Dynamic model of fractional thermoelasticity due to ramp-type heating with two relaxation times

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Abstract. This is an attempt to design a fractional heat conduction model in a bounded cylindrical region exposed to axisymmetric ramp-type heat flux and discuss its thermal behaviour. This model is drafted using the classical theory of fractional thermoelasticity due to the involvement of two relaxation times to the heat conduction equation and the equations of motion. In order to achieve finite thermal wave speed, the heat conduction equation is derived from the viewpoint of Maxwell–Cattaneo law in the context of fractional derivative. Analytical results for the distribution of temperature, displacement and thermal stresses are obtained using integral transforms in the Laplace domain. The Gaver–Stehfest method has been used to invert the results of Laplace domain. The convergence of infinite series solutions has been discussed. As a specific case this model has been applied to a thick circular plate subjected to the axisymmetric ramp-type heat flux. The results obtained for the thermal variations are validated by comparing to coupled and generalized theory of thermoelasticity. These comparisons are shown graphically.

Keywords. Fractional order; thermoelasticity; ramp-type heating; two relaxation times.

1. Introduction

According to the law of heat conduction, heat energy provided to a dynamical system in the absence of physical processes occurring at micro-levels within the heat conductor is linearly dependent on the temperature change of the solid. Since thermal waves cannot transmit at infinite speed, the classical heat conduction law does not hold for short duration micro-level heat conduction problems.

The fractional calculus deals with the theory of integrals and derivatives involving arbitrary order. A differential equation containing fractional-order derivatives is called a fractional differential equation. Thermoelasticity theory based on fractional-order partial differential heat conduction equation is termed as fractional thermoelasticity theory. Fractional derivatives provide an magnificent tool for the interpretation of transmissible characteristics of several types of materials and operations. This is the major merit of fractional derivatives in comparison with classical integer-order models, in which such consequences are actually ignored. It is well known that differential operators of integer order are local operators, whereas fractional ones are non-local. More realistically the non-local properties of fractional derivatives indicate that the next state of a system depends not only upon its current state but also its historical

states. Various heat transfer problems, like modelling of heat exchangers, categorization of various conducting materials to fabricate semiconductors, studying the heating effects of thermoviscoelastic materials and magneto-thermoelastic problems of heat conduction, based on the description of the physical properties in terms of fractional derivatives, lead to boundary value problems involving fractional-order partial differential equations. The theory of fractional thermoelasticity can be effectively employed to solve these problems considering the various associated laws of thermodynamics in the context of non-local properties of fractional derivatives. This must be the reason why many researchers and engineers are exploring new ideas and methodology in this field from the last two decades.

Povstenko [1–3] successfully replaced integer-order time derivatives by Caputo [4] time fractional derivatives in the classical Fourier law of heat conduction and fractional thermoelasticity theory was initiated. Kulkarni and Deshmukh [5] discussed thermal changes in a thick annular disc subjected to axisymmetric heat source in the axial direction. The governing heat conduction equation has been solved using integral transforms and the generalized results for thermal changes are obtained in a series form in terms of Bessel's functions. A few years later, Sherief and co-workers [6, 7] deduced the limiting cases for weak and strong conductivity cases of fractional thermoelasticity theory and applied these limits to a one-dimensional

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thermoelastic fractional heat conduction problem. Youssef and co-workers [8, 9] developed a new thermoelastic model with suitable modifications using fractional theory that supports the uniqueness theorem proposed by Sherief *et al* [6]. The theoretical results were validated through a fractional thermoelastic problem of half-space for ramp-type heating, where variations of change in temperature and thermal stresses were computed numerically and compared to classical and generalized thermoelasticity theories.

Later, Youssef and Elsibai [10] designed a new mathematical model and studied the non-Fourier and coupling effect between the temperature and strain. Following the theory of fractional thermoelasticity proposed by Sherief *et al* [6], Hamza *et al* [11] derived the fractional theory of thermoelasticity for two relaxation times. A uniqueness theorem was proved to validate the proposed results. Hamza *et al* [12] investigated the effect of distribution of thermal stresses in one-dimensional problems of traction-free half-space for various fractional orders and discussed the thermal behaviour in the context of the coupled theory of fractional thermoelasticity associated with two relaxation times. Abbas and Elmaboud [13] compared the optimal homotopy analysis solution to the exact solution of thermoelastic interactions issues in an isotropic medium for a hollow cylinder. The mathematical model was solved by an analytical method and optimal homotopy analysis method (OHAM). Convergence of the obtained homotopy analysis method solution was discussed. Numerical results for the temperature distribution, displacement and radial stress are presented graphically.

Ezzat *et al* [14] designed a model of linear thermo-viscoelasticity for isotropic case by assuming fractional relaxation operators for flowing features of liquids. The formulation is given such that the resulting theorems on coupled and generalized theories of viscoelasticity can be deduced from the governing equations of viscoelasticity. Numerically obtained physical quantities were plotted graphically and compared to approximate outcomes of fractional relaxation operators and some results of thermoelasticity with the ramping parameter of heating. Ezzat *et al* [15] designed a mathematical model of generalized thermoelasticity equations based on memory-dependent derivative (MDD) to obtain some essential theories derived earlier. Exact solutions for different problems in Laplace transform domain were obtained. The numerically inverted results were compared to generalized thermoelasticity, and consequences of time-delay on thermoelasticity for many linear forms of Kernel functions were examined. Ezzat and El-Bary [16] framed a unified mathematical model of the equations of generalized magneto-thermoelasticity based on fractional derivative heat transfer for isotropically perfect conducting media. Mathematical solutions were obtained by applying the method of matrix exponential to the non-dimensional equations. The resulting formulation was applied to a variety of one-dimensional problems, and the results were analysed for some fractional orders. Ezzat

