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Computational method for generalized fractional Benjamin–Bona–Mahony–Burgers equations arising from the propagation of water waves

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Abstract. In this research, by utilizing the concept of the mixed Caputo fractional derivative and left-sided mixed Riemann–Liouville fractional integral, we approximate the solution of generalized fractional Benjamin–Bona–Mahony–Burgers equations (GF-BBMBEs). In addition, using Genocchi polynomial properties, we obtain a new formula to approximate the functions by Genocchi polynomials. In the process of computation, we discuss a method of obtaining the operational matrix of integration and pseudo-operational matrices of the fractional order of derivative. Also, an algorithm of obtaining the mixed fractional integral operational matrix is presented. Using the collocation method and matrices introduced, the proposed equations are converted to a system of nonlinear algebraic equations with unknown Genocchi coefficients. In addition, we discuss the upper bound of the error for the proposed method. Finally, we examine several problems to demonstrate the validity and applicability of the proposed method.

Keywords. Genocchi polynomials; pseudo-operational matrix; Benjamin–Bona–Mahony–Burgers equations; left-sided mixed Riemann–Liouville fractional integral; mixed Caputo fractional derivative; error estimation.

1. Introduction

Many researchers investigated water wave problems, which play an important role in the fields of naval architecture and ship design, offshore structures, physical oceanography and marine hydrodynamics. Water waves appear in various physical phenomena such as solid state physics, fluid mechanics, chemical kinetics, plasma physics, population models, nonlinear optics and drift waves in plasma or the Rossby waves in rotating fluids [1–5]. The Korteweg–de Vries (KdV) equation is the partial differential equation derived by Korteweg and de Vries [6] to describe weakly nonlinear shallow water waves. A general form of the KdV equation is

$$u_t + u_{xxx} + u_x + uu_x = 0, \qquad (1)$$

where $u = u(x,t) : I \times R^+ \to R$ denotes the wave surface and $I \subset R$ is a bounded interval. Benjamin *et al* [7] (1972) study the Benjamin–Bona–Mahony equation for the first time, which is a regularized long-wave equation and is applicable to shallow water waves and to the study of drift waves in plasma or the Rossby waves in rotating fluids. They offered to replace the term u_{xxx} by $-u_{txx}$ in KdV equation to get

$$u_t - u_{txx} + u_x + uu_x = 0. \qquad (2)$$

Also, when the viscosity is taken into account in the modelling of long gravity waves, the Benjamin–Bona–Mahony equation turns into [8–10]

$$u_t - u_{txx} - \nu u_{xx} + u_x + uu_x = 0, \qquad \nu > 0, \qquad (3)$$

which is called the Benjamin–Bona–Mahony–Burgers equation. Burgers is added to the name of the equation because Eq. (3) features a balance between the nonlinear and dispersive effects but takes no account of dissipation. The dispersive effect of Eq. (3) is the same as that of the Benjamin–Bona–Mahony equation due to dispersive term u_{xxt} whereas the dissipative effect due to dissipative term u_{xx} is the same as in the Burgers equation

$$u_t - \nu u_{xx} + u_x + uu_x = 0, \qquad \nu > 0. \qquad (4)$$

The general form of Benjamin–Bona–Mahony–Burgers equation is defined as

$$u_t - u_{txx} - \nu u_{xx} + (f(u))_x = q, \qquad \nu > 0, \qquad (5)$$

where f and q are suitable nonlinear functions. This equation attracted the attention of many researchers and has been studied from various points of view. Zhang *et al*

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[11], by means of an extended tanh-function method with $f(u) = u^2$, obtained its solitary-wave solutions. Kaya [12] presented the Adomian decomposition scheme with $f(u) = u^p$, p as a positive integer, to obtain solitary-wave solutions. Abdollahzadeh *et al* [13] studied Benjamin–Bona–Mahony–Burgers equations by the $f(u) = \frac{u^2}{2}$ by $\frac{G'}{G}$-expansion method. Al-Khaled *et al* [14] used the Adomian decomposition method to approximate wave solutions for generalized Benjamin–Bona–Mahony–Burgers equations. Mekki and Ali [15] employed a finite-difference method to solve Kadomtsev–Petviashvili–Benjamin–Bona–Mahony. Dehghan *et al* [16] applied the meshless method of radial basis functions for solving the nonlinear generalized Benjamin–Bona–Mahony–Burgers equation. Noor *et al* [17] considered the exp-function method to construct some new solitary solutions of the Benjamin–Bona–Mahony and modified Benjamin–Bona–Mahony equations.

Wazwaz and Triki [18] propose some bright-type soliton solutions of the time-dependent coefficient Benjamin–Bona–Mahony equation with a simple damping term. Many other approaches exist for this type of equation, which can be seen in [19–23].

In the present work, we consider the generalized fractional Benjamin–Bona–Mahony–Burgers equations (GF-BBMBEs). According to the mixed Caputo fractional-order derivative, we change the integer-order derivative to the fractional-order derivative. These changes are performed easily, but solving this equation is not convenient. Hence, we present a numerical scheme to obtain an approximate solution of GF-BBMBEs

$$D_t^{\gamma} u - D_0^{r} u - \nu u_{xx} + (f(u))_x = g(x,t), \qquad (6)$$

with initial condition

$$u(x,0) = g_0(x), \qquad x \in [a,b], \qquad (7)$$

and boundary conditions

$$u(0,t) = \varphi_0(t), \quad u(1,t) = \varphi_1(t), \qquad t \in [0,T], \quad T > 0, \qquad (8)$$

where $u = u(x,t)$ is an unknown function and the known functions $g_0(x)$, $\varphi_0(t)$, $\varphi_1(t)$, $\gamma(x,t)$ and f are defined on interval $\Omega = [a,b] \times [0,T]$. D_t^{γ} denotes the fractional derivatives in the Caputo sense of order $0<\gamma\leq 1$ with respect to variable t, and D_{θ}^{r}, $\theta = (0,0)$ denotes the fractional derivatives in the Caputo sense of order $r = (r_1, r_2)$ with respect to variables x and t, where $1<r_1\leq 2$, $0<r_2\leq 1$ and also a, b are positive real constants.

During the years 1817–1889, Angelo Genocchi introduced Genocchi numbers and Genocchi polynomials. Genocchi numbers have been extensively discussed in various contexts in many branches of mathematics [24–30]. In recent years, Genocchi polynomials have received increasing attention. Therefore, the researchers applied these polynomials for solving some problems. For instance, Isah and Phang [31, 32] applied Genocchi polynomials for the solution of delay differential equations and non-linear fractional differential equations, Loh *et al* [33] considered a new operational matrix via Genocchi polynomials for solving the Fredholm-Volterra fractional integro-differential equation and Phang *et al* [34] used Genocchi polynomials for solving fractional optimal control problems. In this research, the Genocchi polynomials with collocation and operational and pseudo-operational matrix are applied to convert the problem into an algebraic system. The superiority of Genocchi polynomials is discussed completely in [39]. Also, one of the advantages of the proposed method is the pseudo-operational matrix. This matrix is more accurate than the operational matrix in the algorithm of approach. The outline of the paper is as follows. In section 2, we describe some definitions and properties of the fractional calculus. In section 3, we present the properties of Genocchi polynomials. Section 4 is devoted to operational and pseudo-operational matrix of integration with integer and fractional order. The main algorithm of the method is described in section 5. In section 6, we consider convergence analysis and error estimates for the proposed method. In section 7, we demonstrate the accuracy of the proposed method by considering several test problems. Section 8 consists of a brief conclusion.

2. Preliminaries

In this section, we focus on some definitions and properties of the variable-order fractional calculus theory.

Definition 1 The Riemann–Liouville fractional integral operator with order $\gamma > 0$ of $u(x, t)$ is defined as [35]

$$I_t^{\gamma} u(x,t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} u(x,s)ds, \qquad (9)$$

where $t > 0$ and $\Gamma(.)$ denotes the Gamma function.

Based on this definition, fractional integration has the following useful property:

$$I_t^{\gamma} t^{\beta} = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)} t^{\beta+\gamma}, & \beta > -1, \\ 0, & \text{otherwise.} \end{cases} \qquad (10)$$

Definition 2 The fractional derivative of $u(x, t)$ in the Caputo sense is defined as [35, 36]

$$ {}_0D_t^{\gamma} u(x,t) = I_t^{q-\gamma} D_t^{q} u(x,t) = \frac{1}{\Gamma(q-\gamma)} \int_0^t (t-s)^{q-\gamma-1} \frac{\partial^q u(x,s)}{\partial s^q} ds, \qquad (11)$$

for $q-1<\gamma\leq q$, $t>0$ and $q\in N$.

It has the following useful property:

$$ {}_0D_t^{\gamma}t^{\beta}=\begin{cases}\dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)}t^{\beta-\gamma}, & q\leq\beta\in N,\\ 0, & \text{otherwise.}\end{cases} \tag{12}$$

Definition 3 Let $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, $\theta=(0,0)$ and $u\in L^1([0,a]\times[0,b])$. The left-sided mixed Riemann–Liouville integral of order $r(x,t)$ of u is defined by [37]

$$(I_\theta^r u)(x,t)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^t(x-\xi)^{r_1-1}(t-\eta)^{r_2-1}u(\xi,\eta)d\eta d\xi. \tag{13}$$

It has the following useful properties:

- $(I_\theta^\theta u)(x,t)=u(x,t)$.
- $(I_\theta^\delta u)(x,t)=\int_0^x\int_0^t u(\xi,\eta)d\xi d\eta \quad \forall(x,t)\in[0,a]\times[0,b],$

$$\delta=(1,1).$$

- If $u\in L^1([0,a]\times[0,b])$, then I_θ^r exists for all $r_1,r_2\in(0,\infty)$. Moreover, $I_\theta^r u\in C([0,a]\times[0,b])$ provided $u\in C([0,a]\times[0,b])$ and

$$(I_\theta^r u)(x,0)=(I_\theta^r u)(0,t)=0. \tag{14}$$

- Let $\lambda,\omega\in(-1,\infty)$ and $r_1,r_2\in(0,\infty)$. Then

$$I_\theta^r x^\lambda t^\omega=\frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\lambda+r_2)}x^{\lambda+r_1}t^{\omega+r_2}. \tag{15}$$

Definition 4 Let $r\in(0,1]\times(0,1]$ and $u\in L^1([0,a]\times[0,b])$. The Caputo fractional-order derivative of order r of u is defined by the expression [37]

$$\begin{aligned}D_\theta^r u(x,t)&=(I_\theta^{1-r}D_{xt}^2u)(x,t)\\&=\frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)}\int_0^x\int_0^t\frac{D_{xt}^2u(\xi,\eta)}{(x-\xi)^{r_1}(t-\eta)^{r_2}}d\xi d\eta.\end{aligned} \tag{16}$$

By $1-r$, we mean $(1-r_1,1-r_2)\in(0,1]\times(0,1]$. Here $D_{xt}^2=\frac{\partial^2}{\partial x\partial t}$, the mixed second-order partial derivative. It has the following useful properties:

- The case $\sigma=(1,1)$ is included, and we have

$$(D_\theta^\delta u)(x,t)=(D_{xt}^2u)(x,t) \qquad \forall(x,t)\in[0,a]\times[0,b]. \tag{17}$$

- Let $\lambda,\omega\in(-1,\infty)$ and $r=(r_1,r_2)\in(0,1]\times(0,1]$. Then

$$D_\theta^r x^\lambda t^\omega=\frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\lambda-r_2)}x^{\lambda-r_1}t^{\omega-r_2}. \tag{18}$$

3. Properties of Genocchi polynomials

In this section, Genocchi polynomials and their properties are presented.

