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A mixed fractional Vasicek model and pricing Bermuda option on zero-coupon bonds

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Abstract. This paper considers the problem of pricing of Bermuda options on zero-coupon bond in which the dynamics of the interest rate model follows the mixed fractional Vasicek model. The strong convergence of the Euler discretization scheme for the mixed fractional Vasicek model is analysed. Specifically, we find an approximate formula for zero-coupon bond price. Numerical experiments are provided and compared for Bermuda-style call and put options with the Monte Carlo simulation approach.

Keywords. Bermuda option; mixed fractional Vasicek model; zero-coupon bond; Monte Carlo simulation.

1. Introduction

Results of Choi and Wirjanto [1] are adapted here to find an approximate formula for zero-coupon bond price. The interest rate is one of the most important control tools in the economy and macroeconomic variables in policy making [2, 3]. This depends on the money and it is based on the preferences of economic units over time. It is possible to justify the maintenance of savings in terms of liquidity. The economies of developing countries are strongly influenced by interest rates and they react quickly according to the changes of the rates. Indeed, the interest rate index is a powerful control tool for managing the market. Based on the importance of interest rate, researchers introduced different models and used them to predict the behaviour of the interest rate paths [4–6]. Vasicek proposed a model in 1977 and used it to forecast the value of the interest rate [5]. Unlike many financial models, such as Cox–Ingersoll–Ross (CIR) and mixed CIR with mixed Wishart volatility process, this model can be used to predict both positive and negative interest rate amounts. The noise of the model is standard Brownian motion. In classical quantitative finance, it is usual to suppose that risky part of the interest rate models is driven by Brownian motion. The Brownian motion is a martingale process and thus the increments of the process have independent and stationary properties. The general form of the process, namely fractional Brownian motion (fBm), is displayed by $B^H, H \in (0,1)$, suggested by Mandelbrot and Van Ness in 1968 as an integral representation of the Brownian motion process [7, 8]. In the case of $H = \frac{1}{2}$, the fBm is a standard Brownian motion process. The increments of the fBm have memory and it is the main difference between fBm and standard Brownian motion. In fact, based on different values of Hurst parameter, the process is divided into three main categories: $0 < H < 0.5$, $H = 0.5$ and $0.5 < H < 1$. Each category has its unique characteristics. When $H \in (0, \frac{1}{2})$, the increments of fBm process have a negative correlation and are utilized to describe the phenomena with intermediate memory. If $H \in (\frac{1}{2}, 1)$, the increments of fBm have a positive correlation [7]. Hence, the process has cumulative behaviour and describes a system containing memory and persistence. There are several long-memory financial data [9, 10]. The interest rate of the countries such as the US, Luxembourg and Switzerland has a long memory. Hence, it is reasonable to predict the interest rate amounts using fBm models. The fBm process is not semi-martingale and it has been shown that applying this process to calculate the price of the financial derivatives, when the dynamic of the underlying asset follows fBm model, can create arbitrage opportunities [11–14]. The researchers presented different strategies to eliminate arbitrage from fractional financial models [15, 16].

Mixed fractional Brownian motion (mfBm) is a linear combination of Brownian motion and fBm with parameter H, defined as follows:

$$M_t^H = \alpha B_t + \gamma B_t^H; H \in (0,1), t \geq 0, \tag{1.1}$$

where α and γ are real constants. Cheridito [17] proves that the mfBm process with Hurst parameter $H \in \left(\frac{3}{4}, 1\right)$ is

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equivalent to a βB_t martingale and hence using this process in financial models creates long-memory property without arbitrage. In recent decades, the mfBm models are applied in many papers and are used to predict the behaviour of the asset price and interest rate amounts [6, 18–21]. Here, the mixed fractional version of the Vasicek model where the standard Brownian motion is replaced by an mfBm with Hurst parameter $H \in \left(\frac{3}{4}, 1\right)$ is presented. To make the model more effective, it is better if the parameters of the models are calibrated by considering real data such as the US interest rate. For this purpose, the maximum likelihood estimation (MLE) method is used to select the values set of the model parameters that maximize the likelihood function. Also, by AIC, we indicate that the mixed fractional Vasicek model with $H \in \left(\frac{3}{4}, 1\right)$ is more appropriate rather than the standard one.

A zero-coupon bond is a debt security that is sold at a discount and does not pay any interest payments to the bondholder. The price of a bond depends on several factors, such as maturity time and interest rate amount [22, 23]. Since the increments of the mixed interest rate models are dependent, researchers use numerical methods to evaluate the zero-coupon bond price. The Monte Carlo simulation is one of the famous method to obtain the price of the financial derivatives. But this method sometimes need more time. Here, we extend the idea presented in [1] to find an analytic approximation formula for pricing the zero-coupon bond and show that this formula is better than the Monte Carlo simulation method.

The option on a bond is one of the most common financial derivatives [1, 6, 9, 24]. An option is a contract that gives the right to the buyer (the owner or holder of the option) without any obligation so that the buyer buys or sells an underlying asset or instrument at a specified strike price prior to or on a specified date, depending on the form of the option. A Bermuda option is a type of exotic option that can be exercised only on predetermined dates, often on one day of each month. Bermuda options are a combination of American and European options [25, 26]. Indeed, Bermuda option is a type of European option with a finite number of expiration times. In some cases, there is no closed form solution to calculate the option price. Thus, the researchers use different ways to approximate the value of the option. The options can be priced by Monte Carlo simulation [27, 28]. In this way, first, the price of the underlying asset is simulated by random number generation for a number of paths. Then, the value of the option is found by calculating the average of discounted returns over the sample paths. Since the option is priced under the risk-neutral measure, the discount rate is the risk-free interest rate.

The rest of this article is organized as follows. In section 2, the mixed fractional Vasicek model is presented. We also show that the Euler discretization scheme of the mixed fractional Vasicek model has strong convergence. In section 3, a zero-coupon bond formula under the mentioned model is given by utilizing the idea in [1]. Finally, the Bermuda call and put options price on the zero-coupon bond is obtained by applying the Monte Carlo simulation method and different values of the model's parameters in section 4.

2. Mixed fractional Vasicek model

One-factor Vasicek model was introduced in 1977 and used to describe interest rate amount [5]. The stochastic differential form of the model can be written as

$$dr_t = \kappa(\theta - r_t)dt + \sigma dB_t, \tag{2.1}$$

where κ, σ and θ are positive constant amounts, and B_t is standard Brownian motion. Over time, the expectation of the model tends to the long-mean parameter θ with speed κ. On applying mfBm instead of Brownian motion, mixed fractional Vasicek model will be obtained. The dynamic of the model is

$$dr_t = \kappa(\theta - r_t)dt + \sigma dM_t^H. \tag{2.2}$$

This model has several advantages over the classical one. First, it is shown that the increments of the model have long-memory property.

Definition 2.1 Let $X = (X_t)_{t \geq 0}$ be a stationary stochastic process and $\rho(n) = Cov(X_k, X_{k+n})$ be an auto-covariance function for $n \geq 1$. The process has a long-memory property if [7]

$$0 \neq \sum_{n=1}^{\infty} |\rho(n)| = +\infty. \tag{2.3}$$

Lemma 2.2 *The increment of the mixed fractional Vasicek model has long memory for* $H \in \left(\frac{3}{4}, 1\right)$.