and El-Bary [17] applied the model of linear magneto-thermo-viscoelasticity for isotropic media to a half-space subjected to two different forms of time-dependent thermal shock, in the presence of a transverse magnetic field. The numerically computed results were plotted and compared to coupled and generalized theories for various fractional orders used. Biswas *et al* [18] investigated the thermal shock response to the propagation of electro-magneto-thermoelastic disturbances in an orthotropic medium using a three-phase-lag model of generalized thermoelasticity theory. Changes of displacement, stress and temperature distribution with and without magnetic effect for various thermoelastic models were presented and compared to different models of thermoelasticity. Loftly and Sarkar [19] applied MDD theory and described a unique design of photothermal therapy using two-temperature thermoelasticity theory. The problem was framed and solved mathematically by assuming an isotopically homogeneous semiconductor material subjected to mechanical and photothermal loading. The Laplace transform technique was applied to the governing equations, general solutions of some physical fields were obtained and the numerically inverted solutions were discussed graphically. Using the two-temperature theory, Sarkar and Loftly [20] designed a problem of thermoelastic response to hydrostatic initial stress. Normal mode analysis technique was used to achieve the mathematical solution. The model, in particular, was applied to a thermally insulated surface that was acted upon by a load. The results for thermal parameters were obtained numerically and compared to those from earlier theories with and without application of hydrostatic initial stress.

Kumar *et al* [21] examined the response of time, thermal and diffusion phase-lags due to axisymmetric heating of a ring. The model was designed following dual-phase-lag models of heat transfer and diffusion. The lateral surfaces of the ring are free of mechanical loading and subjected to an axisymmetric heating. Laplace and Hankel transform techniques were applied to obtain analytical results of commonly used thermal parameters, chemical potential, temperature and mass concentration. Numerically inverted results were shown graphically. Special attention was given to the sound impact of diffusion and thermal phase-lags. Kumar *et al* [22] used a dual-phase-lag in association with two-temperature theory using fractional thermoelasticity to investigate the heating effect of a thick plate forced to axisymmetric heating. Numerically computed results for temperature, displacement, cubical dilatation and thermal stresses are depicted graphically, illustrating the influences of thermal phase-lags and two temperatures. Mittal and Kulkarni *et al* [23] examined thermal variations for a thick circular plate using a dual-phase-lag fractional thermoelastic model associated with one relaxation time in finite cylindrical region. Analytically unknown thermal quantities involved were found in Laplace domain. The numerically inverted results were compared to those from classical and

generalized thermoelasticity theories in axial and radial directions.

Theory of fractional thermoelasticity associated with two relaxation times can be frequently used by mechanical engineers to design heat exchangers, to demagnetize ferromagnetic metallic thin films [24] and to synthesize metals and semiconductor nanomaterials [25], boiler tubes, etc., where temperature-induced elastic deformation occurs. The presented theory can also be used by chemical engineers to observe the fundamental process of chemical reaction dynamics such as bond formation and break [10]. Referring to [24–29], it can be found that the theory discussed in the manuscripts can also be applicable in dynamical systems control and hyperbolic theories in biology.

According to a report by Cotterell and Parkes [30], the post-bending response of a circular plate requires examination especially for heating at the centre and at the edge. However, in the first situation, the plate bends into the form of a saucer, and saddle shapes in the latter case. The plate can be mechanically free or it may be restricted to a variety of edge restraints.

Following the report by Cotterell and Parkes [30], the present manuscript is an attempt to design a mathematical model of two-dimensional solid heat conductor in the form of a thick circular plate in bounded cylindrical domain subjected to axisymmetric ramp-type heat flux and investigate the effects of thermal conduction at micro-levels. Following the thermoelasticity theory with two relaxation times proposed by Hamza *et al* [12], the governing equations have been derived. The model designed has been applied to axisymmetric ramp-type heat flux and analytical solutions for the distribution of temperature, displacement and thermal stresses are obtained in the Laplace transform domain by the application of integral transforms. The inversion of Laplace transform is carried out using the Gaver–Stehfest algorithm [31–33]. The convergence of infinite series solutions obtained using the Gaver–Stehfest algorithm has been discussed as per Kuznetsov [34] convergence criterion. As a special case, the mathematical model is applied for copper material and results are presented graphically for various fractional orders and compared to those from classical theories of thermoelasticity.

To our knowledge, no one has discussed and applied the given methodology for a dynamic axisymmetric fractional thermoelastic problem in finite cylindrical domain for a thick circular plate associated with two relaxation times. This is the new and novel contribution to the field.

2. The mathematical model

2.1 Basic assumptions and governing equations

Consider a thick circular plate whose axis of symmetry is z -axis and origin is located at the centre in the middle plane between upper and lower surfaces. Let b and h be the radius

and thickness of the thick circular plate, respectively. Considering axi-symmetry about z -axis, the convective heat flux described by $\Phi(r, t)$, $\forall t > 0$, a continuous integrable function, is applied on the upper and lower plane boundary surfaces ∂D_1 . The convection due to dissipation takes places into surrounding media through the circular boundary surface ∂D_2 . Initially, the thick circular plate is considered at zero temperature. All the boundaries of the thick circular plate are assumed to be traction free. Considering rotational symmetry about the z -axis, all the thermal quantities are assumed to be independent of the coordinate ϕ . Under these circumstances, the thermal behaviour of the thick circular plate is required to be determined.