3.1 *Genocchi polynomials*

The classical Genocchi polynomials $G_m(x)$ are defined by means of the exponential generating functions as follows [33, 38–41]:

$$\frac{2te^{xt}}{e^t+1}=\sum_{m=0}^{\infty}G_m(x)\frac{t^m}{m!} \quad (|t|<\pi). \tag{19}$$

The Genocchi polynomials of degree m are defined as follows:

$$G_m(x)=\sum_{k=0}^{m}\binom{m}{k}g_{m-k}x^k=2B_m(x)-2^{m+1}B_m\left(\frac{x}{2}\right), \tag{20}$$

where $g_k=2B_k-2^{k+1}B_k$ is the Genocchi number. B_m and $B_m(x)$ are the Bernoulli number and Bernoulli polynomial, respectively. The first few Genocchi polynomials are given as follows:

$$\begin{aligned}G_0(x)&=0, \quad G_1(x)=1, \quad G_2(x)=2x-1,\\G_3(x)&=3x^2-3x, \quad G_4(x)=4x^3-6x^2+1.\end{aligned} \tag{21}$$

We introduce some important properties of Genocchi polynomials

- $G_m(1)+G_m(0)=0, \quad m>1.$
- $\frac{dG_m(x)}{dx}=mG_{m-1}(x), \quad m\geq1.$
- $\frac{d^kG_m(x)}{dx^k}=\begin{cases}0, & m\leq k,\\ k!\binom{m}{k}G_{m-k}(x), & m>k,\end{cases} \qquad k,m\in N\cup0.$
- $\int_0^1G_m(x)G_n(x)dx=\frac{2(-1)^m m!n!}{(m+n)!}g_{m+n}, \qquad m,n\geq1.$
- $\int_a^bG_m(x)dx=\frac{G_{m+1}(b)-G_{m+1}(a)}{m+1}.$
- $\int_0^xG_m(x)dx=\frac{G_{m+1}(x)-g_{m+1}}{m+1}.$

3.2 *Function approximation*

A function f defined over [0, 1] may be expressed in terms of Genocchi polynomials as

$$f(x) \simeq \sum_{m=1}^{M} c_m G_m(x) = C^T G(x). \tag{22}$$

The vector coefficient C is given by

$$C = D^{-1}\langle f(x), G(x)\rangle, \quad D = \langle G(x), G(x)\rangle, \tag{23}$$

where

$$C = [c_1, c_2, \ldots, c_M]^T, \quad G(x) = [G_1(x), G_2(x), \ldots, G_M(x)]^T. \tag{24}$$

Theorem 1 *Let f be a sufficiently smooth function approximated by the truncated Genocchi series $\sum_{m=1}^{M} c_m G_m(x)$. Then the coefficients c_m for $m = 1, 2, \ldots, M$, can be calculated from the following relation* [31]:

$$c_m = \frac{1}{2m!}\left(f^{(m-1)}(1) + f^{(m-1)}(0)\right). \tag{25}$$

Assume that

$$\{\hat{G}_{11}(x,t), \ldots, \hat{G}_{1N}(x,t), \ldots, \hat{G}_{M1}(x,t), \ldots, \hat{G}_{MN}(x,t)\}$$

is a set of 2D-Genocchi functions. Let $f(x, t)$ be an arbitrary element of $L^2([0,1] \times [0,1])$ and

$$S_G = span\{\hat{G}_{11}(x,t), \ldots, \hat{G}_{1N}(x,t), \ldots, \hat{G}_{M1}(x,t), \ldots, \hat{G}_{MN}(x,t)\},$$

where

$$\hat{G}_{mn}(x,t) = G_m(x)G_n(t), \qquad m = 1, 2, \ldots, M, \quad n = 1, 2, \ldots, N. \tag{26}$$

Since S_G is a finite-dimensional vector space, $f(x, t)$ has the unique best approximation out of S_G as

$$f(x,t) \simeq \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}\hat{G}_{mn}(x,t) = G(x)^T A G(t), \tag{27}$$

where the matrix coefficient A is defined as

$$A = D^{-1}\langle\langle f(x,t), G(x)\rangle, G(t)\rangle \hat{D}^{-1}, \quad \hat{D} = \langle G(t), G(t)\rangle. \tag{28}$$

Theorem 2 *Suppose that $f(x, t)$ is a sufficiently smooth function in $\Omega = [0,1] \times [0,1]$ and let*

$$f(x,t) = \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}\hat{G}_{mn}(x,t) \tag{29}$$

be its expansion in terms of 2D-Genocchi functions as described in Eq. (27)*. Then, the coefficients a_{mn} can be calculated as follows:*

$$\begin{aligned} a_{mn} = & \frac{1}{4m!n!}\left(\frac{\partial^{m+n-2} f(1,1)}{\partial x^{m-1}\partial t^{n-1}} + \frac{\partial^{m+n-2} f(1,0)}{\partial x^{m-1}\partial t^{n-1}}\right. \\ & \left. + \frac{\partial^{m+n-2} f(0,0)}{\partial x^{m-1}\partial t^{n-1}} + \frac{\partial^{m+n-2} f(0,1)}{\partial x^{m-1}\partial t^{n-1}}\right), \\ & m = 1, 2, \ldots, M, \qquad n = 1, 2, \ldots, N. \end{aligned} \tag{30}$$

Proof Due to the properties of Genocchi polynomials, we have

$$\begin{aligned} f(0,t) &= \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}G_m(0)G_n(t), \\ f(1,t) &= \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}G_m(1)G_n(t). \end{aligned} \tag{31}$$

Then

$$\begin{aligned} f(0,t) + f(1,t) &= \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}G_m(0)G_n(t) \\ &\quad + \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}G_m(1)G_n(t) \\ &= \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}(G_m(0) + G_m(1))G_n(t) \\ &= \sum_{n=1}^{N} a_{1n}(G_1(0) + G_1(1))G_n(t) \\ &= \sum_{n=1}^{N} 2a_{1n}G_n(t). \end{aligned} \tag{32}$$

Replacing in the variable t in Eq. (32) by 0 and 1, we get

$$\begin{aligned} & f(0,1) + f(1,1) + f(0,0) + f(1,0) \\ & \quad = \sum_{n=1}^{N} 2a_{1n}(G_n(1) + G_n(0)) = 4a_{11}. \end{aligned} \tag{33}$$

As a result, we obtain

$$a_{11} = \frac{1}{4(1!)(1!)}(f(0,1) + f(1,1) + f(0,0) + f(1,0)). \tag{34}$$

According to Eq. (32) and derivative property of Genocchi polynomials, we have

$$\begin{aligned}
\frac{\partial f(0,t)}{\partial t}+\frac{\partial f(1,t)}{\partial t} &= \sum_{n=1}^{N} 2a_{1n}G_n'(t)\\
&= \sum_{n=2}^{N} 2a_{1n}nG_{n-1}(t),\\
\frac{\partial^2 f(0,t)}{\partial t^2}+\frac{\partial^2 f(1,t)}{\partial t^2} &= \sum_{n=1}^{N} 2a_{1n}G_n''(t)\\
&= \sum_{n=3}^{N} 2a_{1n}n(n-1)G_{n-2}(t),\\
&\vdots\\
\frac{\partial^{N-1} f(0,t)}{\partial t^{N-1}}+\frac{\partial^{N-1} f(1,t)}{\partial t^{N-1}} &= \sum_{n=1}^{N} 2a_{1n}G_n^{(N-1)}(t)\\
&= 2a_{1N}N!G_1(t).
\end{aligned} \tag{35}$$

Hence, replacing the variable t in Eq. (35) by 0 and 1, we get

$$\begin{aligned}
&\frac{\partial f(0,1)}{\partial t}+\frac{\partial f(1,1)}{\partial t}+\frac{\partial f(0,0)}{\partial t}+\frac{\partial f(1,0)}{\partial t}\\
&= \sum_{n=2}^{N} 2a_{1n}n(G_{n-1}(1)+G_{n-1}(0))\\
&= 4(1!)(2!)a_{12},\\
&\frac{\partial^2 f(0,1)}{\partial t^2}+\frac{\partial^2 f(1,1)}{\partial t^2}+\frac{\partial^2 f(0,0)}{\partial t^2}+\frac{\partial^2 f(1,0)}{\partial t^2}\\
&= \sum_{n=3}^{N} 2a_{1n}n(n-1)(G_{n-2}(1)+G_{n-2}(0))\\
&= 4(1!)(3!)a_{13},\\
&\vdots\\
&\frac{\partial^{N-1} f(0,1)}{\partial t^{N-1}}+\frac{\partial^{N-1} f(1,1)}{\partial t^{N-1}}+\frac{\partial^{N-1} f(0,0)}{\partial t^{N-1}}+\frac{\partial^{N-1} f(1,0)}{\partial t^{N-1}}\\
&= 2a_{1N}N!(G_1(1)+G_1(0))\\
&= 4(1!)(N!)a_{1N}.
\end{aligned} \tag{36}$$

As a result

$$\begin{aligned}
a_{1n} = &\frac{1}{4(1!)(n!)}\left(\frac{\partial^{n-1} f(0,1)}{\partial t^{n-1}}+\frac{\partial^{n-1} f(1,1)}{\partial t^{n-1}}\right.\\
&\left.+\frac{\partial^{n-1} f(0,0)}{\partial t^{n-1}}+\frac{\partial^{n-1} f(1,0)}{\partial t^{n-1}}\right), \qquad n=1,2,\ldots,N.
\end{aligned} \tag{37}$$

To calculate the other coefficients, we need the derivative of $f(x, t)$ with respect to x:

$$\begin{aligned}
\frac{\partial f(x,t)}{\partial x} &= \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}G_m'(x)G_n(t)\\
&= \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}mG_{m-1}(x)G_n(t).
\end{aligned} \tag{38}$$

Then

$$\begin{aligned}
\frac{\partial f(1,t)}{\partial x}+\frac{\partial f(0,t)}{\partial x} &= \sum_{m=2}^{M}\sum_{n=1}^{N} a_{mn}m(G_{m-1}(1)\\
&\quad +G_{m-1}(0))G_n(t)\\
&= \sum_{n=1}^{N} 2a_{2n}(G_1(1)+G_1(0))G_n(t)\\
&= \sum_{n=1}^{N} 4a_{2n}G_n(t).
\end{aligned} \tag{39}$$

Substituting 0 and 1 for the variable t in Eq. (39), we get

$$\begin{aligned}
&\frac{\partial f(1,1)}{\partial x}+\frac{\partial f(0,1)}{\partial x}+\frac{\partial f(0,0)}{\partial x}+\frac{\partial f(1,0)}{\partial x}\\
&= \sum_{n=1}^{N} 4a_{2n}(G_n(1)+G_n(0)) = 8a_{21}.
\end{aligned} \tag{40}$$

Hence

$$a_{21} = \frac{1}{4(2!)(1!)}\left(\frac{\partial f(1,1)}{\partial x}+\frac{\partial f(0,1)}{\partial x}+\frac{\partial f(0,0)}{\partial x}+\frac{\partial f(1,0)}{\partial x}\right). \tag{41}$$

Using derivative of $\frac{\partial f(x,t)}{\partial x}$ with respect to t, we get

$$\begin{aligned}
\frac{\partial f^2(0,t)}{\partial x\partial t}+\frac{\partial f^2(1,t)}{\partial x\partial t} &= \sum_{n=1}^{N} 4a_{2n}G_n'(t) = \sum_{n=2}^{N} 4a_{2n}nG_{n-1}(t),\\
\frac{\partial^3 f(0,t)}{\partial x\partial t^2}+\frac{\partial^3 f(1,t)}{\partial x\partial t^2} &= \sum_{n=1}^{N} 4a_{2n}G_n''(t)\\
&= \sum_{n=3}^{N} 4a_{2n}n(n-1)G_{n-2}(t),\\
&\vdots\\
\frac{\partial^N f(0,t)}{\partial x\partial t^{N-1}}+\frac{\partial^N f(1,t)}{\partial x\partial t^{N-1}} &= \sum_{n=1}^{N} 4a_{2n}G_n^{(N-1)}(t)\\
&= 4a_{2N}N!G_1(t).
\end{aligned} \tag{42}$$

Therefore, we obtain

$$
\begin{aligned}
&\frac{\partial f^2(0,1)}{\partial x \partial t}+\frac{\partial f^2(1,1)}{\partial x \partial t}+\frac{\partial f^2(0,0)}{\partial x \partial t}+\frac{\partial f^2(1,0)}{\partial x \partial t} \\
&\quad =\sum_{n=2}^{N} 4a_{2n} n(G_{n-1}(1)+G_{n-1}(0)) \\
&\quad =4(2!)(2!)a_{22}, \\
&\frac{\partial^3 f(0,1)}{\partial x \partial t^2}+\frac{\partial^3 f(1,1)}{\partial x \partial t^2}+\frac{\partial^3 f(0,0)}{\partial x \partial t^2}+\frac{\partial^3 f(1,0)}{\partial x \partial t^2} \\
&\quad =\sum_{n=3}^{N} 4a_{2n} n(n-1)(G_{n-2}(1)+G_{n-2}(0)) \\
&\quad =4(2!)(3!)a_{23}, \\
&\quad \vdots \\
&\frac{\partial^N f(0,1)}{\partial x \partial t^{N-1}}+\frac{\partial^N f(1,1)}{\partial x \partial t^{N-1}}+\frac{\partial^N f(0,0)}{\partial x \partial t^{N-1}}+\frac{\partial^N f(1,0)}{\partial x \partial t^{N-1}} \\
&\quad =4a_{2N} N!(G_1(1)+G_1(0)) \\
&\quad =4(2!)(N!)a_{2N}.
\end{aligned}
\tag{43}
$$

According to these calculations, we have

$$
\begin{aligned}
a_{2n} &= \frac{1}{4(2!)(n!)} \\
&\quad \left(\frac{\partial^n f(0,1)}{\partial x \partial t^{n-1}}+\frac{\partial^n f(1,1)}{\partial x \partial t^{n-1}}+\frac{\partial^n f(0,0)}{\partial x \partial t^{n-1}}+\frac{\partial^n f(1,0)}{\partial x \partial t^{n-1}}\right), \\
&\quad n=2,\ldots,N.
\end{aligned}
\tag{44}
$$

Consequently, we obtain the desired result with the continuation of this process. □

Remark 1 It is worth noting that the authors in [42] present these results in a different way.