Proof 2.3 Since the increments of the model have stationary property, applying Definition 2.1, one can show that the model has the long-memory property. Let $n \geq 1$, and $X_k = \Delta R_k = \kappa(\theta - R_k)\Delta t_k + \sigma\left(\alpha \Delta B_k + \gamma \Delta B_k^H\right)$. Then

$$\begin{aligned}
\rho(n) &= Cov(X_k, X_{k+n}) \\
&\simeq \gamma^2 \sigma^2 Cov(\Delta B_k^H, \Delta B_{k+n}^H) \\
&= \frac{h^{2H}\gamma^2\sigma^2}{2}\Big[(n+1)^{2H} \\
&\quad + (n-1)^{2H} - 2n^{2H}\Big] \\
&\simeq \gamma^2\sigma^2 H(2H-1)n^{2H-2},
\end{aligned}$$

where $h = \Delta t_k$ for $k \geq 1$. Thus, for $H \in \left(\frac{3}{4}, 1\right)$ one can write

$$0 \neq \sum_{n=1}^{\infty} |\rho(n)| = +\infty.$$

□

Indeed, since the increments of the mfBm for $H \in \left(\frac{3}{4}, 1\right)$ have long-range memory, the mixed fractional financial models have this property. Furthermore, the model is more general than the classical one. The closed form solution of the mentioned model can be written as

$$r_t = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma \int_0^t e^{-\kappa(t-s)} dM_s^H, H \in \left(\frac{3}{4}, 1\right). \tag{2.4}$$

The mean and variance of the model are given by

$$\mathbb{E}(r_t) = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}),$$

and

$$Var(r_t) = \frac{\sigma^2 \alpha^2}{\kappa}(1 - e^{-\kappa t}) + \sigma^2 \beta^2 e^{-2\kappa t} \int_0^t \int_0^t e^{\kappa(s+u)} |s-u|^{2H-2} ds du.$$

Here, the stochastic calculus related to the fractional Brownian motion is used [7]. When the value of the Hurst parameter increases, the expectation of the model closes on its long mean more quickly over time. In fact, by increasing the value of the Hurst parameter, the model predictions tend to the long-mean amount of the model with high speed.

Calibration of the fractional Vasicek model

Using the MLE method, one can show that the interest rate data of the US have long-memory property. Indeed, it is shown that the fractional Vasicek model with $H \in (3/4, 1)$, where it has long-memory property, can predict the interest data better rather than the classical one. Suppose $0 = t_0 < t_1 < \cdots < t_n$ is a subsequence of the real time such that $\delta_i = t_{i+1} - t_i$ for all $i \geq 0$ is small enough. Then, the closed form of the solution for Euler scheme of the mixed fractional Vasicek model in $[t_i, t_{i+1}]$ can be written as follows:

$$R_{t_{i+1}} = R_{i+1} = e^{-\kappa \delta_i} R_i + \theta\left(1 - e^{-\kappa \delta_i}\right) + \sigma e^{-\kappa \delta_i} \int_0^{\delta_i} e^{\kappa s} dB_s^H. \tag{2.5}$$

The stochastic process $\{R_{i+1}\}_{i \geq 0}$ has normal distribution and the mean and variance of the process are

$$\mu = \mathbb{E}[R_{i+1}] = e^{-\kappa \delta_i} R_i + \theta\left(1 - e^{-\kappa \delta_i}\right), \tag{2.6}$$

and

$$\widehat{\sigma}^2 = Var[R_{i+1}] = \sigma^2 e^{-2\kappa \delta_i} \int_0^{\delta_i} \int_0^{\delta_i} e^{\kappa(u+v)} |u-v|^{2H-2} du dv. \tag{2.7}$$

The integral in this equation is not solvable. Since δ_i is very small for $i \geq 1$, the numerical midpoint method is applied to approximate the integral as

$$\widehat{\sigma}^2 = Var[R_{i+1}] \approx \sigma^2 \delta_i + \frac{2\sigma^2}{\kappa}\left(\frac{\delta_i}{2}\right)^{2H-1}\left(e^{-0.5\kappa\delta_i} - e^{-1.5\kappa\delta_i}\right). \tag{2.8}$$

One of the most important ways to estimate the financial models parameters is the MLE method. The method of maximum likelihood selects the set of values of the model parameters that maximize the likelihood function. The log-likelihood function of a set of observations $R_0, R_1, \ldots, R_n$ from the mixed fractional Vasicek model is as follows:

$$\begin{aligned}\mathcal{L}(\kappa, \theta, H) &= \ln\left(\prod_{i=1}^n f(R_{i+1}|R_i, \kappa, \theta, H)\right) \\ &= \sum_{i=1}^n \ln(f(R_{i+1}|R_i, \kappa, \theta, H)) \\ &= -\frac{n}{2}\ln(2\pi) - n\ln(\widehat{\sigma}) \\ &\quad - \frac{1}{2\widehat{\sigma}^2}\sum_{i=1}^n \left[R_i - e^{-\kappa\delta}R_{i-1} - \theta\left(1 - e^{-\kappa\delta}\right)\right]^2, \delta = \delta_i,\end{aligned} \tag{2.9}$$

where f is conditional probability density of an observation R_{i+1} by considering R_i and given by

$$f(R_{i+1}|R_i; \kappa, \mu, H) = \frac{1}{\sqrt{2\pi\widehat{\sigma}^2}} \exp\left[-\frac{\left(R_i - e^{-\kappa\delta}R_{i-1} - \theta\left(1 - e^{-\kappa\delta}\right)\right)^2}{2\widehat{\sigma}^2}\right], \delta = \delta_i. \tag{2.10}$$

The maximum of this log-likelihood surface can be found at the location where all the partial derivatives are zero. This leads to the following set of constraints:

$$\begin{aligned}&\frac{\partial\mathcal{L}(\kappa, \theta, H)}{\partial\theta} = 0 \\ &\Rightarrow \frac{1}{\widehat{\sigma}^2}\sum_{i=1}^n \left[R_i - e^{-\kappa\delta}R_{i-1} - \theta\left(1 - e^{-\kappa\delta}\right)\right] = 0 \\ &\Rightarrow \theta = \frac{\sum_{i=1}^n \left[R_i - e^{-\kappa\delta}R_{i-1}\right]}{n(1 - e^{-\kappa\delta})},\end{aligned} \tag{2.11}$$

$$\begin{aligned}&\frac{\partial\mathcal{L}(\kappa, \theta, H)}{\partial\kappa} = 0 \\ &\Rightarrow -\frac{\delta e^{-\kappa\delta}}{\widehat{\sigma}^2}\sum_{i=1}^n \left[(R_i - \theta)(R_{i-1} - \theta) - e^{-\kappa\delta}(R_{i-1} - \theta)^2\right] = 0 \\ &\Rightarrow \kappa = -\frac{1}{\delta}\ln\left(\frac{\sum_{i=1}^n (R_i - \theta)(R_{i-1} - \theta)}{\sum_{i=1}^n (R_{i-1} - \theta)^2}\right)\end{aligned} \tag{2.12}$$

and

$$\frac{\partial\mathcal{L}(\kappa,\theta,H)}{\partial\widehat{\sigma}}=0$$
$$\Rightarrow \widehat{\sigma}^2=\frac{\sum_{i=1}^{n}\left[R_i-e^{-\kappa\delta}R_{i-1}-\theta\left(1-e^{-\kappa\delta}\right)\right]^2}{n}. \tag{2.13}$$

