_t \exp(-\tau A)ABu(\tau)d\tau\right]dt \tag{19}$$

that is, x(t) (system 1) = A ∫ [x(t) (system 0)] dt = x(t) (system 0) − B ∫ u(t) dt which constitutes an extended signal processing on input u(t). By repeating this procedure one can define further systems, for example, starting with $dx/dt = Ax + ABu$ and using the metric as in (12) one can define system 2 as $d(dx(t)/dt)/dt = A\ (dx(t)/dt) + A^2\ B\ u(t)$ so that

$$x(t)(\text{system } 2) = A\int_t [x(t)(\text{system } 1)]dt \tag{20}$$

The input matrix $[B\ AB\ A^2B \ldots A^{n-1}B]$ is the controllability matrix with maximum rank n. Extended system responses are not geodesics.

LTI System Controllability: Given an initial and final point in state space there is a geodesic ("straight line") connecting them. At the initial point, specifying an initial velocity (speed + direction) specifies the geodesic. In the left side of (17), the system matrix A by itself may not be able to satisfy the velocity criterion via V = Ax. A constant u can be used to satisfy the criterion since in gABu, $\exp(-tA)$ can be expanded into sum of powers $A^k$, $k = 0,1,\ldots,n-1$ with $A^kB$ linearly independent. This viewpoint is natural in this geometric setting.

A Comment: For LTI/LTV systems, the metric is expressed as a function of t; this does not imply that it is independent of the coordinates since it depends on the system matrices A/A(t) which are defined in a particular coordinate system x. For autonomous systems ($u(t) = 0$), $x(t) = \exp(tA)x(0)$ (LTI); define $g^{-1} = x(t)x^T(t)$; then $g = \exp(-tA^T)\ g(0)\ \exp(-tA)$.

*General Parallel Transport*: Consider vector field **P**($\tau$) (P = dx($\tau$)/d$\tau$) with curve parameter $\tau$ and vector field **Q**(t) (Q = dx(t)/dt) with curve parameter t. If **P** is parallel transported along **Q** then for **P•Q** a true constant

$$[g]dP/dt + (1/2)[dg/dt]P + (1/2)[dg/d\tau]Q$$

$$+(1/2)[\partial Q/\partial x]^T[g]P + (1/2)[\partial P/\partial x]^T[g]Q = 0 \qquad (21)$$

Let Q = Cx, P = Dx assuming constant C, D that commute (CD = DC). If they are linearly independent, $\tau$ is constant along t curve and vice versa since their Lie bracket equals 0 or that dP($\tau$)/dt = dQ(t)/d$\tau$ [3] and they define a two-dimensional coordinate grid. Define $g(t) = \exp(-tC^T)[\exp(-\tau D^T)\exp(-\tau D)]\exp(-tC)$, $g(\tau) = \exp(-\tau D^T)[\exp(-tC^T)\exp(-tC)]\exp(-\tau D)$ for which **P•Q** is a true constant. From (21)

$$dP_\tau(t)/dt = CP_\tau(t), P_\tau(t) = \exp(tC)P_\tau(0) \qquad (22)$$

## 4. Conclusion

By using a metric, a second order structure is imposed on a first order structure and as far as the system response is concerned the procedure is redundant since the response is unaffected. However, using this formulation, extended (distinct) systems have been defined in a geometric context. The inverse problem, that is, for an arbitrary metric, if the corresponding vector field can be found then this field can define a new class of systems.

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## References

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