4. Operational and pseudo-operational matrix of integration of Genocchi polynomials

The purpose of the present section is to introduce the operational matrix of integration and pseudo-operational matrix of mixed fractional integration for Genocchi polynomials.

4.1 *Operational and pseudo-operational matrix of integration*

In the following theorem, we apply integral properties of Genocchi polynomials to obtain the operational matrix of integration.

Method 1

Theorem 3 *Assume that $G(x)$ is the vector of Genocchi polynomials; then the operational matrix of integration is defined by*

$$
\int_0^x G(s)ds \simeq PG(x). \tag{45}
$$

Proof Using integral properties of Genocchi polynomials, we have

$$
\begin{aligned}
&\int_0^x G_{2k-1}(s)ds = \frac{G_{2k}(x)-g_{2k}G_1(x)}{2k}, \\
&\int_0^x G_{2k}(s)ds = \frac{G_{2k+1}(x)}{2k+1}, \qquad k=1,2,\ldots,M,
\end{aligned}
\tag{46}
$$

where g_{2k} denotes the Genocchi number that is defined in Eq. (20). Therefore, if M is even, we get

$$
\begin{aligned}
\int_0^x G(s)ds &= \int_0^x \begin{bmatrix} G_1(s) \\ G_2(s) \\ \vdots \\ G_{M-1}(s) \\ G_M(s) \end{bmatrix} ds \\
&\simeq \begin{bmatrix} \frac{1}{2}(G_2(x)+G_1(x)) \\ \frac{1}{3}G_3(x) \\ \vdots \\ \frac{1}{M}(G_M(x)+g_M G_1(x)) \\ 0 \end{bmatrix} \\
&= PG(x),
\end{aligned}
\tag{47}
$$

where

$$
P=\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
0 & 0 & \frac{1}{3} & 0 & \cdots & 0 \\
\frac{-1}{4} & 0 & 0 & \frac{1}{4} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{-g_M}{M} & 0 & 0 & 0 & \cdots & \frac{1}{M} \\
0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}.
$$

If M is odd, we have

$$
\begin{aligned}
\int_0^x G(s)ds = \int_0^x \begin{bmatrix} G_1(s) \\ G_2(s) \\ \vdots \\ G_{M-1}(s) \\ G_M(s) \end{bmatrix} ds \simeq \begin{bmatrix} \frac{1}{2}(G_2(x)+G_1(x)) \\ \frac{1}{3}G_3(x) \\ \vdots \\ \frac{1}{M}G_M(x) \\ 0 \end{bmatrix} \\
=PG(x),
\end{aligned}
\tag{48}
$$

where

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \cdots & 0 \\ \frac{-1}{4} & 0 & 0 & \frac{1}{4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{M} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

□

Method 2

To calculate the pseudo-operational matrix of integration, we can use the transfer matrix of Genocchi polynomials to Taylor polynomials, that is

$$G(x) = D_1 T(x), \tag{49}$$

where

$$T(x) = [T_i(x)]^T, \quad T_i(x) = x^i, \qquad i = 0, 1, \ldots, M-1. \tag{50}$$

If the dimension of D_1 is even, we have

$$D_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -3 & 3 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -6 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ g_M & 0 & \frac{M! g_{M-2}}{2!(M-2)!} & 0 & \cdots & M & 0 \end{bmatrix}.$$

If the dimension of D_1 is odd, we have

$$D_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -3 & 3 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -6 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & g_M & 0 & \frac{M! g_{M-3}}{3!(M-3)!} & \cdots & M & 0 \end{bmatrix}.$$

Then

$$\int_0^x G(s)ds = \int_0^x D_1 T(s)ds = D_1 \int_0^x T(s)ds = x D_1 H_1 T(x) \\ = x D_1 H_1 D_1^{-1} G(x) = x Q_1 G(x), \tag{51}$$

where $Q_1 = D_1 H_1 D_1^{-1}$ is called the pseudo-operational matrix of integration and

$$H_1 = \left[h_{ij}^1\right], \quad h_{ij}^1 = \begin{cases} \frac{1}{i+1}, & i = j, \\ 0, & i \neq j, \end{cases} \qquad i,j = 0, 1, \ldots, M-1. \tag{52}$$

Consequently, considering operational and pseudo-operational matrices of integration, we have

$$\int_0^x G(s)ds = x Q_1 G(x) \simeq P G(x). \tag{53}$$

Remark 2 The integral pseudo-operational matrix with respect to variable t is calculated by the afore-mentioned process. We express this matrix as follows:

$$\int_0^t G(s)ds = t Q_2 G(t). \tag{54}$$

4.2 *Mixed pseudo-operational matrix of fractional integration*

The main objective of this section is to introduce the mixed fractional integral pseudo-operational matrix using Riemann–Liouville integral and Genocchi polynomials properties. First, we obtain the pseudo-operational matrix of fractional order in the following form:

$$I_x^{r_1} G(x) \simeq x^{r_1} \xi_1^{r_1} G(x). \tag{55}$$

$\xi_1^{r_1}$ is a pseudo-operational matrix of fractional integration with respect to x, which is calculated as follows:

$$\begin{aligned} I_x^{r_1} G_m(x) &= I^{r_1}\left(\sum_{k=0}^{m} \binom{m}{k} g_{m-k} x^k\right) = \sum_{k=0}^{m} \binom{m}{k} g_{m-k} I^{r_1}(x^k) \\ &= \sum_{k=0}^{m} \binom{m}{k} g_{m-k} \frac{\Gamma(k+1)}{\Gamma(k+r_1+1)} x^{k+r_1} \\ &= x^{r_1} \sum_{k=0}^{m} \binom{m}{k} g_{m-k} \frac{\Gamma(k+1)}{\Gamma(k+r_1+1)} x^k \\ &= x^{r_1} \sum_{k=0}^{m} a_{m,k}^{r_1} x^k, \end{aligned} \tag{56}$$

where

$$a_{m,k}^{r_1} = \binom{m}{k} g_{m-k} \frac{\Gamma(k+1)}{\Gamma(k+r_1+1)}. \tag{57}$$

Expanding x^k with Genocchi polynomials, we have

$$x^k \simeq \sum_{j=1}^{M} b_{k,j}^{r_1} G_j(x). \tag{58}$$

Hence, we obtain

$$\begin{aligned} I_x^{r_1} G_m(x) \simeq& x^{r_1} \sum_{k=0}^{m} a_{m,k}^{r_1} \left(\sum_{j=1}^{M} b_{k,j}^{r_1} G_j(x) \right) \\ =& x^{r_1} \sum_{j=1}^{M} \sum_{k=0}^{m} a_{m,k}^{r_1} b_{k,j}^{r_1} G_j(x) \\ =& x^{r_1} \sum_{j=1}^{M} \sum_{k=0}^{m} c_{m,k,j}^{r_1} G_j(x), \end{aligned} \tag{59}$$

where $c_{m,k,j}^{r_1} = a_{m,k}^{r_1} b_{k,j}^{r_1}$. As a result

$$I_x^{r_1} G_m(x) \simeq x^{r_1} \sum_{j=1}^{M} \xi_{m,k,j}^{r_1} G_j(x), \qquad \xi_{m,k,j}^{r_1} = \sum_{k=0}^{m} c_{m,k,j}^{r_1}, \tag{60}$$

where

$$\xi_1^{r_1} = \begin{bmatrix} \sum_{k=0}^{1} c_{1,k,1}^{r_1} & \sum_{k=0}^{1} c_{1,k,2}^{r_1} & \cdots & \sum_{k=0}^{1} c_{1,k,M}^{r_1} \\ \sum_{k=0}^{2} c_{2,k,1}^{r_1} & \sum_{k=0}^{2} c_{2,k,2}^{r_1} & \cdots & \sum_{k=0}^{2} c_{2,k,M}^{r_1} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=0}^{M} c_{M,k,1}^{r_1} & \sum_{k=0}^{M} c_{M,k,2}^{r_1} & \cdots & \sum_{k=0}^{M} c_{M,k,M}^{r_1} \end{bmatrix}.$$

Due to this process, we have

$$I_t^{r_2} G(t) \simeq t^{r_2} \xi_2^{r_2} G(t), \tag{61}$$

where

$$\xi_2^{r_2} = \begin{bmatrix} \sum_{l=0}^{1} c_{1,l,1}^{r_2} & \sum_{l=0}^{1} c_{1,l,2}^{r_2} & \cdots & \sum_{l=0}^{1} c_{1,l,N}^{r_2} \\ \sum_{l=0}^{2} c_{2,l,1}^{r_2} & \sum_{l=0}^{2} c_{2,l,2}^{r_2} & \cdots & \sum_{l=0}^{2} c_{2,l,N}^{r_2} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{l=0}^{N} c_{N,l,1}^{r_2} & \sum_{l=0}^{N} c_{N,l,2}^{r_2} & \cdots & \sum_{l=0}^{N} c_{N,l,N}^{r_2} \end{bmatrix}.$$

By means of the pseudo-operational matrix of fractional integration, which is defined in Eqs. (60) and (61), we obtain the mixed pseudo-operational matrix of fractional integration of the two-dimensional Genocchi polynomials as follows:

$$\begin{aligned} I_0^r \hat{G}_{mn}(x,t) =& I_0^{(r_1,r_2)}(G_m(x)G_n(t)) = (I_x^{r_1} G_m(x))(I_t^{r_2} G_n(t)) \\ \simeq& x^{r_1} t^{r_2} \sum_{i=1}^{M} \xi_{m,k,i}^{r_1} G_i(x) \sum_{j=1}^{N} \xi_{n,l,j}^{r_2} G_j(t) \\ =& x^{r_1} t^{r_2} \sum_{i=1}^{M} \sum_{j=1}^{N} \xi_{m,k,i}^{r_1} \xi_{n,l,j}^{r_2} G_i(x) G_j(t) \\ =& x^{r_1} t^{r_2} \sum_{i=1}^{M} \sum_{j=1}^{N} \xi_{m,k,i}^{r_1} \xi_{n,l,j}^{r_2} \hat{G}_{ij}(x,t). \end{aligned} \tag{62}$$

In other words, the matrix form of this statement is expressed as

$$I_0^r \hat{G}(x,t) \simeq x^{r_1} t^{r_2} \vartheta_{MN}^{r_1,r_2} \hat{G}(x,t), \tag{63}$$

where $\vartheta_{MN}^{r_1,r_2} = \xi_1^{r_1} \otimes \xi_2^{r_2}$.