Let $(R_0, R_1, \ldots, R_n)$ be an interest rate data set. Then, one can write the model parameters with regard to following notations as

$$R_x=\sum_{i=1}^{n}R_{i-1}, R_y=\sum_{i=1}^{n}R_i,$$
$$R_{xx}=\sum_{i=1}^{n}R_{i-1}^2,$$
$$R_{yy}=\sum_{i=1}^{n}R_i^2, R_{xy}=\sum_{i=1}^{n}R_{i-1}R_i.$$

Then

$$\theta=\frac{R_yR_{xx}-R_xR_{xy}}{n\left(R_{xx}-R_{xy}\right)-\left(R_x^2-R_xR_y\right)} \tag{2.14}$$

$$\kappa=-\frac{1}{\delta}\ln\frac{R_{xy}-\theta R_x-\theta R_y+n\theta^2}{R_{xx}-2\theta R_x+n\theta^2} \tag{2.15}$$

and

$$\widehat{\sigma}^2=\frac{Ryy+e^{-2\kappa\delta}Rxx+n\left(\theta(1-e^{-\kappa\delta})\right)^2-2e^{-\kappa\delta}Rxy}{n}-\frac{2\theta(1-e^{-\kappa\delta})Ry-2\theta(1-e^{-\kappa\delta})e^{-\kappa\delta}Rx}{n}. \tag{2.16}$$

Akaike Information Criterion (AIC): In recent decades, researchers have presented various information criteria to compare different models for given data. The AIC was first introduced by Akaike and used in order to identify the most effective model by comparing the different models for each event [29]. The most effective model is the model with the lowest AIC score and is calculated as

$$AIC=2K-2\log\left(\mathcal{L}\left(\widehat{\theta}|y\right)\right),$$

where K is the number of estimable parameters and $\log\left(\mathcal{L}\left(\widehat{\theta}|y\right)\right)$ is the log likelihood at its maximum likelihood estimator $\widehat{\theta}$ based on y observations. Now, two interest rate models are compared by considering the US interest rate data. These models have the same deterministic and different random parts. Therefore, the MLE method produces the same amounts for the same parameters in their deterministic parts. In this note, the formula presented in [29] is used to estimate the parameters of the standard Vasicek model. In table 1, by applying AIC for the US interest rate, one can deduce that the mixed fractional Vasicek model is more effective than the standard Vasicek one. This implies that the data of the US have long-memory property.

Furthermore, the model is compared to the famous interest rate models such as MCIR model and MCIR with mixed Wishart process model. The dynamics of the MCIR model is as follows:

$$dR_t=\kappa(\theta-R_t)dt+\sigma\sqrt{R_t}dM_t^H,$$

where κ, θ and σ are the speed of the mean reverting property, the long-mean model and the volatility id the interest rate, respectively. Also, the dynamics of the MCIR model with mixed Wishart process is

$$\begin{aligned}dR_t&=\kappa(\theta-R_t)dt\\&\quad+c_2\sqrt{c_1R_t+c_2(trace(GX_t))}dM_{1,t}^H,\\dX_t&=\left(\beta^2\alpha Q^TQ+KX_t+X_tK^T\right)dt\\&\quad+\alpha Q^TQdt^{2H}\\&\quad+\sqrt{X_t}dM_{1,t}^HQ\\&\quad+Q^T\left(dM_{2,t}^H\right)^T\sqrt{X_t},\end{aligned}$$

where c_1 and c_2 are constants, K is a symmetric positive definite matrix, Q is invertible symmetric matrix, trace is the operator of the diagonal matrix, and M_1^H and M_2^H are two independent mfBm processes. Here, these models are compared with respect to the interest rate of Luxembourg and Switzerland countries. When the rate of interest rate is positive, the three models with low error have been able to predict the amounts of interest rates. However, for negative values of interest rates, the results of the simulation proposed model are more efficient than two other famous models. Indeed, given the structure of the MCIR model and MCIR with mixed Wishart process model, it is clear that they cannot predict the negative values of the interest rate as well as the proposed model. These results can be seen in tables 2 and 3.

Table 1. Comparison of standard and mixed fractional version of the Vasicek model for the US interest rate from 01:2015 to 05:2019.

Parameters	κ	θ	σ^2	H	AIC
Standard Vasicek model	0.5484	2.3658	0.1746	0.5	−38.9998
Mixed fractional Vasicek model	0.5484	2.3658	0.2671	0.8170	−65.8751

Table 2. Comparison of mixed Vasicek model, MCIR model and MCIR with mixed Wishart process for Luxembourg country interest rate from 01:2015 to 07:2017.

Time	01/2015	07/2015	01/2016	07/2016	01/2017	07/2017
Real data	0.4733	0.5606	0.1839	−0.3646	−0.0760	0.6927
MCIR model	0.5184	0.5711	0.2366	0.0032	0.0191	0.7833
MCIR with mixed Wishart process	0.4832	0.5521	0.1906	0.0001	0.0083	0.7412
Mixed Vasicek model	0.5025	0.5442	0.1565	−0.3381	−0.1327	0.6283

Table 3. Comparison of mixed Vasicek model, FCIR model and MCIR with mixed Wishart process for Switzerland country interest rate from 01:2015 to 07:2017.

Time	01/2015	07/2015	01/2016	07/2016	01/2017	07/2017
Real data	−0.07	−0.04	−0.299	−0.543	−0.072	0.06
FCIR model	0.0019	0.0008	0.0002	0	0.0005	0.101
MCIR with mixed Wishart process	0.0008	0	0	0	0.0001	0.078
Mixed Vasicek model	−0.016	−0.011	−0.2566	−0.411	−0.1332	− 0.023

Strong convergence of the Euler discretization scheme

Since the Bermuda option has several expiration times, the interest rate data are found at these times. For this purpose, the Euler method is used. In the following, the strong convergence of the Euler method is discussed for the mixed fractional Vasicek model. The general form of the strong convergence concept for the stochastic differential equations with non-Lipschitz coefficient is presented in [6, 18]. Here, we present a sketch of the proof of the stochastic differential equation with Lipschitz coefficient. Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of [0, T]. The discrete and continuous time versions of the proposed model equation are as follows:

$$\begin{aligned} X_{k+1} =& X_k + \kappa(\theta - X_k)\Delta t \\ &+ \sigma\left(\alpha\Delta B_k + \gamma\Delta B_k^H\right) \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \overline{X}_t :=& X_k + \kappa\int_{t_k}^{t}(\theta - X_s)ds \\ &+ \sigma\Bigg[\alpha(B(t) - B(t_k)) \\ &+ \gamma\left(B^H(t) - B^H(t_k)\right)\Bigg], \end{aligned} \tag{2.18}$$

where for all $t \in [t_k, t_{k+1})$, $X_t = X_k$.