Following Sherief *et al* [6], in order to achieve finite speed of thermal wave propagation, the Maxwell–Cattaneo law in the context of theory of generalized thermoelasticity based on fractional derivative has been used as

$$q_i + \tau_1 \frac{\partial^\alpha}{\partial t^\alpha} q_i = -\kappa_{ij} \theta_{,j}, \quad 0 < \alpha \leq 1. \quad (1)$$

From the Maxwell–Cattaneo law of heat conduction defined earlier, the governing heat conduction equation with the mixed boundary conditions has the form

$$\kappa \Delta^2 \theta = \rho C_e \left(\frac{\partial}{\partial t} + \frac{\tau_1^\alpha}{\Gamma(\alpha+1)} \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}} \right) \theta + \gamma \theta_0 \frac{\partial e}{\partial t}, \quad \alpha \in (0, 1], \quad (2)$$

where Δ^2 is two-dimensional Laplacian operator in cylindrical coordinate system and θ_0 is reference temperature, assumed to be such that $|\theta - \theta_0| < 1$.

For displacement vector $\vec{u}(r, z, t) = (u_1, 0, u_2)$, the equations of motion can be written as

$$\begin{aligned} \mu \Delta^2 u_1 - \frac{\mu}{r^2} u_1 + (\lambda + \mu) \frac{\partial e}{\partial r} - \gamma \left(1 + \frac{\tau_2^\alpha}{\Gamma(\alpha+1)} \frac{\partial^\alpha}{\partial t^\alpha} \right) \frac{\partial \theta}{\partial r} \\ = \rho \frac{\partial^2 u_1}{\partial t^2}, \end{aligned} \quad (3)$$

$$\begin{aligned} \mu \Delta^2 u_2 + (\lambda + \mu) \frac{\partial e}{\partial z} - \gamma \left(1 + \frac{\tau_2^\alpha}{\Gamma(\alpha+1)} \frac{\partial^\alpha}{\partial t^\alpha} \right) \frac{\partial \theta}{\partial z} \\ = \rho \frac{\partial^2 u_2}{\partial t^2}, \end{aligned} \quad (4)$$

$$e = \frac{u_1}{r} + \frac{\partial u_1}{\partial r} + \frac{\partial u_2}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} (r u_1) + \frac{\partial u_2}{\partial z}. \quad (5)$$

The normal and shear stresses constitutive relations in terms of temperature change and displacement supplement the preceding equations:

$$\sigma_{rr} = 2\mu \frac{\partial u_1}{\partial r} + \lambda e - \gamma \left(1 + \frac{\tau_2^\alpha}{\Gamma(\alpha+1)} \frac{\partial^\alpha}{\partial t^\alpha} \right) (\theta - \theta_0), \quad (6)$$

$$\sigma_{zz} = 2\mu \frac{\partial u_2}{\partial z} + \lambda e - \gamma \left(1 + \frac{\tau_2^\alpha}{\Gamma(\alpha+1)} \frac{\partial^\alpha}{\partial t^\alpha} \right) (\theta - \theta_0), \quad (7)$$

$$\sigma_{rz} = \mu \left(\frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial r} \right). \quad (8)$$

The quantities τ_1 and τ_2 are time relaxation parameters, which satisfy the inequalities $\tau_2 \geq \tau_1 > 0$.

Equations (1)–(8) constitutes the mathematical formulation of the problem in the context of Caputo fractional differentiation.

2.2 Non-dimensional formulation

In order to convert the afore-mentioned mathematical formulation into non-dimensional form, the following dimensionless variables are used:

$$\begin{aligned} r' &= avr, \quad z' = avz, \quad u'_1 = avu_1, \quad u'_2 = avu_2, \quad t' = a^2vt, \\ \tau'_1 &= a^2v\tau_1, \quad \tau'_2 = a^2v\tau_2, \quad \sigma'_{ij} = \frac{\sigma_{ij}}{\mu}, \quad \Theta' = \frac{\gamma(\theta - \theta_0)}{(\lambda + 2\mu)}, \\ \epsilon &= \frac{\gamma^2\theta_0}{\kappa a^2v\rho}, \quad v = \frac{\rho C_e}{\kappa}, \quad a = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \end{aligned}$$

Using these dimensionless variables and neglecting primes, the governing equations (2)–(8) are converted as follows:

$$\begin{aligned} \Delta^2 u_1 - \frac{u_1}{r^2} + (\eta^2 - 1) \frac{\partial e}{\partial r} \\ - \eta^2 \left(1 + \frac{\tau_2^\alpha}{\Gamma(\alpha+1)} \frac{\partial^\alpha}{\partial t^\alpha} \right) \frac{\partial \Theta}{\partial r} = \eta^2 \frac{\partial^2 u_1}{\partial t^2}, \end{aligned} \quad (9)$$

$$\begin{aligned} \Delta^2 u_2 + (\eta^2 - 1) \frac{\partial e}{\partial z} \\ - \eta^2 \left(1 + \frac{\tau_2^\alpha}{\Gamma(\alpha+1)} \frac{\partial^\alpha}{\partial t^\alpha} \right) \frac{\partial \Theta}{\partial z} = \eta^2 \frac{\partial^2 u_2}{\partial t^2}, \end{aligned} \quad (10)$$