4.3 *Pseudo-operational matrix of fractional derivative*

Here, we present the pseudo-operational matrix of the fractional derivative as follows:

$$D_t^{\gamma}(t^r G(t)) \simeq t^{r-\gamma} \Theta^{\gamma,r} G(t), \qquad 0 < \gamma(x,t) \leq 1, \quad r = 0,1,2,\ldots, \tag{64}$$

$\Theta^{\gamma,r}$ is called the pseudo-operational matrix of fractional derivative. Each component of the proposed matrix is computed as follows:

$$\begin{aligned} D_t^{\gamma}(t^r G_n(t)) =& D_t^{\gamma}\left(\sum_{k=0}^{n} \binom{n}{k} g_{n-k} t^{k+r} \right) \\ =& \sum_{k=0}^{n} \binom{n}{k} g_{n-k} D_t^{\gamma}(t^{k+r}) \\ =& \sum_{k=1}^{n} \binom{n}{k} g_{n-k} \frac{\Gamma(k+r+1)}{\Gamma(k+r-\gamma+1)} t^{k+r-\gamma} \\ =& t^{r-\gamma} \sum_{k=1}^{n} \binom{n}{k} g_{n-k} \frac{\Gamma(k+r+1)}{\Gamma(k+r-\gamma+1)} t^{k}. \end{aligned} \tag{65}$$

Approximating t^k in terms of Genocchi polynomials, we get

$$t^k \simeq \sum_{j=1}^{N} d_{k,j}^{\gamma,r} G_j(t). \tag{66}$$

Putting this formula in Eq. (65), we obtain

$$D_t^{\gamma} G_n(t) = t^{r-\gamma} \sum_{j=1}^{N} \left(\sum_{k=1}^{n} \binom{n}{k} g_{n-k} \frac{\Gamma(k+r+1)}{\Gamma(k+r-\gamma+1)} d_{k,j}^{\gamma,r} \right) G_j(t)$$
$$= t^{r-\gamma} \sum_{j=1}^{N} \sum_{k=1}^{n} \theta_{n,k,j}^{\gamma,r} G_j(t), \tag{67}$$

where

$$\theta_{n,k,j}^{\gamma,r} = \binom{n}{k} g_{n-k} \frac{\Gamma(k+r+1)}{\Gamma(k+r-\gamma+1)} d_{k,j}^{\gamma,r}. \tag{68}$$

Therefore, due to this relation, each row of the pseudo-operational matrix is obtained as

$$D_t^{\gamma} G_n(t) = t^{r-\gamma} \left[\sum_{k=1}^{n} \theta_{n,k,1}^{\gamma,r}, \sum_{k=1}^{n} \theta_{n,k,2}^{\gamma,r}, \ldots, \sum_{k=1}^{n} \theta_{n,k,N}^{\gamma,r} \right] G(t),$$
$$n = 1, 2, \ldots, N. \tag{69}$$

5. Genocchi collocation method

Consider the GF-BBMBEs given in Eq. (6) with initial condition as in Eq. (7) and boundary conditions in Eq. (8). In our construction, we assume that

$$\frac{\partial^3 u(x,t)}{\partial x^2 \partial t} \simeq G^T(x) U G(t), \tag{70}$$

where $U_{M \times N}$ is an unknown matrix:

$$U = [u_{mn}], \qquad m = 1, 2, \ldots, M, \qquad n = 1, 2, \ldots, N. \tag{71}$$

Integrating Eq. (70) with respect to t and using initial conditions, we obtain

$$\frac{\partial^2 u(x,t)}{\partial x^2} \simeq t G^T(x) U Q_2 G(t) + g_0''(x). \tag{72}$$

Integrating Eq. (72) of order 2 with respect to x, we have

$$\frac{\partial u(x,t)}{\partial x} \simeq xt G^T(x) Q_1^T U Q_2 G(t) + (g_0'(x) - g_0'(0)) + \frac{\partial u(0,t)}{\partial x}. \tag{73}$$

Thus

$$u(x,t) \simeq x^2 t G^T(x) \hat{Q}_1^T Q_1^T U Q_2 G(t) + (g_0(x) - g_0(0) - x g_0'(0))$$
$$+ x \frac{\partial u(0,t)}{\partial x} + \varphi_0(t), \tag{74}$$

where

$$\int_0^x s G(s) ds = \int_0^x s D_1 T(s) ds = D_1 \int_0^t s T(s) ds$$
$$= x^2 D_1 \hat{H}_1 T(x) = x^2 D_1 \hat{H}_1 D_1^{-1} G(x)$$
$$= x^2 \hat{Q}_1 G(x), \tag{75}$$

and [43]

$$\hat{H}_1 = \left[\hat{h}_{ij}^1 \right], \quad \hat{h}_{ij}^1 = \begin{cases} \frac{1}{i+2}, & i = j, \\ 0, & i \neq j, \end{cases} \qquad i,j = 0, 1, \ldots, M-1.$$

As seen, $\frac{\partial u(0,t)}{\partial x}$ is an unknown function in Eqs. (73) and (74). Hence, by integrating Eq. (73) with respect to x from 0 to 1, we have

$$u(1,t) - u(0,t) = t \left(\int_0^1 x G^T(x) dx \right) Q_1^T U Q_2 G(t)$$
$$+ (g_0(1) - g_0(0) - g_0'(0)) + \frac{\partial u(0,t)}{\partial x}, \tag{76}$$

where

$$\int_0^1 x G^T(x) dx = \int_0^1 x D_1 T^T(x) dx = S^T D_1^T, \tag{77}$$

and [43]

$$S = [s_{i1}], \quad s_{i1} = \frac{1}{i+2}, \qquad i = 0, 1, \ldots, M-1. \tag{78}$$

Then, we obtain

$$\frac{\partial u(0,t)}{\partial x} = \varphi_1(t) - \varphi_0(t) - t S^T D_1^T Q_1^T U Q_2 G(t)$$
$$+ (g_0(1) - g_0(0) - g_0'(0)). \tag{79}$$

Substituting Eq. (79) in Eqs. (73) and (74), the approximate of $u(x,\ t)$ and $\frac{\partial u(x,t)}{\partial x}$ is obtained using Genocchi polynomials. Finally, using Eqs. (74) and (79), we have

$$\begin{aligned} D_t^{\gamma}u(x,t) \simeq & x^2 t^{1-\gamma} G^T(x)\hat{Q}_1^T Q_1^T U Q_2 \Theta^{\gamma,1} G(t) \\ & + x\big(D_t^{\gamma}\varphi_1(t) - D_t^{\gamma}\varphi_0(t) \\ & - t^{1-\gamma} S^T D_1^T Q_1^T U Q_2 \Theta^{\gamma,1} G(t)\big)\big) \\ & + D_t^{\gamma}\varphi_0(t). \end{aligned} \tag{80}$$

Also, using the mixed pseudo-operational matrix of fractional integration in section 4.2, we obtain

$$\begin{aligned} D_0^r u(x,t) &= I_0^{(2-r_1,1-r_2)}\left(\frac{\partial^3 u(x,t)}{\partial x^2 \partial t}\right) \\ &= I_0^{(2-r_1,1-r_2)}\big(G^T(x) U G(t)\big) \\ &= \big(I_x^{2-r_1} G^T(x)\big) U \big(I_t^{1-r_2} G(t)\big) \\ &\simeq x^{2-r_1} t^{1-r_2} G^T(x) (\xi_1^{2-r_1})^T U \xi_2^{1-r_2} G(t). \end{aligned} \tag{81}$$

Substituting this outcome in Eq. (6) and collocating this equation in nodal points of Newton–Cotes [43], we get a system of algebraic equations. Solving the proposed system and using Newton's iterative method, the unknown matrix U is determined. Finally, substituting the obtained matrix in Eq. (74), we get the approximate solution.

6. Error estimation

In this section, we present convergence analysis of the approximate solution to the exact solution.

6.1 *Convergence and error estimation*

Theorem 4 *Suppose that u is a smooth function on $[0,1]\times[0,1]$, so that*

$$u(x,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\hat{G}_{mn}(x,t). \tag{82}$$

If

$$u_{MN}(x,t) = \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}\hat{G}_{mn}(x,t) \tag{83}$$

is the 2D-Genocchi expansion of the exact solution $u(x,t)$, then we have the following estimation:

$$\|u(x,t) - u_{MN}(x,t)\|_2 \le \left(\sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty} \frac{\theta_{mn}\delta_k^m \delta_j^n}{16(m!)^2(n!)^2}\right)^{\frac{1}{2}}, \tag{84}$$

where

$$\begin{aligned} \theta_{m,n} = & \Big|\frac{\partial^{m+n-2}u(1,1)}{\partial x^{m-1}\partial t^{n-1}}\Big|^2 + \Big|\frac{\partial^{m+n-2}u(1,0)}{\partial x^{m-1}\partial t^{n-1}}\Big|^2 \\ & + \Big|\frac{\partial^{m+n-2}u(0,0)}{\partial x^{m-1}\partial t^{n-1}}\Big|^2 + \Big|\frac{\partial^{m+n-2}u(0,1)}{\partial x^{m-1}\partial t^{n-1}}\Big|^2, \end{aligned} \tag{85}$$

and

$$\delta_k^m = \sum_{k=1}^{m}\binom{m}{k}^2 \frac{g_{m-k}^2}{2k+1}, \quad \delta_j^n = \sum_{j=1}^{n}\binom{n}{j}^2 \frac{g_{n-j}^2}{2j+1}. \tag{86}$$

Proof According to the assumptions, we have

$$\begin{aligned} \|u(x,t) - u_{MN}(x,t)\|_2^2 = & \int_0^1\int_0^1 |u(x,t) \\ & - u_{MN}(x,t)|^2 dxdt \\ = & \int_0^1\int_0^1 \Big|\sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty} a_{mn}\hat{G}_{mn}(x,t)\Big|^2 dxdt \\ = & \int_0^1\int_0^1 \Big|\sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty} a_{mn}G_m(x)G_n(t)\Big|^2 dxdt \\ \le & \int_0^1\int_0^1 \sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty} |a_{mn}G_m(x)G_n(t)|^2 dxdt \\ = & \sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty} |a_{mn}|^2\left(\int_0^1 |G_m(x)|^2dx\right)\left(\int_0^1 |G_n(t)|^2dt\right), \end{aligned} \tag{87}$$

given that [33]

$$\begin{aligned} \int_0^1 |G_m(x)|^2 dx \le \sum_{k=1}^{m}\binom{m}{k}^2 \frac{g_{m-k}^2}{2k+1}, \\ \int_0^1 |G_n(t)|^2 dt \le \sum_{j=1}^{n}\binom{n}{j}^2 \frac{g_{n-j}^2}{2j+1}. \end{aligned} \tag{88}$$

Due to the assumptions and Eqs. (87) and (88), we obtain

$$\begin{aligned} \|u(x,t) - u_{MN}(x,t)\|_2^2 \le & \sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty} |a_{mn}|^2 \\ & \left(\sum_{k=1}^{m}\binom{m}{k}^2 \frac{g_{m-k}^2}{2k+1}\right) \\ & \times\left(\sum_{j=1}^{n}\binom{n}{j}^2 \frac{g_{n-j}^2}{2j+1}\right) \\ = & \sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty} |a_{mn}|^2 \delta_k^m \delta_j^n. \end{aligned} \tag{89}$$

Using Eq. (30), we have

$$\begin{aligned}
&\|u(x,t)-u_{MN}(x,t)\|_2^2\\
&\leq \sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty}\frac{\delta_k^m\delta_j^n}{16(m!)^2(n!)^2}\Big|\frac{\partial^{m+n-2}u(1,1)}{\partial x^{m-1}\partial t^{n-1}}\\
&\quad+\frac{\partial^{m+n-2}u(1,0)}{\partial x^{m-1}\partial t^{n-1}}+\frac{\partial^{m+n-2}u(0,0)}{\partial x^{m-1}\partial t^{n-1}}+\frac{\partial^{m+n-2}u(0,1)}{\partial x^{m-1}\partial t^{n-1}}\Big|^2\\
&\leq \sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty}\frac{\delta_k^m\delta_j^n}{16(m!)^2(n!)^2}\left(\Big|\frac{\partial^{m+n-2}u(1,1)}{\partial x^{m-1}\partial t^{n-1}}\Big|^2\right.\\
&\quad+\Big|\frac{\partial^{m+n-2}u(1,0)}{\partial x^{m-1}\partial t^{n-1}}\Big|^2+\Big|\frac{\partial^{m+n-2}u(0,0)}{\partial x^{m-1}\partial t^{n-1}}\Big|^2\\
&\quad\left.+\Big|\frac{\partial^{m+n-2}u(0,1)}{\partial x^{m-1}\partial t^{n-1}}\Big|^2\right).
\end{aligned} \tag{90}$$

Simplifying this expression, we have

$$\|u(x,t)-u_{MN}(x,t)\|_2^2\leq \sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty}\frac{\theta_{mn}\delta_k^m\delta_j^n}{16(m!)^2(n!)^2}. \tag{91}$$

This completes the proof. □

It is seen from Theorem 4 that when we increase the terms of 2D-Genocchi functions M and N, the approximate solution is convergent to the exact solution:

$$\|u(x,t)-u_{MN}(x,t)\|\rightarrow 0,\quad M,N\rightarrow\infty. \tag{92}$$

Theorem 5 *Assume that the hypotheses in Theorem 4 hold. If $\bar{u}_{MN}(x,t)$, is an approximate solution of Eq.* (1) *obtained by the method in section 5 as*

$$\bar{u}_{MN}(x,t)=\sum_{m=1}^{M}\sum_{n=1}^{N}\bar{a}_{mn}\hat{G}_{mn}(x,t), \tag{93}$$

then

$$\begin{aligned}
\|u(x,t)-\bar{u}_{MN}(x,t)\|_2 &\leq \left(\sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty}\frac{\theta_{mn}\delta_k^m\delta_j^n}{16(m!)^2(n!)^2}\right)^{\frac{1}{2}}\\
&+\left(\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{1}{4m!n!}(e_{mn}^{11}+e_{mn}^{10}+e_{mn}^{00}+e_{mn}^{01})\right)^{\frac{1}{2}}\\
&\times\left(\sum_{m=1}^{M}\delta_k^m\right)^{\frac{1}{2}}\left(\sum_{n=1}^{N}\delta_j^n\right)^{\frac{1}{2}},
\end{aligned} \tag{94}$$

where

$$e_{mn}^{ab}=\Big|\frac{\partial^{m+n-2}u(a,b)}{\partial x^{m-1}\partial t^{n-1}}-\frac{\partial^{m+n-2}\bar{u}(a,b)}{\partial x^{m-1}\partial t^{n-1}}\Big|^2,\qquad a,b=0,1. \tag{95}$$