Theorem 2.3 *The Euler solution of Eq. (2.17) with continuous time extension Eq. (2.18)satisfies*

$$\lim_{\Delta t \longrightarrow 0}\mathbb{E}\left[\sup_{0\le t\le T}|\overline{X}_t - r_t|^2\right] = 0. \tag{2.19}$$

Proof 2.5 Let r_t satisfy Eq. (2.2); then $\mathbb{E}[|r_t|^p] < C$ for each $p > 2$ and large enough amount constant C. Therefore, we conclude that for all $p > 2$, $\mathbb{E}\left[\sup_{0\le t\le T}|\overline{X}_t|^p\right] < C$. Consider the following stopping times:

$$\begin{aligned} \eta_n &= \inf\{t \ge 0 : |r_t| \ge n\} \\ \tau_n &= \inf\left\{t \ge 0 : |\overline{X}_t| \ge n\right\} \\ \xi_n &= \eta_n \wedge \tau_n. \end{aligned} \tag{2.20}$$

Let $e(t) = \overline{X}_t - r_t$. Then for any $\delta > 0$

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\right] \\ &\le \mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2 1_{\{\xi_n > T\}}\right] \\ &\quad + \frac{2\delta}{p}\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^p\right] \\ &\quad + \frac{1-\frac{2}{p}}{\delta^{\frac{2}{p-2}}}\mathbb{P}[\eta_n \le T \, or \, \tau_n \le T]. \end{aligned} \tag{2.21}$$

Now, one can write

$$\begin{aligned} &\mathbb{P}[\eta_n \le T \, or \, \tau_n \le T] \\ &\quad \le \mathbb{P}[\eta_n \le T] + \mathbb{P}[\tau_n \le T] \le \frac{2C}{n^p}. \end{aligned} \tag{2.22}$$

Using these results, one can conclude that

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^p\right] \\ &\le 2^{p-1}\mathbb{E}\left[\sup_{0\le t\le T}\left(|\overline{X}_t|^p + |r_t|^p\right)\right] \le 2^p C. \end{aligned} \tag{2.23}$$

Equation (2.21) gives

$$\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\right] \le \mathbb{E}\left[\sup_{0\le t\le T}|\overline{X}_{t\wedge\eta_n} - r_{t\wedge\eta_n}|^2\right] + \frac{2^{p+1}\delta C}{p} + \frac{2(p-2)C}{p\delta^{\frac{2}{p-2}}n^p}. \tag{2.24}$$

Now, a bound for the first term on the right hand side of inequality (2.24) is obtained. For all $t \in [0, T]$

$$\mathbb{E}\left[\sup_{0\le t\le T}|\overline{X}_{t\wedge\xi_n} - r_{t\wedge\xi_n}|^2\right] \le \kappa^2 T\left[\mathbb{E}\left[\int_0^{t\wedge\xi_n}|X_s - \overline{X}_s|^2 ds\right] + \int_0^T \mathbb{E}\left[\sup_{0\le s\le T}|\overline{X}_{s\wedge\xi_n} - r_{s\wedge\xi_n}|^2\right]ds\right]. \tag{2.25}$$

Therefore

$$\mathbb{E}[\int_0^{t\wedge\xi_n}|X_s - \overline{X}_s|^2 ds] \le 2T\Big(\big(2\kappa^2\theta^2 + 2\kappa^2 C^{\frac{2}{p}}\big)(\Delta t)^2 + \sigma^2(\Delta t)^{2H}\mathbb{E}\Big[|B^H(1)|^2\Big]\Big), \tag{2.26}$$

where $\Delta t = \max\{\Delta t_i : i = 1, \ldots\}$. In this way

$$\mathbb{E}\Big[sup_{0\le t\le T}|\overline{X}_{t\wedge\xi_n} - r_{t\wedge\xi_n}|^2\Big] \le 2\kappa^2 T^2\Big(2\kappa^2\theta^2 + 2\kappa^2 C^{\frac{2}{p}}\Big)(\Delta t)^2 \tag{2.27}$$

$$+ \sigma^2(\Delta t)^{2H})\mathbb{E}\Big[|B^H(1)|^2\Big] + \kappa^2 T^2\int_0^T \mathbb{E}\left[\sup_{0\le t\le T}|\overline{X}_{s\wedge\xi_n} - r_{s\wedge\xi_n}|^2\right]ds. \tag{2.28}$$

Thus, using the Gronwall inequality, we deduce that

$$\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\right] < \epsilon. \tag{2.29}$$

□

3. Zero-coupon bond formula

Let $(\Omega, \mathcal{F}, Q)$ be a probability space where Q is a risk-neutral measure. The value of the zero-coupon bond with maturity time T at time $0 \le t \le T$ is denoted by $P_{t,T}$ and

$$P_{t,T} = \mathbb{E}_t^Q\left[exp\left(-\int_t^T r_s ds\right)\right], \tag{3.1}$$

where $r_t, t \ge 0$, satisfies the mixed fractional Vasicek model. Since the integral in Eq. (3.1) is not solvable, the Trapezoidal method is used to estimate the integral. Here, an approximate solution to estimate the value of the zero-coupon bond is presented.

Theorem 3.1 *The zero-coupon bond price with maturity time Tat $t = 0$can be estimated as*

$$P_{0,T} = \exp\Big(-\theta(T - B(T)) - r_0 B(T) + \frac{\sigma^2\alpha^2 T}{2\kappa^2} - \frac{\sigma^2\alpha^2 T}{\kappa^2}B(T) + \frac{\sigma^2\alpha^2}{4\kappa^3}\left(1 - e^{-2\kappa T}\right) + \xi(T, H)\Big) \tag{3.2}$$

where $B(T) = B(0, T) = \frac{1}{\kappa}(1 - e^{-\kappa T})$,

$$\xi(h, H) = \frac{\sigma^2 T}{2\kappa^2}\left(\gamma^2 h^{2H-1}\right) - \frac{\sigma^2 T}{\kappa^2}\left(\gamma^2 h^{2H}\right)\left(1 - \frac{\kappa h}{2}\right)\left(\frac{1 - (1-\kappa h)^n}{\kappa h}\right) + \frac{\sigma^2}{2\kappa^2}\left(\gamma^2 h^{2H}\right)\left(1 - \frac{\kappa h}{2}\right)^2\left(\frac{1 - (1-\kappa h)^{2n}}{1 - (1-\kappa h)^2}\right), \tag{3.3}$$

and, for sufficiently small h, $\xi(T, H) \approx \xi(h, H)$.

Proof 3.2 Let $0 = t_0 < t_1 \cdots < t_n = T$ be a partition for [0, T] such that $h = \frac{T}{n}$ and $t_k = kh$. By the Trapezoidal method to zero-coupon formula at $t = 0$, we get

$$\int_0^T r_s ds \approx h\left[\frac{r_0}{2} + r_1 + \cdots + r_{n-1} + \frac{r_n}{2}\right]. \tag{3.4}$$