$$\Delta^2 \Theta = \left(\frac{\partial}{\partial t} + \frac{\tau_1^\alpha}{\Gamma(\alpha+1)} \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}} \right) \Theta + \epsilon \dot{e}, \quad (11)$$

$$\sigma_{rr} = 2 \frac{\partial u_1}{\partial r} + (\eta^2 - 2)e - \eta^2 \left(1 + \frac{\tau_2^\alpha}{\Gamma(\alpha+1)} \frac{\partial^\alpha}{\partial t^\alpha} \right) \Theta, \quad (12)$$

$$\sigma_{zz} = 2 \frac{\partial u_2}{\partial z} + (\eta^2 - 2)e - \eta^2 \left(1 + \frac{\tau_2^\alpha}{\Gamma(\alpha+1)} \frac{\partial^\alpha}{\partial t^\alpha} \right) \Theta, \quad (13)$$

$$\sigma_{rz} = \left(\frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial r} \right), \quad (14)$$

where $\eta^2 = \frac{\lambda + 2\mu}{\mu}$.

Combining Eqs. (9) and (10), and making use of Eq. (5), one obtains the following equation:

$$\Delta^2 e - \left(1 + \frac{\tau_2^\alpha}{\Gamma(\alpha+1)} \frac{\partial^\alpha}{\partial t^\alpha} \right) \Delta^2 \Theta = \frac{\partial^2 e}{\partial t^2}. \quad (15)$$

Equations (9)–(15) constitute the dimensionless version of governing equations.

2.3 Special cases

The governing dimensionless heat conduction equation (11) can be categorized for various thermoelastic theories as follows:

Case-01: $\alpha = 0$

$$\Delta^2 \Theta = \frac{\partial}{\partial t} (2\Theta + \epsilon e). \quad (16)$$

Equation (16) is a parabolic equation, which is the governing heat conduction equation of classical theory of thermoelasticity.

Case-02: $\alpha = 1; \tau_1 > 0, \tau_2 \geq \tau_1 > 0$

$$\Delta^2 \Theta = \frac{\partial \Theta}{\partial t} + \tau_1 \frac{\partial^2 \Theta}{\partial t^2} + \epsilon \frac{\partial e}{\partial t}. \quad (17)$$

Equation (17) is the governing heat conduction equation of generalized theory of thermoelasticity.

Case-03: $\alpha \in (0, 1); \tau_1 > 0, \tau_2 \geq \tau_1 > 0$.

Following Povstenko [35], the governing heat conduction equation (11) is identified as fractional generalization of Cattaneo approach.

Following Ignaczak and Ostoj-Starzewski [36], one may note, from case 2, that the governing heat conduction equation (17) becomes hyperbolic as and when fractional-order $\alpha \rightarrow 1$. Therefore one can always attain the goal of finite speed of thermal wave propagation.

2.4 Initial and boundary conditions

Considering axisymmetry about z -axis, the convective heat flux function Φ of space variable r and time t is applied to boundaries ∂D_1 as follows:

$$\pm \kappa \frac{\partial \Theta}{\partial z} + h_1 \Theta = \Phi(r, t) \quad \text{for } z = \pm \frac{h}{2}. \quad (18)$$

The convection due to dissipation takes places into surrounding media through the circular boundary surface ∂D_2 described by

$$\kappa \frac{\partial \Theta}{\partial r} + h_2 \Theta = 0 \quad \text{for } r = b, \quad (19)$$

where h_1 and h_2 are heat transfer coefficient of the plane and circular boundary surface, respectively.

Assuming there is no mechanical loading at the boundaries, mathematically

$$\sigma_{zz} = 0; \sigma_{rz} = 0, \quad z = \pm \frac{h}{2}, \quad (20)$$

$$\sigma_{rr} = 0; \sigma_{rz} = 0, \quad r = b. \quad (21)$$

Assuming quiescent state, the initial conditions for the model are given as

$$\Theta(r, z, 0) = \dot{\Theta}(r, z, 0) = \vec{u}(r, z, 0) = \vec{\dot{u}}(r, z, 0) = 0. \quad (22)$$

Equations (18)–(22) show initial and boundary conditions for thick circular plate.

3. The solution

Analytical solutions of the governing dimensionless equations have been achieved by the application of integral transforms. The Laplace transform is applied with respect to time variable t and Hankel transforms is used with respect to space variable r . The inversions of the equations in the Laplace domain has been achieved numerically using the Gaver–Stehfest algorithm.

3.1 Transformation in the Laplace domain

Following Povstenko [3], let the function $f(t)$ and its successive $(n-1)^{\text{th}}$ -order derivatives be continuous for $0 \leq t < \infty$ and of exponential order. Moreover the α^{th} -order Caputo fractional derivative of a function $f(t)$ is piecewise continuous on $[0, \infty)$. Then its Laplace transform is denoted and defined as

$$\mathcal{L}\{D_C^\alpha f(t)\} = s^\alpha f^*(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad (23)$$

where $(n-1) < \alpha < n$ and $\mathcal{L}\{f(t)\} = f^*(s)$. Moreover, the Laplace transform of fractional integral is defined as

$$\mathcal{L}\{I^\alpha f(t)\} = \frac{1}{s^\alpha} \bar{f}(s). \quad (24)$$

Assume that the function $f(x, t)$ and all of its corresponding time derivatives are zero for $t = 0$. Using Eq. (23) in view of zero initial condition, the governing equations (9)–(15) are transformed in Laplace domain to