Proof This can be written as

$$\begin{aligned}
\|u(x,t)-\bar{u}_{MN}(x,t)\|_2 \leq \|u(x,t)-u_{MN}(x,t)\|_2\\
+\|u_{MN}(x,t)-\bar{u}_{MN}(x,t)\|_2,
\end{aligned} \tag{96}$$

which is an upper bound for the first part of this inequality obtained in Theorem 6.1. For the second part, we have

$$\begin{aligned}
&\|u_{MN}(x,t)-\bar{u}_{MN}(x,t)\|_2\\
&=\left(\int_0^1\int_0^1\Big|\sum_{m=1}^{M}\sum_{n=1}^{N}(a_{mn}-\bar{a}_{mn})\hat{G}_{mn}(x,t)\Big|^2dxdt\right)^{\frac{1}{2}}\\
&\leq\left(\int_0^1\int_0^1\left[\sum_{m=1}^{M}\sum_{n=1}^{N}|a_{mn}-\bar{a}_{mn}|^2\right]\left[\sum_{m=1}^{M}\sum_{n=1}^{N}|\hat{G}_{mn}(x,t)|^2\right]dxdt\right)^{\frac{1}{2}}\\
&\leq\left(\int_0^1\int_0^1\left[\sum_{m=1}^{M}\sum_{n=1}^{N}|a_{mn}-\bar{a}_{mn}|^2\right]\left[\sum_{m=1}^{M}\sum_{n=1}^{N}|G_m(x)|^2|G_n(t)|^2\right]dxdt\right)^{\frac{1}{2}}\\
&\leq\left(\sum_{m=1}^{M}\sum_{n=1}^{N}|a_{mn}-\bar{a}_{mn}|^2\right)^{\frac{1}{2}}\left(\sum_{m=1}^{M}\int_0^1|G_m(x)|^2dx\right)^{\frac{1}{2}}\\
&\quad\times\left(\sum_{n=1}^{N}\int_0^1|G_n(t)|^2dt\right)^{\frac{1}{2}}.
\end{aligned} \tag{97}$$

According to the previous assumptions

$$\begin{aligned}
&\|u_{MN}(x,t)-\bar{u}_{MN}(x,t)\|_2\leq\\
&\left(\sum_{m=1}^{M}\sum_{n=1}^{N}|a_{mn}-\bar{a}_{mn}|^2\right)^{\frac{1}{2}}\left(\sum_{m=1}^{M}\delta_k^m\right)^{\frac{1}{2}}\left(\sum_{n=1}^{N}\delta_j^n\right)^{\frac{1}{2}}.
\end{aligned} \tag{98}$$

On the other hand

$$
\begin{aligned}
\sum_{m=1}^{M}\sum_{n=1}^{N}|a_{mn}-\bar{a}_{mn}|^2 &\le \sum_{m=1}^{M}\sum_{n=1}^{N}\frac{1}{4m!n!}\\
&\times\Bigg(\Big|\frac{\partial^{m+n-2}u(1,1)}{\partial x^{m-1}\partial t^{n-1}}-\frac{\partial^{m+n-2}\bar{u}(1,1)}{\partial x^{m-1}\partial t^{n-1}}\Big|^2\\
&+\Big|\frac{\partial^{m+n-2}u(1,0)}{\partial x^{m-1}\partial t^{n-1}}-\frac{\partial^{m+n-2}\bar{u}(1,0)}{\partial x^{m-1}\partial t^{n-1}}\Big|^2\\
&+\Big|\frac{\partial^{m+n-2}u(0,0)}{\partial x^{m-1}\partial t^{n-1}}-\frac{\partial^{m+n-2}\bar{u}(0,0)}{\partial x^{m-1}\partial t^{n-1}}\Big|^2\\
&+\Big|\frac{\partial^{m+n-2}u(0,1)}{\partial x^{m-1}\partial t^{n-1}}-\frac{\partial^{m+n-2}\bar{u}(0,1)}{\partial x^{m-1}\partial t^{n-1}}\Big|^2\Bigg),\\
&=\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{1}{4m!n!}(e_{mn}^{11}+e_{mn}^{10}+e_{mn}^{00}+e_{mn}^{01}).
\end{aligned}
\tag{99}
$$

Hence, we have

$$
\begin{aligned}
&\|u_{MN}(x,t)-\bar{u}_{MN}(x,t)\|_2\\
&\le\left(\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{1}{4m!n!}(e_{mn}^{11}+e_{mn}^{10}+e_{mn}^{00}+e_{mn}^{01})\right)^{\frac{1}{2}}\\
&\times\left(\sum_{m=1}^{M}\delta_k^m\right)^{\frac{1}{2}}\left(\sum_{n=1}^{N}\delta_j^n\right)^{\frac{1}{2}}.
\end{aligned}
\tag{100}
$$

Finally, substituting the results of Theorem 4 and Eq. (100) in Eq. (96), the proof is completed. □

6.2 *Upper bound of error in the mixed fractional integral*

In the following theorem, we obtain the upper bound of error for left-sided mixed Riemann–Liouville integral of the approximate solution.

Theorem 6 *Assume that $u_{MN}(x,t)$ is an approximation of $u(x,t)$. Then, we have upper bound of error for the left-sided mixed Riemann–Liouville integral of $u_{MN}(x,t)$ as*

$$
\begin{aligned}
&\|I_\theta^r u(x,t)-I_\theta^r u_{M,N}(x,t)\|_2\\
&\le\frac{1}{\Gamma(r_1)\Gamma(r_2)}\left(\sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty}\frac{\theta_{mn}\delta_k^m\delta_j^n}{16(m!)^2(n!)^2}\right)^{\frac{1}{2}}.
\end{aligned}
\tag{101}
$$

Proof In order to obtain the upper bound of error for left-sided mixed Riemann–Liouville integral, we apply Theorem 4 and Definition 3 as

$$
\begin{aligned}
&\|I_\theta^r u(x,t)-I_\theta^r u_{M,N}(x,t)\|_2\\
&=\Big\|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^t(x-\xi)^{r_1-1}(t-\eta)^{r_2-1}\\
&\quad\times[u(\xi,\eta)-u_{M,N}(\xi,\eta)]d\eta d\xi\Big\|_2\\
&\le\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^t\|u(\xi,\eta)-u_{M,N}(\xi,\eta)\|_2 d\eta d\xi\\
&\le\frac{1}{\Gamma(r_1)\Gamma(r_2)}\|u(\xi,\eta)-u_{M,N}(\xi,\eta)\|_2\int_0^x\int_0^t d\eta d\xi\\
&=\frac{xt}{\Gamma(r_1)\Gamma(r_2)}\|u(\xi,\eta)-u_{M,N}(\xi,\eta)\|_2\\
&\le\frac{1}{\Gamma(r_1)\Gamma(r_2)}\|u(\xi,\eta)-u_{M,N}(\xi,\eta)\|_2\\
&\le\frac{1}{\Gamma(r_1)\Gamma(r_2)}\left(\sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty}\frac{\theta_{mn}\delta_k^m\delta_j^n}{16(m!)^2(n!)^2}\right)^{\frac{1}{2}}.
\end{aligned}
\tag{102}
$$

Therefore, the proof of this theorem is completed. □

7. Illustrative examples

In this section, we will employ our method to obtain approximate solutions of some examples. The computations associated with the examples were performed using MATLAB. In some examples, we express the following error norms:

$$
\begin{aligned}
l_2&=\sqrt{\sum_{i=1}^{MN}|u(x_i,1)-u_{MN}(x_i,1)|^2},\\
l_\infty&=\max_{1\le i\le MN}|u(x_i,1)-u_{MN}(x_i,1)|.
\end{aligned}
\tag{103}
$$

Example 1 We consider the linear GF-BBMBE in the following form:

$$
\begin{aligned}
&D_t^\gamma u(x,t)-D_\theta^r u(x,t)-\frac{\partial^2 u(x,t)}{\partial x^2}=g(x,t),\qquad 0<\gamma\le 1,\\
&1<r_1\le 2,\qquad 0<r_2\le 1,
\end{aligned}
\tag{104}
$$

with initial condition $u(x,0)=0,\ x\in R$, and boundary conditions

$$
u(0,t)=0,\quad u(1,t)=t^3,\qquad 0\le t\le 1,
\tag{105}
$$

with

$$
\begin{aligned}
g(x,t)&=\frac{\Gamma(4)}{\Gamma(4-\gamma)}x^2t^{3-\gamma}-2t^3\\
&-\frac{\Gamma(3)\Gamma(4)}{\Gamma(3-r_1)\Gamma(4-r_2)}x^{2-r_1}t^{3-r_2}.
\end{aligned}
\tag{106}
$$

The exact solution of this example is $u(x,t)=x^2t^3$. Considering the proposed method for $M=2,N=2$ with $\gamma=1,r_1=2,r_2=1$, we obtain the following system:

$$\begin{cases} 0.79320987u_{12} - 1.23611111u_{11} + 0.5256558u_{13} + 0.8518518518u_{21} - 0.547325102u_{22} \\ -0.36201131u_{23} + 0.4826388u_{31} - 0.3088991u_{32} - 0.205520190u_{33} + 0.1875 = 0, \\ 0.953125u_{31} - 1.291666666u_{11} + 0.888888u_{21} - 0.65625u_{23} + 0.5057870370u_{31}\ 0.372829861u_{33} + 0.1967592592 = 0, \\ 0.5256558u_{13} - 0.7932098u_{12} - 1.2361111u_{11} + 0.851851851851u_{21} + 0.547325102u_{22} \\ -0.362011316u_{23} + 0.48263888u_{31} + 0.3088991769u_{32} - 0.20552019u_{33} + 0.1875 = 0, \\ 1.015432098u_{12} - 1.56944444u_{11} + 0.66454475308u_{13} + 0.25u_{21} - 0.1666666u_{22} \\ -0.1041666u_{23} + 1.05208333u_{31} - 0.678240740u_{32} - 0.4463252314u_{33} + 1.85416666 = 0, \\ 1.203125u_{13} - 1.625u_{11} + 0.25u_{21} - 0.1875u_{23} + 1.09375u_{31} - 0.80859u_{33} + 1.9375 = 0, \\ 0.664544753u_{13} - 1.015432098u_{12} - 1.569444444u_{11} + 0.25u_{21} + 0.1666666u_{22} \\ -0.10416666u_{23} + 1.0520833u_{31} + 0.67824074u_{32} - 0.4463252u_{33} + 1.8541666 = 0, \\ 1.237654320u_{12} - 1.9027777u_{11} + 0.803433641u_{13} - 0.5740740u_{21} + 0.362139917u_{22} \\ +0.24627057u_{23} + 0.90856481u_{31} - 0.59284979u_{32} - 0.382989326u_{33} + 5.613425925 = 0, \\ 1.453125u_{13} - 1.9583333u_{11} - 0.611111111u_{21} + 0.447916666u_{23} \\ +0.93171296u_{31} - 0.6922743055u_{33} + 5.84490740 = 0, \\ 0.803433641u_{13} - 1.2376543u_{12} - 1.9027777u_{11} - 0.57407407u_{21} - 0.3621399176u_{22} \\ +0.246270u_{23} + 0.908564814u_{31} + 0.592849u_{32} - 0.38298932u_{33} + 5.6134259 = 0. \end{cases}$$

Applying Newton's iterative method, each element of the unknown coefficient matrix is obtained:

$$\begin{aligned} &u_{11} = 3.000000000000001, u_{12} = -9.474964 \times 10^{-16}, \\ &u_{13} = 2.2123522 \times 10^{-15}, u_{21} = 2.999999999999999, \\ &u_{22} = 1.2908828 \times 10^{-16}, u_{23} = -2.4112797 \times 10^{-15}, \\ &u_{31} = 2.0000000000000013, u_{32} = -1.8968498 \times 10^{-15}, \\ &u_{33} = 3.9029096 \times 10^{-15}. \end{aligned}$$

As a result, the approximate solution is given as follows:

$$\begin{aligned} u(x,t) =& t^3x^2 - 1.48718154 \times 10^{-15}x^2t^2 \\ &+ 9.7647719 \times 10^{-16}xt \\ &+ 5.3829236 \times 10^{-16}tx^2 - 1.5706838 \\ &\times 10^{-15}xt^2 \\ &+ 6.5958576 \times 10^{-15}xt^3 - 2.0664110 \\ &\times 10^{-15}t^2x^4 \\ &+ 9.7572740 \times 10^{-15}t^3x^4 - 2.67067754 \\ &\times 10^{-15}tx^3 \\ &+ 1.15590798 \times 10^{-15}tx^4 + 5.1242764 \\ &\times 10^{-15}t^2x^3 \\ &- 2.58373810 \times 10^{-15}x^3t^3. \end{aligned} \tag{107}$$

Also, the numerical results of this example are displayed in table 1 and figure 1. Table 1 shows the absolute errors for various choices of r_1, r_2 with $M = 2, N = 3$ using the present method. As we can see, with different choices of r_1, r_2, we achieve good approximate solutions. Also, the absolute error and approximate solution for $M = 2, N = 3$ on interval $[-10, 10] \times [0, 1]$ are plotted in figure 1(left) and 1(right), respectively.