Thus, the conditional expectation based on the information until t_{n-1} is obtained as

$$\begin{aligned}\mathbb{E}_{n-1}&\left[\exp\left(-\int_0^T r_s ds\right)\right]\\ &= \exp\left(-h\left[\frac{r_0}{2} + r_1 + \cdots + r_{n-2}\right] - hr_{n-1}\right)\\ &\times \mathbb{E}_{n-1}\left[\exp\left(-\frac{h}{2}[r_{n-1} + \kappa(\theta - r_{n-1})h + \sigma\Delta M^H_{n-1}(h)]\right)\right]\\ &= \exp\Big(-h\left[\frac{r_0}{2} + r_1 + \cdots + r_{n-2}\right] - \frac{\kappa\theta h^2}{2} - \frac{h}{2}[(3-\kappa h)r_{n-1}] + AF_1\Big),\end{aligned} \tag{3.5}$$

where r_n satisfies Eq. (2.17),

$AF_u = \exp\left(\frac{\sigma^2 h^2}{8} a_u^2(\alpha^2 h + \gamma^2 h^{2H})\right)$, $u \geq 2$, a_u satisfies the recursive relation

$$\begin{aligned} a_1 &= 1 \\ a_u &= 2 + (1-\kappa h)a_{u-1}, \end{aligned} \tag{3.6}$$

and it can be explicitly rewritten as

$$a_u = -\frac{2}{\kappa h}\left[(1-\kappa h)^{u-1}\left(1-\frac{\kappa h}{2}\right) - 1\right]. \tag{3.7}$$

By considering a_n frequently, the general formula for $\mathbb{E}_{n-u}$ can be written as follows:

$$\begin{aligned} &\mathbb{E}_{n-u}\left[\exp\left(-\int_0^T r_s ds\right)/\prod_{i=1}^{u-1} AF_i\right] \\ &= \exp\left(-h\left[\frac{r_0}{2} + r_1 + \cdots + r_{n-u-1}\right]\right. \\ &\left. -\frac{\kappa\theta h^2}{2}\sum_{i=1}^{u} a_i - \frac{h}{2} r_{n-u} a_{u+1} + AF_u\right). \end{aligned} \tag{3.8}$$

Therefore

$$\begin{aligned} &\mathbb{E}_0\left[\exp\left(-\int_0^T r_s ds\right)/\prod_{i=1}^{n-1} AF_i\right] \\ &= \exp\left(-\frac{\kappa\theta h^2}{2}\sum_{i=1}^{n} a_i\right. \\ &\left. -\frac{h}{2} r_0[1 + a_n(1-\kappa h)] + AF_n\right). \end{aligned} \tag{3.9}$$

Using Eqs. (3.6) and (3.7) in Eq. (3.9), one can conclude that

$$\begin{aligned} \frac{h}{2}[1 + a_n(1-\kappa h)] &= \frac{h}{2} + \frac{1}{\kappa}[1 \\ &-(1-\kappa h)^{n-1}](1-\kappa h) \\ &+\frac{h}{2}(1-\kappa h)^{n+1}. \end{aligned}$$

If $n \to \infty$, then

$$\frac{h}{2}[1 + a_n(1-\kappa h)] = \frac{1}{\kappa}\left(1 - e^{-\kappa T}\right). \tag{3.10}$$

Also, $\sum_{i=1}^{n} a_i$ can be written as follows:

$$\begin{aligned} \sum_{i=1}^{n} a_i &= \frac{2}{\kappa h}\sum_{i=1}^{n}\left[1 - \left(1-\frac{\kappa h}{2}\right)(1-\kappa h)^{i-1}\right]. \\ &= \frac{2T}{\kappa h^2} - \frac{2}{\kappa h}\left(1-\frac{\kappa h}{2}\right)\frac{(1-(1-\kappa h)^n)}{\kappa h}. \end{aligned} \tag{3.11}$$

Thus

$$\frac{\kappa\theta h^2}{2}\sum_{i=1}^{n} a_i = \theta(T - B(T)). \tag{3.12}$$

Finally, $\ln\left(\prod_{i=1}^{n-1} AF_i\right)$ can be estimated by

$$\begin{aligned} &\ln\left(\prod_{i=1}^{n-1} AF_i\right) \\ &= \frac{\sigma^2}{8}\left(\alpha^2 h^3 + \gamma^2 h^{2+2H}\right)\sum_{i=1}^{n} a_i^2 \\ &= \frac{\sigma^2}{8}\left(\alpha^2 h^3 + \gamma^2 h^{2+2H}\right)\left[\frac{4}{\kappa^2 h^2}\sum_{i=1}^{n}\left(1\right.\right. \\ &- 2\left(1-\frac{\kappa h}{2}\right)\times(1-\kappa h)^{i-1} \\ &\left.\left.+\left(1-\frac{\kappa h}{2}\right)^2(1-\kappa h)^{2i-2}\right)\right] \\ &= \frac{\sigma^2 T}{2\kappa^2}\left(\alpha^2\right) - \frac{\sigma^2 T}{\kappa^2}\left(\alpha^2 h\right) \\ &\left(1-\frac{\kappa h}{2}\right)\left(\frac{1-(1-\kappa h)^n}{\kappa h}\right) \\ &+\frac{\sigma^2}{2\kappa^2}\left(\alpha^2 h\right)\left(1-\frac{\kappa h}{2}\right)^2 \\ &\left(\frac{1-(1-\kappa h)^{2n}}{1-(1-\kappa h)^2}\right) + \xi(h,H), \end{aligned}$$

where

$$\begin{aligned} \xi(h,H) = &\frac{\sigma^2 T}{2\kappa^2}\left(\gamma^2 h^{2H-1}\right) - \frac{\sigma^2 T}{\kappa^2}\left(\gamma^2 h^{2H}\right) \\ &\times\left(1-\frac{\kappa h}{2}\right)\left(\frac{1-(1-\kappa h)^n}{\kappa h}\right) \\ &+\frac{\sigma^2}{2\kappa^2}\left(\gamma^2 h^{2H}\right) \\ &\times\left(1-\frac{\kappa h}{2}\right)^2\left(\frac{1-(1-\kappa h)^{2n}}{1-(1-\kappa h)^2}\right). \end{aligned}$$

When $h \to 0$ and for $H \in \left(\frac{3}{4}, 1\right)$

$$\begin{aligned} \ln\left(\prod_{i=1}^{n-1} AF_i\right) &= \frac{\sigma^2\alpha^2 T}{2\kappa^2} - \frac{\sigma^2\alpha^2 T}{\kappa^2} B(T) \\ &+\frac{\sigma^2\alpha^2}{4\kappa^3}\left(1 - e^{-2\kappa T}\right) + \xi(T,H) \end{aligned} \tag{3.13}$$