$$\Delta^2 \bar{u}_1 - \frac{\bar{u}_1}{r^2} + (\eta^2 - 1) \frac{\partial \bar{e}}{\partial r} - \eta^2 \left(1 + \frac{\tau_2^\alpha s^\alpha}{\Gamma(\alpha + 1)}\right) \frac{\partial \bar{\Theta}}{\partial r} = \eta^2 s^2 \bar{u}_1, \quad (25)$$

$$\Delta^2 \bar{u}_2 + (\eta^2 - 1) \frac{\partial \bar{e}}{\partial z} - \eta^2 \left(1 + \frac{\tau_2^\alpha s^\alpha}{\Gamma(\alpha + 1)}\right) \frac{\partial \bar{\Theta}}{\partial z} = \eta^2 s^2 \bar{u}_2, \quad (26)$$

$$\left(\Delta^2 - s - \frac{\tau_1^\alpha s^{\alpha+1}}{\Gamma(\alpha + 1)}\right) \bar{\Theta} = \epsilon s \bar{e}, \quad (27)$$

$$(\Delta^2 - s^2) \bar{e} = \left(1 + \frac{\tau_2^\alpha s^\alpha}{\Gamma(\alpha + 1)}\right) \Delta^2 \bar{\Theta}, \quad (28)$$

$$\bar{\sigma}_{rr} = 2 \frac{\partial \bar{u}_1}{\partial r} + (\eta^2 - 2) \bar{e} - \eta^2 \left(1 + \frac{\tau_2^\alpha s^\alpha}{\Gamma(\alpha + 1)}\right) \bar{\Theta}, \quad (29)$$

$$\bar{\sigma}_{zz} = 2 \frac{\partial \bar{u}_2}{\partial z} + (\eta^2 - 2) \bar{e} - \eta^2 \left(1 + \frac{\tau_2^\alpha s^\alpha}{\Gamma(\alpha + 1)}\right) \bar{\Theta}, \quad (30)$$

$$\bar{\sigma}_{rz} = \left(\frac{\partial \bar{u}_1}{\partial z} + \frac{\partial \bar{u}_2}{\partial r}\right). \quad (31)$$

Equations (18) and (19) are transformed in Laplace domain to

$$\pm k \frac{\partial \bar{\Theta}}{\partial z} + h_1 \bar{\Theta} = \bar{\Phi}(r, s), \quad (32)$$

$$k \frac{\partial \bar{\Theta}}{\partial r} + h_2 \bar{\Theta} = 0, \quad (33)$$

$$\bar{\sigma}_{zz} = \bar{\sigma}_{rz} = \bar{\sigma}_{rz} = \bar{\sigma}_{rr} = 0, \quad (34)$$

where $\bar{\Phi}(r, s)$ is transformed function in Laplace domain.

Eliminating $\bar{e}$ in Eqs. (27) and (28), one gets the following equation in $\bar{\Theta}$:

$$\left\{ \Delta^4 - \Delta^2 \left[s^2 + s \left(1 + \frac{\tau_1^\alpha s^\alpha}{\Gamma(\alpha + 1)}\right) + \epsilon s \left(1 + \frac{\tau_1^\alpha s^\alpha}{\Gamma(\alpha + 1)}\right) \left(1 + \frac{\tau_2^\alpha s^\alpha}{\Gamma(\alpha + 1)}\right) \right] + s^3 \left(1 + \frac{\tau_1^\alpha s^\alpha}{\Gamma(\alpha + 1)}\right) \right\} \bar{\Theta} = 0. \quad (35)$$

Equation (35) can be expressed in the following manner:

$$(\Delta^2 - l_1^2)(\Delta^2 - l_2^2) \bar{\Theta} = 0, \quad (36)$$

where l_1^2, l_2^2 are the positive roots of real parts of the following characteristic equation:

$$l^4 - l^2 \left[s^2 + s \left(1 + \frac{\tau_1^\alpha s^\alpha}{\Gamma(\alpha + 1)}\right) + \epsilon s \left(1 + \frac{\tau_1^\alpha s^\alpha}{\Gamma(\alpha + 1)}\right) \left(1 + \frac{\tau_2^\alpha s^\alpha}{\Gamma(\alpha + 1)}\right) \right] + s^3 \left(1 + \frac{\tau_1^\alpha s^\alpha}{\Gamma(\alpha + 1)}\right) = 0. \quad (37)$$

Solving Eq. (36), one obtains

$$\bar{\Theta} = \bar{\Theta}_1 + \bar{\Theta}_2, \quad (38)$$

where $\bar{\Theta}_i$ is a solution of the following homogeneous equation:

$$(\Delta^2 - l_i^2)\bar{\Theta}_i = 0, \quad i = 1, 2. \quad (39)$$

Equations (25)–(39) mathematically describe the problem in Laplace domain.