Example 2 We consider the nonlinear GF-BBMBE in the following form:

$$\frac{\partial u(x,t)}{\partial t} + D_0^r u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial x} u(x,t) = g(x,t), \quad 1 < r_1 \leq 2, \quad 0 < r_2 \leq 1, \tag{108}$$

initial condition $u(x,0) = 0$, $x \in R$, boundary conditions

$$u(0,t) = 0, \quad u(1,t) = \sin(t), \qquad 0 \leq t \leq 1,$$

with $g(x,t) = (x^2+2)\cos(t) - 2\sin(t) + 2x^3\sin^2(t)$. The exact solution of this example when $r_1 = 2$ and $r_2 = 1$ is $u(x,t) = x^2\sin(t)$. The numerical results of this example are displayed in table 2 and figures 2 and 3. Table 2 shows the absolute error for various values of N with $r_1 = 2, r_2 = 1, M = 2$ on interval [0, 1] and the convergence rate of present method. As seen, on increasing N, the approximate solution obtained by the present method converges to the exact solution. Figure 2 shows graphs of approximate solution for various choices of r with $N = 5, M = 2$ and

Table 1. Absolute error for various choices of r_1, r_2 with $M = 2, N = 3$ and $\gamma = 0.8$ of Example 1.

(x_i, t_i)	$r_1 = 2, r_2 = 1$	$r_1 = 1.8, r_2 = 0.75$	$r_1 = 1.5, r_2 = 0.5$
(0, 0)	0	0	0
(0.1, 0.1)	3.15×10^{-18}	1.02×10^{-18}	1.16×10^{-18}
(0.2, 0.2)	7.03×10^{-18}	1.88×10^{-18}	3.35×10^{-18}
(0.3, 0.3)	7.62×10^{-18}	1.54×10^{-18}	4.51×10^{-18}
(0.4, 0.4)	4.75×10^{-18}	5.99×10^{-19}	3.35×10^{-18}
(0.5, 0.5)	0	0	0
(0.6, 0.6)	4.65×10^{-18}	2.66×10^{-19}	4.02×10^{-18}
(0.7, 0.7)	7.71×10^{-18}	1.14×10^{-18}	6.42×10^{-18}
(0.8, 0.8)	8.38×10^{-18}	4.38×10^{-18}	5.36×10^{-18}
(0.9, 0.9)	6.16×10^{-17}	1.30×10^{-18}	1.40×10^{-18}
(1, 1)	1.83×10^{-40}	0	0
CPU	4.51×10^{-2}	8.54×10^{-2}	4.55×10^{-2}

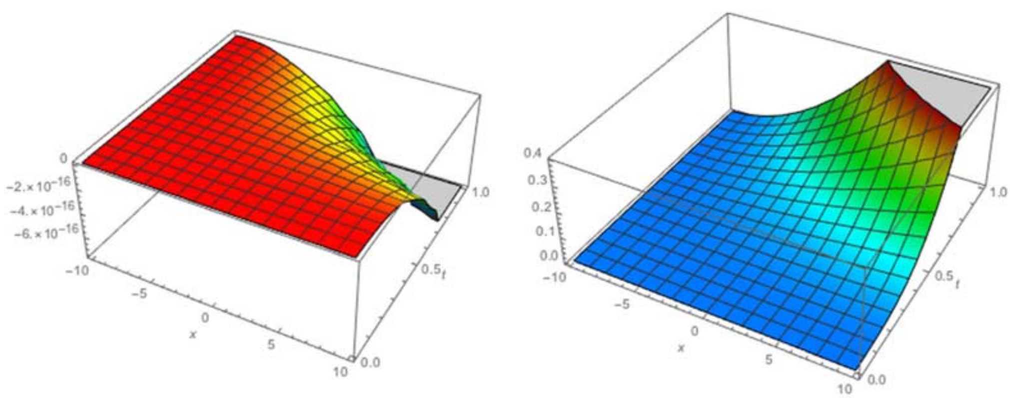

Figure 1. Absolute error (left) and approximate solution (right) for $r_1 = 1.75,\ r_2 = 0.75$, on the interval $[-10, 10] \times [0, 1]$ with $M = 2, N = 3$ and $\gamma = 0.9$ of Example 1.

$t = 1$. Also, the absolute error and approximate solution for $M = N = 3$ on interval $[-10, 10] \times [0, 1]$ are plotted in figure 3(left) and 3(right), respectively. From table 2 and figures 2 and 3, it is clear that we achieved a good approximation of the exact solution.

Table 2. Absolute error for different values of N with $r_1 = 2, r_2 = 1$ on interval $[0, 1] \times [0, 1]$ of Example 2.

(x_i, t_i)	$M = 2, N = 3$	$M = 2, N = 5$	$M = 2, N = 6$
(0, 0)	0	0	0
(0.1, 0.1)	1.98×10^{-5}	9.12×10^{-8}	9.79×10^{-9}
(0.2, 0.2)	3.63×10^{-5}	9.22×10^{-8}	8.72×10^{-9}
(0.3, 0.3)	2.79×10^{-5}	6.02×10^{-8}	7.47×10^{-9}
(0.4, 0.4)	8.08×10^{-6}	7.43×10^{-8}	6.27×10^{-9}
(0.5, 0.5)	1.85×10^{-6}	7.76×10^{-8}	1.97×10^{-10}
(0.6, 0.6)	8.59×10^{-6}	2.34×10^{-8}	7.35×10^{-9}
(0.7, 0.7)	2.96×10^{-5}	2.57×10^{-8}	1.05×10^{-8}
(0.8, 0.8)	3.68×10^{-5}	4.29×10^{-9}	1.42×10^{-8}
(0.9, 0.9)	1.44×10^{-5}	2.77×10^{-8}	1.50×10^{-8}
(1, 1)	2.46×10^{-40}	1.06×10^{-40}	1.06×10^{-40}
CPU	4.56×10^{-2}	5.33×10^{-2}	5.66×10^{-2}

Example 3 We consider the nonlinear GF-BBMBE in the following form:

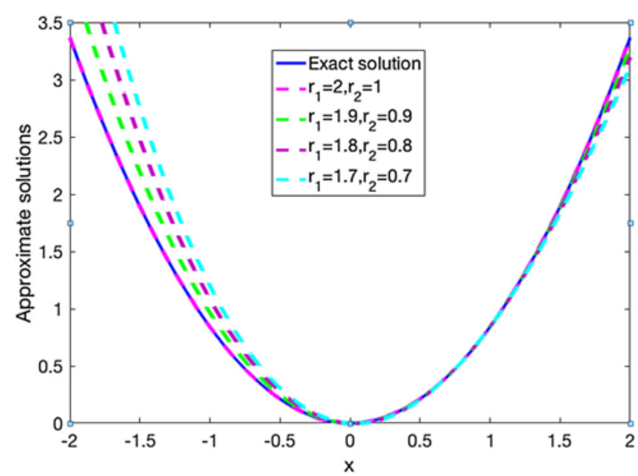

Figure 2. Approximate solutions for various values of r with $M = 2, N = 5$, on the interval $x \in [-2, 2]$ and $t = 1$ of Example 2.

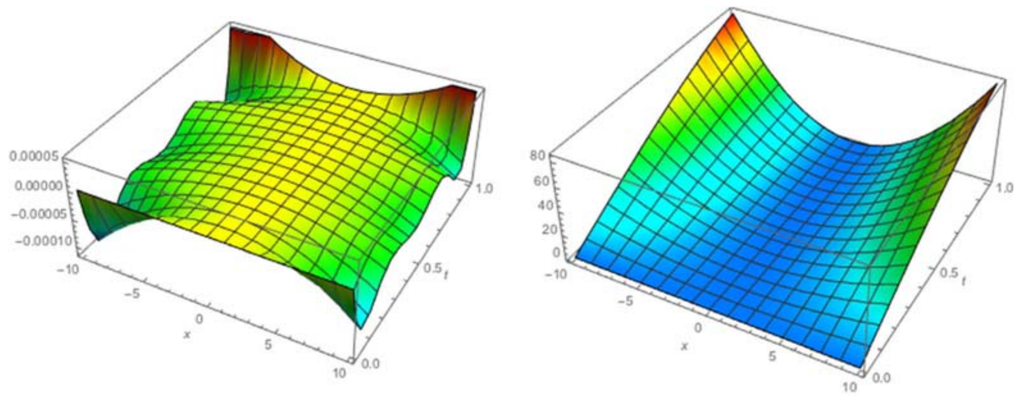

Figure 3. Absolute error (left) and approximate solution (right) for $r_1 = 2, r_2 = 1$, with $M = 2, N = 5$ on the interval $(x, t) \in [-10, 10] \times [0, 1]$ of Example 2.

Table 3. Error for different values of γ, r_1, r_2, M, N with $v = 0.5$ on interval $[0, 1] \times [0, 1]$ for Example 3.

M	N	γ	r_1	r_2	L_2	L_∞
3	3	1	2	1	2.6443×10^{-4}	1.7855×10^{-4}
		0.5	2	0.5	2.4052×10^{-4}	1.4480×10^{-4}
		0.75	2	0.5	2.4221×10^{-4}	1.4662×10^{-4}
6	3	1	2	1	5.6059×10^{-7}	3.0708×10^{-7}
		0.5	2	0.5	1.2003×10^{-7}	5.9843×10^{-8}
		0.75	2	0.5	1.2853×10^{-7}	6.3887×10^{-8}

$$D_t^\gamma u(x,t) + D_\theta^r u(x,t) - v\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial x} u(x,t) = g(x,t),$$
$$0 < \gamma \leq 1, \quad 1 < r_1 \leq 2, \quad 0 < r_2 \leq 1, \tag{109}$$

with initial condition $u(x, 0) = \exp(x)$, $x \in R$, and boundary conditions

$$u(0,t) = (t+1)^3, \quad u(1,t) = (t+1)^3 \exp(1),$$
$$0 \leq t \leq 1,$$

and

$$g(x,t) = \left(\frac{6}{\Gamma(4-\gamma)} t^{3-\gamma} + \frac{6}{\Gamma(3-\gamma)} t^{2-\gamma} + \frac{3}{\Gamma(2-\gamma)} t^{1-\gamma}\right) \times \exp(x) + \left(\frac{6}{\Gamma(4-r_2)} t^{3-r_2} + \frac{6}{\Gamma(3-r_2)} t^{2-r_2} + \frac{3}{\Gamma(2-r_2)} t^{1-r_2}\right) \times \exp(x) - v(t+1)^3 \exp(x) + (t+1)^6 \exp(2x).$$

The exact solution of this example when $r_1 = 2$ is $u(x,t) = (t+1)^3 \exp(x)$. The numerical results of this example are displayed in table 3 and figures 4–6. Table 3 shows error for different values of γ, r_1, r_2, M, N with $v = 0.5$ on using the present method. Figure 4 shows graphs of absolute error for various values of r_1 with $r_2 = 1$, $\gamma = 0.8$, $v = 0.5$, $t = 1$, $M = 5, N = 2$ on the interval $x \in [0, 1]$.

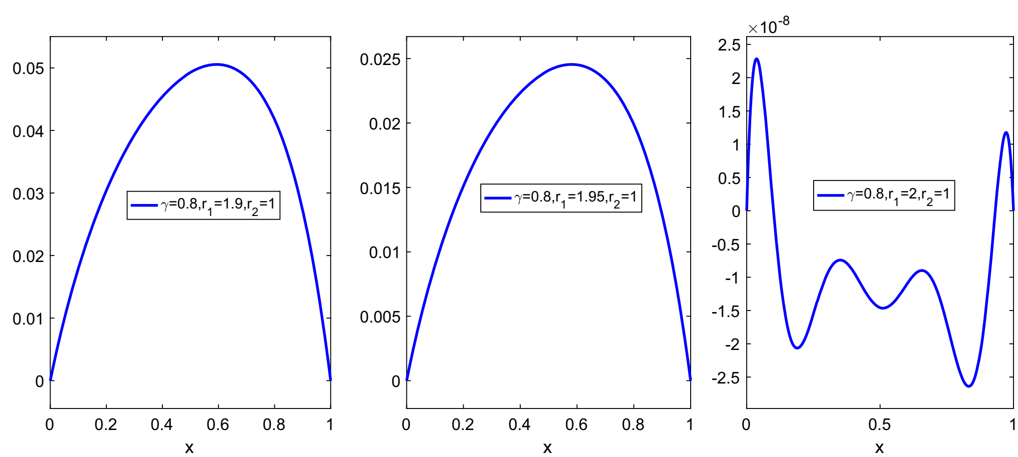

Figure 4. Absolute error between the approximate solution and the exact solution for various values of $r_1 = 1.9, 1.95, 2$ with $r_2 = 1$, $\gamma = 0.8$, $v = 0.5$, $t = 1$ and $M = 5, N = 2$ of Example 3.