where $\xi(T,H) \approx \xi(h,H)$ for small values of h. Therefore, we can write

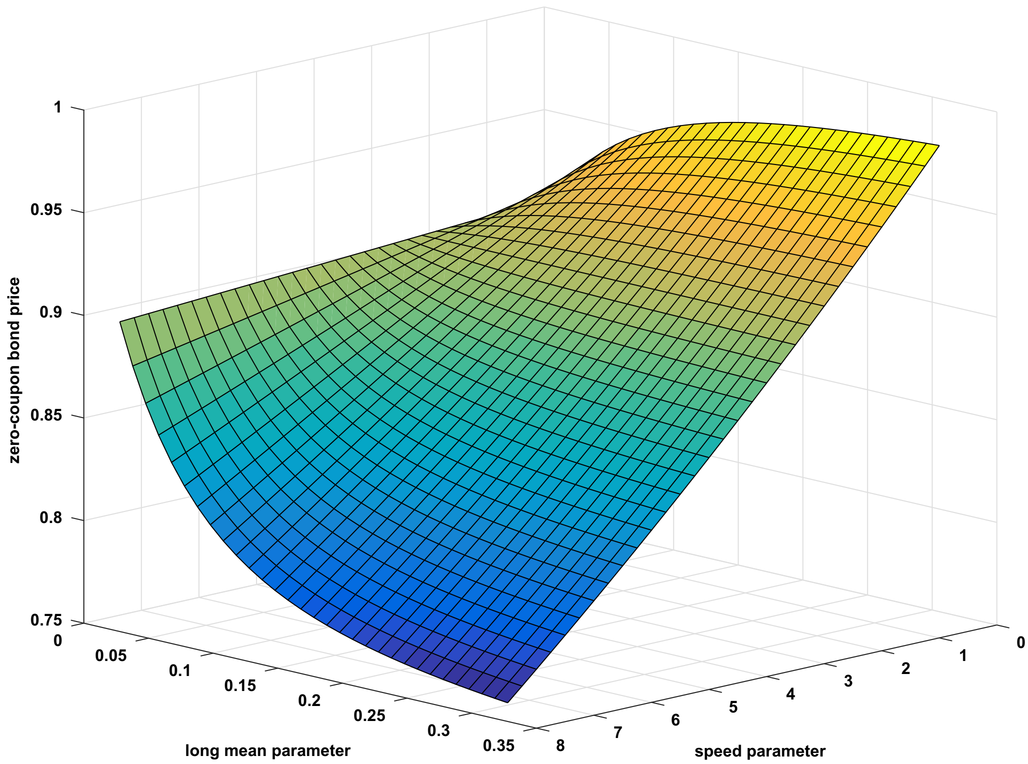

Figure 1. Impact of changing κ and θ parameters on the zero-coupon bond under mixed fractional Vasicek model for parameters $r_0 = 0.1$, $\sigma = 0.1$, $H = 0.8$, $\gamma = 0.3$, $T = 1$ and $\alpha = 0.1$.

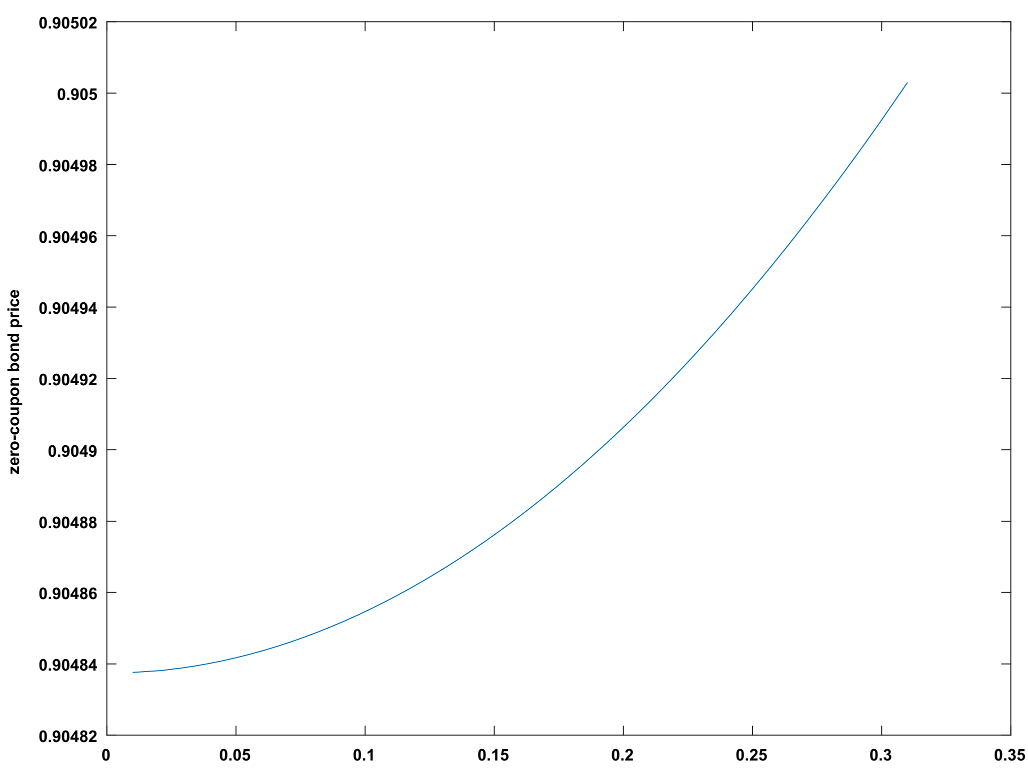

Figure 2. Impact of changing σ parameter on the zero-coupon bond under mixed fractional Vasicek model for parameters $\kappa = 1$, $\theta = 0.1$, $r_0 = 0.1$, $H = 0.8$, $\gamma = 0.3$, $T = 1$ and $\alpha = 0.1$.

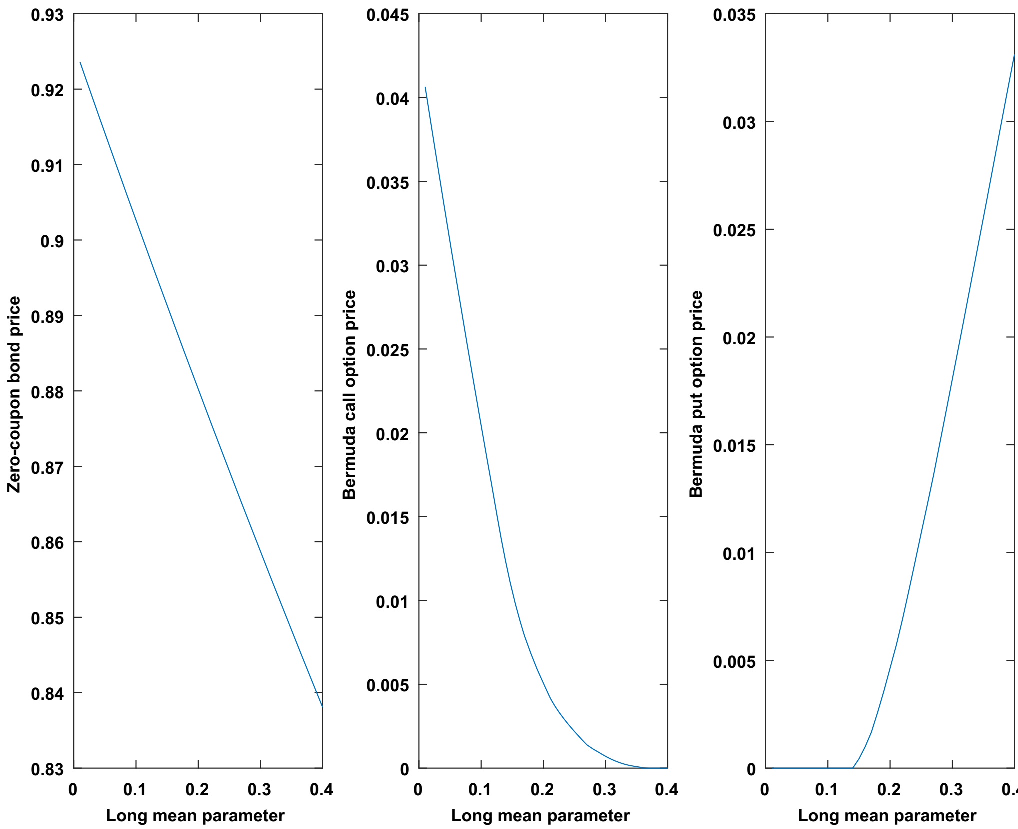

Figure 3. Impact of changing θ parameter on the zero-coupon bond, Bermuda call and put options under mixed fractional Vasicek model for parameters $\kappa = 1$, $r_0 = 0.1$, $\sigma = 0.1$, $\gamma = 0.3$, $T = 1$, $K = 0.85$, $H = 0.8$ and $\alpha = 0.1$.