3.2 Transformation in the Hankel domain

Following Özisik [37], the finite Hankel integral transform $\psi^*(\zeta_n, z, t)$ and corresponding inverse transform of a function $\psi(r, z, t)$ are given by

$$\psi^*(\zeta_n, z, t) = \int_0^b r R_0(\zeta_n, r) \bar{\psi}(r, z, t) dr, \quad (40)$$

$$\psi(r, z, t) = \sum_{n=1}^{\infty} R_0(\zeta_n, r) \psi^*(\zeta_n, z, t). \quad (41)$$

Here, the sum is defined over the eigenvalues, $\zeta_n > 0$, which are positive roots of the transcendental equation

$$\zeta_n J'_0(\zeta_n b) + \frac{h_2}{\kappa} J_0(\zeta_n b) = 0, \quad n \in N. \quad (42)$$

$R_0(\zeta_n, r)$ denotes the kernel of Hankel transform given by

$$R_0(\zeta_n, r) = \frac{\sqrt{2\kappa\zeta_n} J_0(\zeta_n r)}{b J_0(\zeta_n b) \sqrt{[h_2^2 + \kappa^2 \zeta_n^2]}}. \quad (43)$$

3.3 Results in Laplace domain

Applying Hankel integral transform and the corresponding inverse transforms to the dimensionless equations (25)–(31), in addition to (39), one obtains the analytical solutions of thermal parameters in Laplace transforms as follows:

$$\bar{\Theta} = \sum_{n=1}^{\infty} \sum_{i=1}^{i=2} R_0(\zeta_n, r) C_i(\zeta_n, s) (l_i^2 - s^2) \cosh m_i z, \quad (44)$$

$$\bar{e} = \sum_{n=1}^{\infty} \sum_{i=1}^{i=2} R_0(\zeta_n, r) C_i(\zeta_n, s) l_i^2 \left(1 + \frac{\tau_2^\alpha s^\alpha}{\Gamma(\alpha + 1)} \right) \cosh m_i z,$$

$$\text{where } m_i = \sqrt{\zeta_n^2 + l_i^2}, \quad i = 1, 2,$$

(45)

$$\begin{aligned} \bar{u}_1 = & \sum_{n=1}^{\infty} \left[\sum_{i=1}^{i=2} R_0(\zeta_n, r) C_i(\zeta_n, s) \right. \\ & \left. \cosh m_i z \left(1 + \frac{\tau_2^\alpha s^\alpha}{\Gamma(\alpha + 1)} \right) \right. \\ & \left. \left[\frac{2l_i^2 m_i^2 + l_i^2 m_3^2 - \eta^2 s^2 m_i^2}{(m_i^2 + m_3^2)} \right] - D(\zeta_n, s) m_3 \cosh m_3 z \right], \end{aligned} \quad (46)$$

$$\begin{aligned} \bar{u}_2 = & \sum_{n=1}^{\infty} \left[\sum_{i=1}^{i=2} R_0(\zeta_n, r) C_i(\zeta_n, s) m_i \sinh m_i z \right. \\ & \left. \left\{ \frac{\left(1 + \frac{\tau_2^\alpha s^\alpha}{\Gamma(\alpha + 1)} \right) [\eta^2 s^2 - l_i^2]}{(m_i^2 + m_3^2)} \right\} + D(\zeta_n, s) \sinh m_3 z \right], \\ & m_3 = \sqrt{\zeta_n^2 + \eta^2 s^2}, \end{aligned} \quad (47)$$

$$\begin{aligned} \bar{\sigma}_{rr} = & \sum_{n=1}^{\infty} \sum_{i=1}^{i=2} R_0(\zeta_n, r) C_i(\zeta_n, s) \left(1 + \frac{\tau_2^\alpha s^\alpha}{\Gamma(\alpha + 1)} \right) \\ & (\eta^2 s^2 - 2l_i^2) \cosh m_i z + \sum_{n=1}^{\infty} \sum_{i=1}^{i=2} R_1(\zeta_n, r) C_i(\zeta_n, s) \\ & \cosh m_i z \left(1 + \frac{\tau_2^\alpha s^\alpha}{\Gamma(\alpha + 1)} \right) \left[\frac{2l_i^2 m_i^2 + l_i^2 m_3^2 - \eta^2 s^2 m_i^2}{(m_i^2 + m_3^2)} \right] \\ & - 2 \sum_{n=1}^{\infty} R_1(\zeta_n, r) D(\zeta_n, s) m_3 \cosh m_3 z, \\ & R_1(\zeta_n, r) = \frac{d}{dr} R_0(\zeta_n, r), \end{aligned} \quad (48)$$

$$\begin{aligned} \bar{\sigma}_{zz} = & \sum_{n=1}^{\infty} \sum_{i=1}^{i=2} R_0(\zeta_n, r) C_i(\zeta_n, s) \cosh m_i z \\ & \left[\frac{3m_i^2 \eta^2 s^2 - 4m_i^2 l_i^2 + m_3^2 (\eta^2 s^2 - 2l_i^2)}{(m_i^2 + m_3^2)} \right] \\ & + 2 \sum_{n=1}^{\infty} R_0(\zeta_n, r) m_3 D(\zeta_n, s) \cosh m_3 z, \end{aligned} \quad (49)$$

$$\begin{aligned} \bar{\sigma}_{rz} = & \sum_{n=1}^{\infty} \sum_{i=1}^{i=2} R_1(\zeta_n, r) C_i(\zeta_n, s) \\ & \left(1 + \frac{\tau_2^\alpha s^\alpha}{\Gamma(\alpha + 1)} \right) \frac{[\eta^2 s^2 - l_i^2]}{(m_i^2 + m_3^2)} m_i \sinh m_i z \\ & + \sum_{n=1}^{\infty} \sum_{i=1}^{i=2} R_0(\zeta_n, r) C_i(\zeta_n, s) m_i \sinh m_i z \\ & \left(1 + \frac{\tau_2^\alpha s^\alpha}{\Gamma(\alpha + 1)} \right) \left[\frac{2l_i^2 m_i^2 + l_i^2 m_3^2 - \eta^2 s^2 m_i^2}{(m_i^2 + m_3^2)} \right] \\ & + \sum_{n=1}^{\infty} R_1(\zeta_n, r) D(\zeta_n, s) \sinh m_3 z. \end{aligned} \quad (50)$$