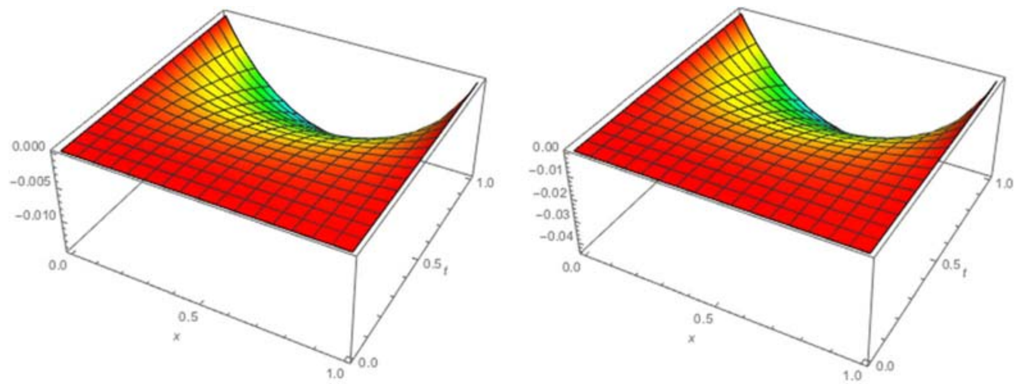

Figure 5. Absolute errors between the approximate solution and the exact solution for $r_1 = 1.95$ (left) and $r_1 = 1.85$ (right) with $r_2 = 1$, $\gamma = 0.7$ and $v = 1$ on the interval $(x, t) \in [0, 1] \times [0, 1]$ and $M = 6, N = 3$ of Example 3.

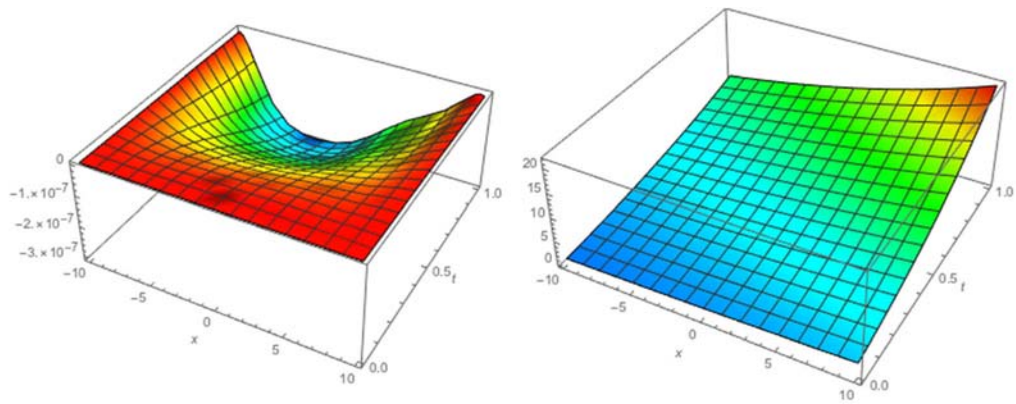

Figure 6. Absolute error (left) and approximate solution (right) for $r_1 = 2, r_2 = 0.5,$ and $\gamma = 0.5$ with $M = 6, N = 3$ on interval $(x, t) \in [-10, 10] \times [0, 1]$ of Example 3.

It is seen in figure 4 that as r_1 approaches 2, absolute error tends to zero. Figures 5(left) and 5(right) illustrate the behaviour of the absolute error for various choices of r_1 and r_2 with $M = 6, N = 3$. Also, the absolute error and approximate solution for $M = 6, N = 3$ on interval $[-10, 10] \times [0, 1]$ are plotted in figure 6(left) and 6(right), respectively.

Example 4 We consider the nonlinear GF-BBMBE in the following form:

$$\frac{\partial u(x,t)}{\partial t} - D_\theta^r u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial x} = \frac{\partial u(x,t)}{\partial x} u(x,t) + g(x,t), \quad 1 < r_1 \leq 2, \quad 0 < r_2 \leq 1, \tag{110}$$

with initial condition $u(x, 0) = \sec(hx)$, $0 \leq x \leq 1$, and boundary condition

$$u(0,t) = \sec(-ht), \quad u(1,t) = \sec(h(1-t)), \qquad 0 \leq t \leq 1,$$

Table 4. Absolute errors for different values of h with $N = M = 3$ on interval $[0, 1] \times [0, 1]$ for Example 4.

	Present method	
h	L_2	L_∞
$\frac{1}{5}$	7.2333×10^{-4}	3.8117×10^{-4}
$\frac{1}{10}$	6.3973×10^{-4}	3.2713×10^{-4}
$\frac{1}{20}$	6.3527×10^{-4}	3.2383×10^{-4}
$\frac{1}{30}$	2.0421×10^{-5}	1.0737×10^{-5}
$\frac{1}{40}$	1.1538×10^{-5}	6.0698×10^{-6}

with

$$g(x,t) = \frac{h\sin(h(x-t))}{\cos(h(x-t))^3} - \frac{5h^3\sin(h(t-x))}{\cos(h(t-x))^2} - \frac{2h^2\sin(h(t-x))^2}{\cos(h(t-x))^3} - \frac{6h^3\sin(h(t-x))^3}{\cos(h(t-x))^4} - \frac{h^3}{\cos(h(t-x))}. \tag{111}$$

Table 5. Absolute errors for different choices of r_1, r_2 with $h = \frac{1}{30}$, $M = N = 4$ and $t = 1$ of Example 4.

x	$r_1 = 1.9, r_2 = 0.9$	$r_1 = 1.7, r_2 = 0.8$	$r_1 = 1.5, r_2 = 0.6$
0	1.21×10^{-40}	1.21×10^{-40}	1.21×10^{-40}
0.1	1.67×10^{-5}	9.80×10^{-6}	1.26×10^{-5}
0.2	2.89×10^{-5}	1.51×10^{-5}	1.99×10^{-5}
0.3	3.70×10^{-5}	1.70×10^{-5}	2.22×10^{-5}
0.4	4.14×10^{-6}	1.65×10^{-5}	2.36×10^{-5}
0.5	4.20×10^{-6}	1.42×10^{-5}	2.18×10^{-5}
0.6	3.91×10^{-6}	1.09×10^{-5}	1.85×10^{-5}
0.7	3.28×10^{-5}	7.28×10^{-6}	1.43×10^{-5}
0.8	2.34×10^{-5}	3.87×10^{-6}	9.61×10^{-6}
0.9	1.18×10^{-5}	1.27×10^{-6}	4.76×10^{-6}
1	0	0	0

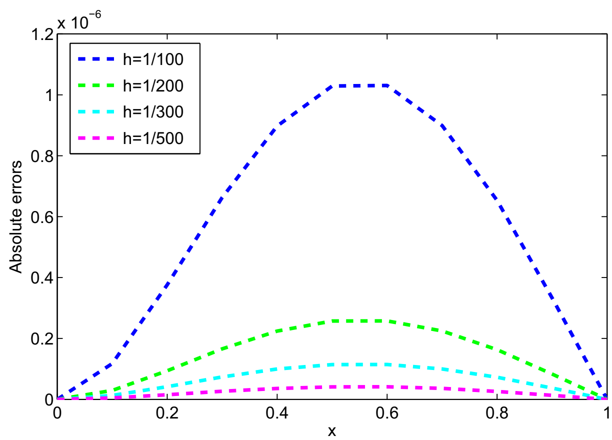

Figure 7. Absolute error for various choices of h with $N = M = 4$, $r_1 = 2, r_2 = 1$ and $t = 1$ of Example 4.

The exact solution when $r_1 = 2, r_2 = 1$ is $u(x,t) = \sec(h(x-t))$. The numerical results of this example are displayed in tables 4 and 5 and figure 7. In table 4, errors l_2 and l_∞ of the proposed method with $N = M = 3$, $r_1 = 2, r_2 = 1$ and different values of h are shown. Figure 7 shows graphs of absolute error for various values h with $N = M = 4$ and $r_1 = 2, r_2 = 1$. Table 5 shows the absolute error for various choices of r with $M = N = 4$, $h = \frac{1}{30}$ and $t = 1$. From tables 4 and 5 and figure 7, it is clear that we achieve a good approximation of the exact solution using a few terms of Genocchi polynomials.

Example 5 We consider the nonlinear GF-BBMBE in the following form [44]:

$$\frac{\partial u(x,t)}{\partial t} - D_\theta^r u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial x} = g(x,t), \quad 1 < r_1 \le 2, \quad 0 < r_2 \le 1, \tag{112}$$

with initial condition $u(x,0) = \sin(x)$, $x \in R$ and boundary conditions

$$u(0,t) = 0, \quad u(1,t) = \exp(-t)\sin(1), \qquad 0 \le t \le 1,$$

where $g(x,t) = \exp(-t)(\cos(x) - \sin(x) + \frac{1}{2}\exp(-t)\sin(2x))$. The exact solution when $r_1 = 2, r_2 = 1$ is $u(x,t) = \exp(-t)\sin(x)$. In this example, using Theorems 4 and 5, we calculate the upper bound of error. Hence, for $r_1 = 2, r_2 = 1$ and $M = N = 4$, we obtain

$$\|u(x,t) - u_{44}(x,t)\|_2 \le 9.4188 \times 10^{-3} \tag{113}$$

and

$$\|u_{44}(x,t) - \bar{u}_{44}(x,t)\|_2 \le 1.6228 \times 10^{-7}. \tag{114}$$

As a result, we get

$$\|u(x,t) - \bar{u}_{44}(x,t)\|_2 \le 9.4188 \times 10^{-3}. \tag{115}$$

The numerical results of this example are displayed in tables 6 and 7 and figures 8 and 9. In table 6, we compare the absolute errors between the exact and approximate solutions using the present method for different values of N, M and $r_1 = 2, r_2 = 1$, by the method in [44]. In this table, it is clear that, on increasing the number of Genocchi polynomials M, N, the approximate solutions converge to the exact solution. Also, numerical results for different choices of r_1, r_2 with $M = N = 5$ are given in table 7. In table 7, we see that as r_1 approaches 2 and r_2 approaches 1, the numerical solutions converge to the solution of integer-

Table 6. Absolute errors for different values of N, M with $r_1 = 2, r_2 = 1$ and $t = 0.01$ on interval $[0,1] \times [0,1]$ of Example 5.