$$
\begin{aligned}
P_{0,T} =& \mathbb{E}\left[\exp\left(-\int_0^T r_s ds\right)\right] \\
=& \exp\Bigg(-\theta(T - B(T)) - r_0 B(T) \\
&+ \frac{\sigma^2\alpha^2 T}{2\kappa^2} - \frac{\sigma^2\alpha^2 T}{\kappa^2} B(T) \\
&+ \frac{\sigma^2\alpha^2}{4\kappa^3}\left(1 - e^{-2\kappa T}\right) + \xi(T,H)\Bigg).
\end{aligned}
$$

□

The price of the zero-coupon bond has a reverse relationship with the amount of interest rate. Countries with high interest rate have a lower zero-coupon bond price. Equation (3.2) shows that on increasing the value of the long-mean parameter, the zero-coupon price decreases. Furthermore, when the κ parameter increases, the speed of the mean reverting property in the mentioned model increases and the value of the interest rate closes on the value of the long-mean parameter θ. Therefore, the value of the zero-coupon bond closes on $e^{-\theta T}$. The volatility parameter of the interest rate model is represented by σ. By considering Eq. (3.2), one can conclude that the coefficients of the parameter are small. Since in the real interest data the volatility is not large, by changing the parameter, the zero-coupon bond price is not changed significantly. These results are seen in figures 1, 2 and 3.

In table 4, for measuring the efficiency of this formula, the value of the zero-coupon bond is calculated. The results show that the formula is very quick when the number of iterations is $n < 10000$. The proposed formula is fast and accurate for $n > 10000$.

Table 4. Comparison of the mentioned formula and Monte Carlo method regarding the mixed fractional Vasicek model and parameters $r_0 = 0.1$, $T = 1$, $\kappa = 0.5484$, $\theta = 0.2365$, $\sigma = 0.2671$, $H = 0.8170$, $\alpha = 0.1$ and $\gamma = 0.65$.

Time expiration	$n = 1$	$n = 10$	$n = 100$	$n = 1000$	$n = 10000$	$n = 100000$
Mentioned method	$\frac{price=0.8757}{time=0.006934}$	$\frac{price=0.8757}{time=0.006934}$	$\frac{price=0.8757}{time=0.006934}$	$\frac{price=0.8757}{time=0.006934}$	$\frac{price=0.8757}{time=0.006934}$	$\frac{price=0.8757}{time=0.006934}$
Monte Carlo method	$\frac{price=0.8568}{time=0.024509}$	$\frac{price=0.8791}{time=0.039366}$	$\frac{price=0.8773}{time=0.094657}$	$\frac{price=0.8761}{time=0.686305}$	$\frac{price=0.8759}{time=6.517385}$	$\frac{price=0.87598}{time=65.564529}$

Table 5. Impact of changing θ parameter on the zero-coupon bond, call and put Bermuda options price under mixed fractional Vasicek model for parameters $\kappa = 1$, $r_0 = 0.1$, $\sigma = 0.1$, $\gamma = 0.3$, $T = 1$, $H = 0.8$, $K = 0.85$ and $\alpha = 0.1$.

θ	0.01	0.1	0.2	0.3	0.4
Call Bermuda option	0.0406	0.0204	0.0051	0.0007	0
Put Bermuda option	0	0	0.0046	0.0180	0.0331
Zero-coupon bond	0.9236	0.9026	0.8802	0.8587	0.8381

Table 6. Impact of changing κ parameter on the zero-coupon bond, call and put Bermuda options price under mixed fractional Vasicek model for parameters $\theta = 0.2$, $r_0 = 0.1$, $\sigma = 0.1$, $\gamma = 0.3$, $T = 1$, $H = 0.8$, $K = 0.8$ and $\alpha = 0.1$.

κ	0.5	1	2	5	10
Call Bermuda option	0.0743	0.0637	0.0489	0.0189	0.0037
Put Bermuda option	0	0	0.0011	0.0051	0.0129
Zero-coupon bond	0.8840	0.8802	0.8551	0.8216	0.7911

4. Bermuda option pricing

In this section, the Bermuda option on a zero-coupon bond is studied. Options are derivative financial products that allow buying and selling of risks related to future price variations. Derivative financial products imply that they derive their value from another underlying asset, such as a stock or zero-coupon bond. The option gives the buyer the right, but no obligation, to buy or sell the underlying asset at a particular price on or before a specified date in the future. An option to buy an underlying asset is referred to as a call option. An option to sell an underlying asset is referred to as a put option. Bermuda options are a fusion of American and European options. American options are exercisable at any time between the purchase date and the date of expiration. European options are exercised only at the date of expiration. A Bermuda option is an option where the buyer has the right to exercise at set times (always discretely spaced). Later, the payoff of the Bermuda call and put options by maturity time T is as follows:

$$\max_{\tau_1 \in \Lambda} \left(P_{\tau_1,T} - K \right)^+,$$

and

$$\max_{\tau_2 \in \Lambda} \left(K - P_{\tau,T} \right)^+,$$

where $\Lambda = \{0 = t_0 < t_1 < \cdots < t_n \leq T\}$. The Bermuda calls option at the contract inception $t = 0$ given by the discounted risk-neutral expectation of its payoff at expiration $t = T$ is obtained as