Applying the finite Hankel transform to the boundary conditions (32) and (33), one obtains the following simultaneous equations in unknown constants $C_1(\zeta_n, s)$, $C_2(\zeta_n, s)$ and $D(\zeta_n, s)$ as follows:

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{i=1}^{i=2} \zeta_n J_1(\zeta_n b) C_i(\zeta_n, s) m_i \sinh\left(m_i \frac{h}{2}\right) \\
& \frac{\left(1 + \frac{\tau_2^{\alpha} s^{\alpha}}{\Gamma(\alpha+1)}\right) [\eta^2 s^2 - l_i^2]}{(m_i^2 + m_3^2)} \\
& - \sum_{n=1}^{\infty} \sum_{i=1}^{i=2} J_0(\zeta_n b) C_i(\zeta_n, s) m_i \left(1 + \frac{\tau_2^{\alpha} s^{\alpha}}{\Gamma(\alpha+1)}\right) \quad (51) \\
& \left[\frac{2l_i^2 m_i^2 + l_i^2 m_3^2 - \eta^2 s^2 m_i^2}{(m_i^2 + m_3^2)} \right] \sinh\left(m_i \frac{h}{2}\right) \\
& + \sum_{n=1}^{\infty} D(\zeta_n, s) [\zeta_n J_1(\zeta_n b) + J_0(\zeta_n b) m_3^2] \\
& \sinh\left(m_3 \frac{h}{2}\right) = 0.
\end{aligned}$$

$$\begin{aligned}
& \sum_{i=1}^{i=2} \left[\frac{3m_i^2 \eta^2 s^2 - 4m_i^2 l_i^2 + m_3^2 (\eta^2 s^2 - 2l_i^2)}{(m_i^2 + m_3^2)} \right] \\
& \left(1 + \frac{\tau_2^{\alpha} s^{\alpha}}{\Gamma(\alpha+1)}\right) C_i(\zeta_n, s) \quad (52) \\
& \cosh\left(m_i \frac{h}{2}\right) + 2m_3 D(\zeta_n, s) \cosh\left(m_3 \frac{h}{2}\right) = 0, \\
& \sum_{i=1}^{i=2} C_i(\zeta_n, s) (l_i^2 - s^2) [\kappa m_i \sinh m_i \left(\frac{h}{2}\right) \\
& + h_1 \cosh m_i \left(\frac{h}{2}\right)] = \Phi^*(\zeta_n, s). \quad (53)
\end{aligned}$$

On solving Eqs. (51)–(53) simultaneously, one obtains the unknown constants $C_1(\zeta_n, s)$, $C_2(\zeta_n, s)$ and $D(\zeta_n, s)$.

3.4 Inversion of Laplace transforms

Apply the Gaver–Stehfest algorithm [31–33] to approximate the time domain solution using the following equations:

$$f(t) \approx f_N(t) = \frac{\ln(2)}{t} \sum_{k=1}^{2N} \left[\Omega_k \bar{f}\left(\frac{k \ln(2)}{t}\right) \right], \quad (54)$$

$$\Omega_k = (-1)^{N+k} \left[\sum_{p=\left\lfloor \frac{k+1}{2} \right\rfloor}^{p=\min(k, N)} \frac{p^{N+1}}{N!} \binom{N}{p} \binom{2p}{p} \binom{p}{k-p} \right], \quad (55)$$

where $\lfloor y \rfloor = y - \{y\}$, $\forall y \in R$, is the flooring function and $2N$ is an even integer whose value depends on the word length of the computer used.

3.5 Kuznetsov criterion for the convergence of Gaver–Stehfest algorithm

Referring to Kuznetsov [34], since the sequence $f_N(t) \approx f(t)$ defined by Eqs. (54) and (55), where $f: R^+ \rightarrow R$ is a continuously integrable and exponentially ordered function of t whose corresponding Laplace transform $\bar{f}(s)$ exists $\forall s > 0$, the sequence $f_N(t)$ is absolutely convergent in the neighbourhood of t . Also, the bounded variations of $f(t)$ show the convergence of infinite series solutions.

4. Numerical scheme

4.1 Illustrative example

As a special case, a thick circular plate is exposed to ramp-type axisymmetric heat flux defined for $r_b \in (0, r)$ and $z = \pm(h/2)$, expressed in the form of Heaviside unit step function as follows:

$$\Phi(r, t) = \begin{cases} \frac{\Theta_1}{t_0} t [H(r) - H(r - r_b)] & t \in (0, t_0) \\ \Theta_1 [H(r) - H(r - r_b)] & t \in (t_0, \infty) \end{cases}, \quad (56)$$

$$\bar{\Phi}^*(\zeta_n, s) = \frac{\Theta_1 r_b J_1(\zeta_n r_b) e^{-st_0}}{s^2 t_0 \zeta_n} [1 - e^{-st_0}], \quad (57)$$

where Θ_1 is the strength of ramp-type heat flux.

4.2 Dimensions used for numerical calculations

Radius of the plate: $b = 1.0$ m.

Radius of inner annular region of the plate: $r_b = 0.5$ m.

Thickness of the plate: $h = 0.3$ m.