	Present method		Method in [44]
x	$M = N = 4$	$M = N = 6$	$M = 16$
0.1	9.39×10^{-8}	3.86×10^{-10}	4.8×10^{-7}
0.2	1.76×10^{-7}	7.34×10^{-10}	8.6×10^{-7}
0.3	2.52×10^{-7}	1.01×10^{-9}	1.1×10^{-6}
0.4	3.15×10^{-7}	1.22×10^{-9}	1.3×10^{-6}
0.5	3.54×10^{-7}	1.33×10^{-9}	1.3×10^{-6}
0.6	3.60×10^{-7}	1.34×10^{-9}	1.3×10^{-6}
0.7	3.31×10^{-7}	1.23×10^{-9}	1.1×10^{-6}
0.8	2.61×10^{-7}	9.82×10^{-10}	8.6×10^{-7}
0.9	1.62×10^{-7}	5.69×10^{-10}	4.8×10^{-7}
CPU	6.35×10^{-2}	9.04×10^{-2}	–

Table 7. Absolute errors for different values of r_1, r_2 with $M = 5, N = 5$, $x \in [0, 1]$ and $t = 1$ of Example 5.

x	$r_1 = 1.7, r_2 = 0.7$	$r_1 = 1.8, r_2 = 0.8$	$r_1 = 1.9, r_2 = 0.9$	$r_1 = 2, r_2 = 1$
0	0	0	0	0
0.1	1.10×10^{-3}	7.79×10^{-4}	4.23×10^{-4}	2.29×10^{-8}
0.2	2.13×10^{-3}	1.49×10^{-3}	8.10×10^{-4}	7.96×10^{-9}
0.3	2.96×10^{-3}	2.07×10^{-3}	1.12×10^{-3}	3.20×10^{-8}
0.4	3.53×10^{-3}	2.47×10^{-3}	1.32×10^{-3}	5.55×10^{-8}
0.5	3.78×10^{-3}	2.64×10^{-3}	1.42×10^{-3}	5.72×10^{-8}
0.6	3.70×10^{-3}	2.58×10^{-3}	1.38×10^{-3}	5.35×10^{-8}
0.7	3.26×10^{-3}	2.27×10^{-3}	1.22×10^{-3}	6.17×10^{-8}
0.8	2.49×10^{-3}	1.73×10^{-3}	9.32×10^{-4}	6.31×10^{-8}
0.9	1.39×10^{-3}	9.71×10^{-3}	5.21×10^{-4}	2.02×10^{-8}
1	1.12×10^{-18}	1.12×10^{-18}	1.12×10^{-18}	1.12×10^{-18}

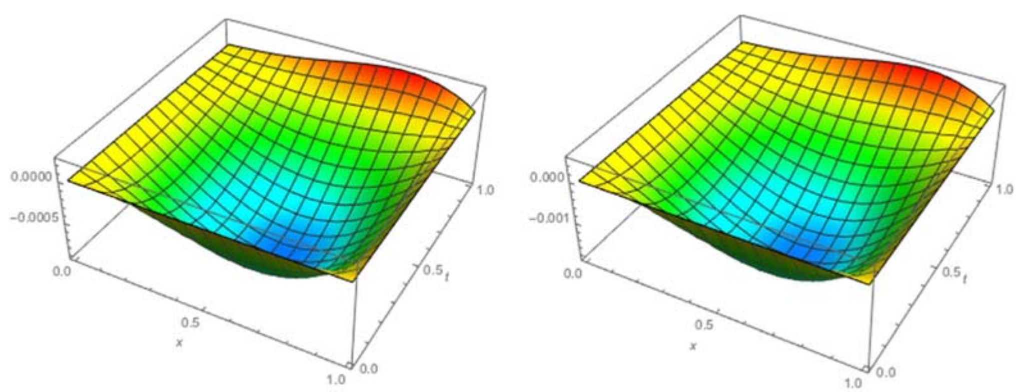

Figure 8. Absolute errors between the approximate solution and the exact solution for $r_1 = 1.95, r_2 = 0.95$ (left) and $r_1 = 1.9, r_2 = 0.9$ (right) on the interval $(x, t) \in [0, 1] \times [0, 1]$ with $M = 5, N = 5$ of Example 5.

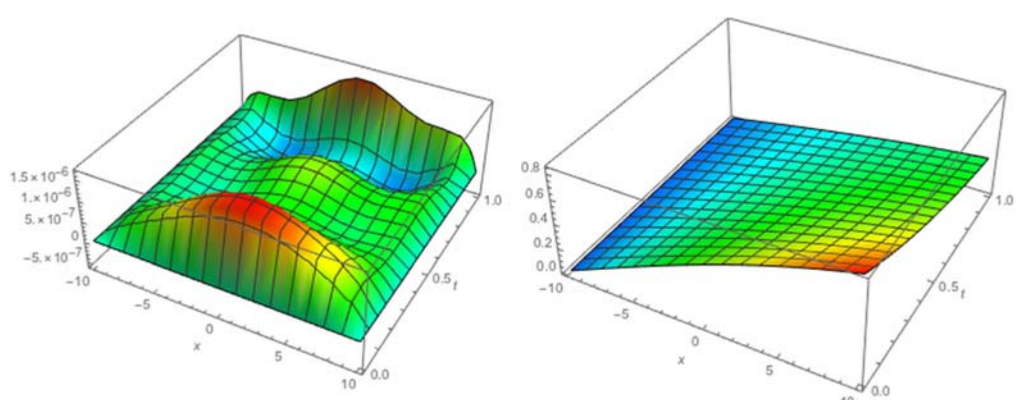

Figure 9. Absolute error (left) and approximate solution (right) for $r_1 = 2$, $r_2 = 1$, on the interval $(x, t) \in [-10, 10] \times [0, 1]$ and $M = 4, N = 4$ of Example 5.

order differential equation. In figure 8a and b, we show the behaviour of the absolute errors for various choices of r_1 and r_2 with $M = N = 5$ on interval $[0, 1] \times [0, 1]$. These figures show that the absolute errors tend to zero, when r_1 approaches 2 and r_2 approaches 1. Moreover, the absolute error and approximate solution for $M = N = 4$ on interval $[-10, 10] \times [0, 1]$ are plotted in figure 9(left) and 9(right), respectively.

8. Conclusion

In this paper, we showed a way of approximating the solution of GF-BBMBEs. We apply the Genocchi polynomials and their properties to achieve the results with high accuracy. The properties of 2D-Genocchi basis functions and pseudo-operational matrices have a direct influence on the computation method in obtaining an accurate approximate solution. The combination of the collocation method and 2D-Genocchi functions transform the GF-BBMBEs into a system of algebraic equations. Finally, to illustrate the superiority of the proposed method, some test problems were examined.

Nomenclature

u Fluid velocity in the horizontal direction
D_t^{γ} Caputo fractional derivative operator
D_{θ}^{r} Mixed Caputo fractional derivative operator

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References

[1] Eilenberger G 1983 *Solitons*. Berlin: Springer-Verlag
[2] Whitham G 1974 *Linear and nonlinear waves*. New York: Wiley
[3] Gray P and Scott S 1990 *Chemical waves and instabilities*. Oxford: Clarendon
[4] Hasegawa A 1975 *Nonlinear effects and plasma instabilities*. Berlin: Springer-Verlag
[5] Meiss J and Horton W 1982 Fluctuation spectra of a drift wave soliton gas. *Phys. Fluids.* 25: 1838–1843
[6] Korteweg D and De Vries G 1895 XLI. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Lond. Edinb. Dubl. Philos. Mag. J. Sci.* 39: 422–443
[7] Benjamin T, Bona J and Mahony J 1972 Model equations for long waves in nonlinear dispersive systems. *Philos. Trans. R. Soc. Lond. A* 272(1220): 47–78
[8] Byatt-Smith J 1971 The effect of laminar viscosity on the solution of the undular bore. *J. Fluid Mech.* 48: 33–40
[9] Dutykh D 2009 Visco-potential free-surface flows and long wave modelling. *Eur. J. Mech. B: Fluids* 28: 430–443
[10] Kakutani T and Matsuuchi K 1975 Effect of viscosity on long gravity waves. *J. Phys. Soc. Jpn.* 39(1): 237–246
[11] Zhang H, Wei G and Gao Y 2002 On the general form of the Benjamin–Bona–Mahony equation in fluid mechanics. *Czech. J. Phys.* 52: 373–377
[12] Kaya D 2004 A numerical simulation of solitary-wave solutions of the generalized regularized long-wave equation. *Appl. Math. Comput.* 149: 833–841
[13] Abdollahzadeh M, Ghanbarpour M, Hosseini A and Kashani S 2010 Exact travelling solutions for Benjamin–Bona–Mahony–Burgers equations by (G'/G)-expansion method. *Int. J. Appl. Math. Comput.* 3: 70–76
[14] Al-Khaled K, Momani S and Alawneh A 2005 Approximate wave solutions for generalized Benjamin–Bona–Mahony–Burgers equations. *Appl. Math. Comput.* 171: 281–292
[15] Mekki A and Ali M 2013 Numerical simulation of Kadomtsev–Petviashvili–Benjamin–Bona–Mahony equations using finite difference method. *Appl. Math. Comput.* 219: 11214–11222
[16] Dehghan M, Abbaszadeh M and Mohebbi A 2014 The numerical solution of nonlinear high dimensional generalized Benjamin–Bona–Mahony–Burgers equation via the meshless method of radial basis functions. *Comput. Math. Appl.* 68: 212–237
[17] Noor M, Noor K, Waheed A and Al-Said E 2011 Some new solitonary solutions of the modified Benjamin–Bona–Mahony equation. *Comput. Math. Appl.* 62: 2126–2131
[18] Wazwaz A and Triki H 2011 Soliton solutions for a generalized KdV and BBM equations with time-dependent coefficients. *Commun. Nonlinear Sci. Numer. Simul.* 16: 1122–1126
[19] Singh K, Gupta R and Kumar S 2011 Benjamin–Bona–Mahony (BBM) equation with variable coefficients: similarity reductions and Painleve analysis. *Appl. Math. Comput.* 217: 7021–7027
[20] Yin H and Hu J 2010 Exponential decay rate of solutions toward traveling waves for the Cauchy problem of generalized Benjamin–Bona–Mahony–Burgers equations. *Nonlin. Anal.: Theor.* 73: 1729–1738
[21] Tari H and Ganji D 2007 Approximate explicit solutions of nonlinear BBMB equations by He's methods and comparison with the exact solution. *Phys. Lett. A* 367: 95–101
[22] Achouri T, Khiari N and Omrani K 2006 On the convergence of difference schemes for the Benjamin–Bona–Mahony (BBM) equation. *Appl. Math. Comput.* 182: 999–1005
[23] Abbasbandy S and Shirzadi A 2010 The first integral method for modified Benjamin–Bona–Mahony equation. *Commun. Nonlin. Sci. Numer. Simul.* 15: 1759–1764
[24] Araci S, Acikgoz M and Aen E 2013 On the extended Kim's p-adic q-deformed fermionic integrals in the p-adic integer ring. *J. Number Theory* 133: 3348–3361
[25] Bayad A and Kim T 2010 Identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials. *Adv. Stud. Contemp. Math.* 20: 247–253
[26] Araci S 2012 Novel identities for q-Genocchi numbers and polynomials. *J. Funct. Space Appl.* 2012
[27] Srivastava H, Kurt B and Simsek Y 2012 Some families of Genocchi type polynomials and their interpolation functions. *Integr. Transf. Spec. F.* 23: 919–938
[28] Araci S, Acikgoz M, Bagdasaryan A and Sen E 2013 The Legendre polynomials associated with Bernoulli, Euler, Hermite and Bernstein polynomials. arXiv preprint arXiv: 1312.7838
[29] Araci S 2014 Novel identities involving Genocchi numbers and polynomials arising from applications of umbral calculus. *Appl. Math. Comput.* 233: 599–607
[30] Lim D 2016 Some identities of degenerate Genocchi polynomials. *Bull. Korean Math. Soc.* 53: 569–579

[31] Isah A and Phang C 2018 Operational matrix based on Genocchi polynomials for solution of delay differential equations. *Ain Shams Eng. J.* 9: 2123–2128

[32] Isah A and Phang C 2019 New operational matrix of derivative for solving non-linear fractional differential equations via Genocchi polynomials. *J. King Saud. Univ. Sci.* 31: 1–7

[33] Loh J R, Phang C and Isah A 2017 New operational matrix via Genocchi polynomials for solving Fredholm–Volterra fractional integro-differential equations. *Adv. Math. Phys.* 2017, Article ID 3821870, 12 pp.

[34] Phang C, Ismail N, Isah A and Loh J 2018 A new efficient numerical scheme for solving fractional optimal control problems via a Genocchi operational matrix of integration. *J. Vib. Control* 24: 3036–3048

[35] Podlubny I 1999 Fractional differential equations. In: *Mathematics in Science and Engineering*, vol. 198

[36] Agheli B and Firozja M A 2019 Approximate solution for high-order fractional integro-differential equations via trigonometric basic functions. *Sadhana* 44: 77

[37] Vityuk A and Golushkov A 2004 Existence of solutions of systems of partial differential equations of fractional order. *Nonlin. Oscil.* 7: 318–325

[38] Dehestani H, Ordokhani Y and Razzaghi M 2019 Application of the modified operational matrices in multiterm variable-order time-fractional partial differential equations. *Math. Meth. Appl. Sci.* 1–18

[39] Dehestani H, Ordokhani Y and Razzaghi M 2019 Hybrid functions for numerical solution of fractional Fredholm–Volterra functional integro-differential equations with proportional delays. *Int. J. Numer. Model.* 42: 7296–7313

[40] Isah A and Phang C 2016 Genocchi wavelet-like operational matrix and its application for solving non-linear fractional differential equations. *Open Phys.* 14: 463–472

[41] Dehestani H, Ordokhani Y and Razzaghi M 2019 A numerical technique for solving various kinds of fractional partial differential equations via Genocchi hybrid functions. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Madr.* 113: 3297–3321

[42] Loh J R and Phang C 2018 A new numerical scheme for solving system of Volterra integro-differential equation. *Alex. Eng. J.* 57: 1117–1124

[43] Dehestani H, Ordokhani Y and Razzaghi M 2018 Fractional-order Legendre–Laguerre functions and their applications in fractional partial differential equations. *Appl. Math. Comput.* 336: 433–453

[44] Singh I and Kumar S 2017 Haar wavelet methods for numerical solutions of Harry Dym (HD), BBM Burgers' and 2D diffusion equations. *Bull. Braz. Math. Soc. New Ser.* 49: 1–26