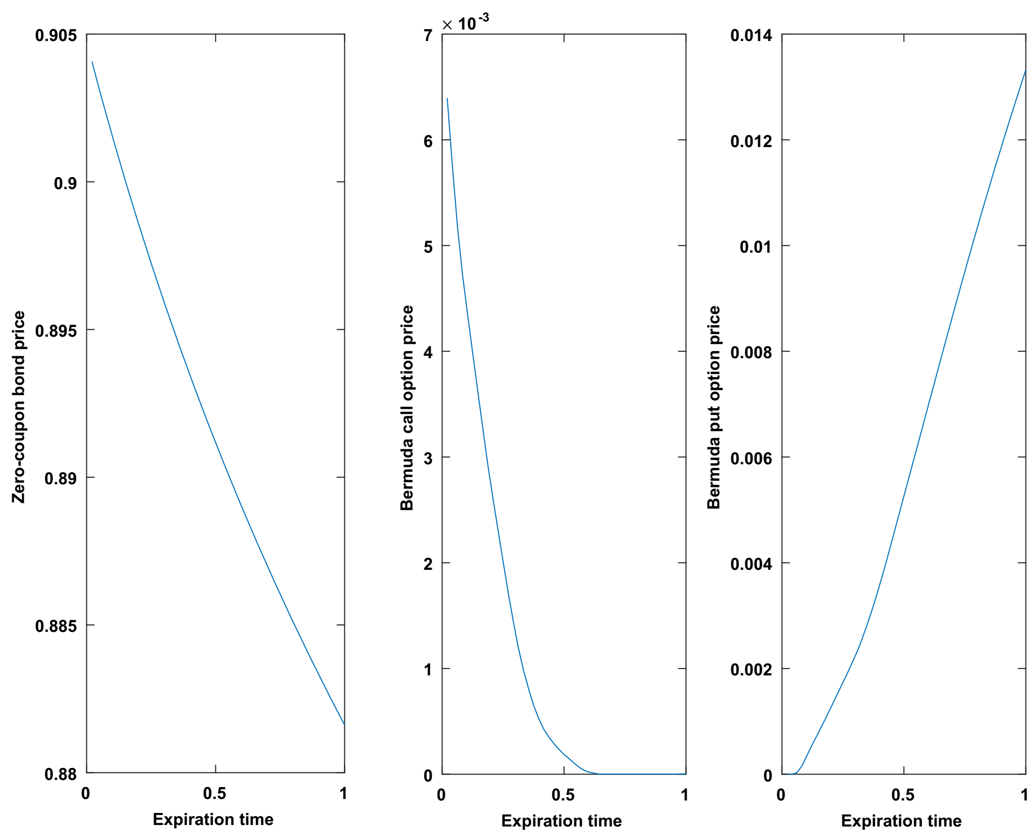

Figure 4. Impact of changing κ parameter on the zero-coupon bond, Bermuda call and put options under mixed fractional Vasicek model for parameters $\theta = 0.2$, $r_0 = 0.1$, $\sigma = 0.1$, $\alpha = 0.3$, $\gamma = 0.3$, $T = 1$, $H = 0.8$, $K = 0.897$ and $\alpha = 0.1$.

Table 7. Impact of changing maturity time parameter on the zero-coupon bond, call and put Bermuda options price under mixed fractional Vasicek model for parameters $\kappa = 1$, $r_0 = 0.1$, $\sigma = 0.1$, $\gamma = 0.3$, $\theta = 0.2$, $H = 0.8$, $K = 0.897$ and $\alpha = 0.1$.

Time	1/12	4/12	7/12	10/12	1
Call Bermuda option	0.0047	0.0010	0	0	0
Put Bermuda option	0.0002	0.0026	0.0067	0.0105	0.0129
Zero-coupon bond	0.9020	0.8951	0.8894	0.8845	0.8802

$$Call(0,T) = \max_{\tau_1 \in \Lambda} \mathbb{E}\left[P_{0,\tau_1} \exp(P(0,\tau_1) - K)^+\right], \tag{4.1}$$

and the Bermuda put option formula can be written as

$$Put(0,T) = \max_{\tau_2 \in \Lambda} \mathbb{E}\left[P_{0,\tau_2} \exp(K - P(0,\tau_2))^+\right]. \tag{4.2}$$

In mathematical finance, the Monte Carlo simulation method is a way for pricing the option, if the option does not have a closed form solution. The Monte Carlo simulation obtains the payoff associated with the data for each simulated path; using the average discounted payoff, it approaches the expected discounted payoff, which is the value of path-dependent option.

5. Numerical results

We now find the value of the option under the proposed model using the Monte Carlo simulation method for different values of the model parameters.

Table 5 and figure 3 present the price of the Bermuda call and put options and zero-coupon bond for various values of the long-mean parameter θ. The results show that on decreasing the long-mean parameter, the prices of the call option and zero-coupon bond increase.

Table 6 and figure 4 present the prices of the Bermuda call and put options and zero-coupon bond for various values of the speed parameter κ. The results show that on increasing the value of the parameter, the value of the zero-coupon bond at time $t = 0$ with time maturity T tends to $e^{-\theta T}$. Thus, the behaviour of the zero-coupon bond price paths strongly depends on the value of the initial interest rate r_0. If $r_0 > \theta$, then on increasing the κ parameter, the value of the zero-coupon bond increases and the converse.

Table 7 and figure 5 present the values of the Bermuda call and put options and zero-coupon bond estimated for

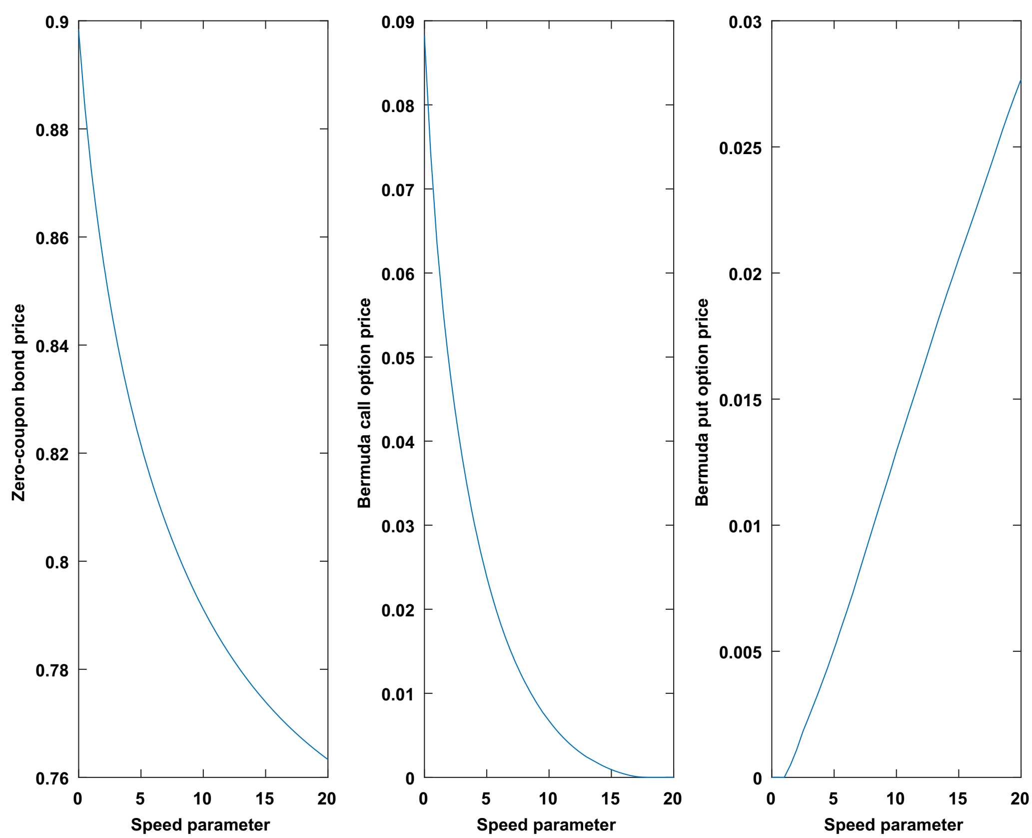

Figure 5. Impact of changing maturity time parameter on the zero-coupon bond, call and put Bermuda options price under mixed fractional Vasicek model for parameters $\kappa = 1$, $r_0 = 0.1$, $\sigma = 0.1$, $\gamma = 0.3$, $\theta = 0.2$, $H = 0.8$, $K = 0.897$ and $\alpha = 0.1$.

different values of the time maturity parameter. From Eq. (3.2), it is obvious that when time to maturity decreases, the value of the zero-coupon bond increases; thus, from Eqs. (4.1) and (4.2), it is obvious that values of the call option increase and put option price decreases.

6. Conclusion

In this paper, an analytic approximation formula for pricing zero-coupon bond is derived when the dynamics of the short-term interest rate is driven by a mixed fractional version of the Vasicek model. The Bermuda call and put option prices on the zero-coupon bond are calculated and the influence of the model parameters on these financial security prices is discussed.

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