4.3 Material properties

Following Hamza *et al* [12], the numerical scheme has been applied to the copper material with the following physical properties:

$$\begin{aligned}
& \alpha_t = 1.78 \times 10^{-5} \text{ K}^{-1}, \quad \eta^2 = 4, \quad \epsilon = 0.0168 \text{ Nm J}^{-1}, \\
& \Theta_0 = 293 \text{ K}, \quad \Theta_1 = 373.15 \text{ K}, \quad \kappa = 386 \text{ J K}^{-1} \text{ m}^{-1} \text{ s}^{-1}, \\
& h_1 = h_2 = k = 10 \text{ W m}^{-2} \text{ K}^{-1} \text{ J K}^{-1} \text{ s}^{-1}, \\
& \lambda = 7.76 \times 10^{10} \text{ Nm}^{-2}, \quad \mu = 3.86 \times 10^{10} \text{ Nm}^{-2}, \\
& \rho = 8954 \text{ kg m}^{-3}, \quad \tau_1 = 0.25 \text{ ps}, \quad \tau_2 = 0.31 \text{ ps}, \\
& C_e = 383.1 \text{ J kg}^{-1} \text{ K}^{-1}, \quad a = 4.158 \times 10^3 \text{ ms}^{-1}.
\end{aligned}$$

The first ten positive roots of the transcendental equation $\zeta_n J_0'(\zeta_n b) + \frac{h_2}{\kappa} J_0(\zeta_n b) = 0$ are given as

Figure 1. Distribution of temperature $\Theta(r, t)$ in radial direction.

Figure 4. Distribution of displacement $u(r, t)$ in axial direction.

Figure 2. Distribution of temperature $\Theta(r, t)$ in axial direction.

Figure 5. Distribution of radial stress $\sigma_{rr}(r, t)$ in radial direction.

Figure 3. Distribution of displacement $u(r, t)$ in radial direction.

Figure 6. Distribution of radial stress $\sigma_{rr}(r, t)$ in axial direction.

$$\zeta_n = [0.2278, 3.8389, 7.0189, 10.1760, 13.3253, 16.4720, 19.6169, 22.7609, 25.9049, 29.0468].$$

To observe the numerical variations of dimensionless values of temperature change Θ , displacement u , radial stress σ_{rr} , axial stress σ_{zz} and shear stress σ_{rz} in radial direction at

$z = 0.0$ m and for $0 \leq r \leq 1$ and in axial direction at $r = 0.5$ m and for $z \in [-0.15, 0.15]$, the numerical calculations are done for $t = 0.10$ s and $t_0 = 0.02$ s.

Figure 7. Distribution of axial stress $\sigma_{zz}(r, t)$ in radial direction.

Figure 8. Distribution of axial stress $\sigma_{zz}(r, t)$ in axial direction.

Figure 9. Distribution of shear stress $\sigma_{rz}(r, t)$ in radial direction.

5. Conclusion

The work presented in this article is an attempt to design a fractional-order heat conduction problem of a thick circular plate subjected to ramp-type heat flux, following the theory

Figure 10. Distribution of shear stress $\sigma_{rz}(r, t)$ in axial direction.

of fractional thermoelasticity associated with two relaxation times. The results of the distribution of temperature Θ , displacement u and thermal stresses σ_{rr} , σ_{zz} , σ_{rz} are computed numerically and plotted in radial and axial directions. These plots show mixed patterns and every thermal parameter attains its maximum and minimum values within the specified dimensions in radial and axial sense.

This work establishes the fact that in general thermal waves propagate with finite speed, by introducing relaxation times $\tau_1 > 0$, $\tau_2 > 0$ to the proposed governing equations of heat conduction and motion proposed in this model.

Figures 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 show the temperature, displacement and stresses for $0 \leq r \leq 1$ radially and for $-0.15 \leq z \leq 0.15$ axially at small value of time $t = 0.1$ s and fixed ramping time $t_0 = 0.02$ s for various fractional-order $\alpha \in [0, 1]$ that cover the cases of weak $\alpha \in (0, 1)$ and normal conductivity $\alpha = 1.0$.

It is found that increased value of fractional-order α causes increased speed of wave propagation, which vanishes more rapidly for increasing values of variables r and z .

Following Hamza *et al* [12], the heat conduction equation (2) becomes hyperbolic due to the presence of hyperbolic correction term $\left\{ \frac{\tau_1^\alpha}{\Gamma(\alpha + 1)} \right\}$. This results in a slower

heat propagation speed equal to
$$\sqrt{\frac{\text{thermal diffusivity}}{\frac{\tau_1^\alpha}{\Gamma(\alpha + 1)}}},$$

which represents more heat energy accumulated near the boundary surfaces.

Results depicted in plots 1 – 10 exhibit the fact that each thermal parameter Θ , u , σ_{rr} , σ_{zz} , σ_{rz} fulfills the initial and boundary conditions established for all the fractional orders considered in the heat conduction model. It has been observed that as $\alpha \rightarrow 0$ the fractional-order thermoelasticity theory approaches the classical theory and as and when

$\alpha \rightarrow 1$ the fractional theory of thermoelasticity tends to the generalized LS-theory [38]. Numerically computed results also satisfy the equations of equilibrium and compatibility of thermoelasticity theory.

The presented theory is limited to only those mathematical models that satisfy the proposed initial and boundary conditions. As per fractional theory of thermoelasticity proposed by Sherief *et al* [6], this fractional theoretical mathematical model can be applied only when the absolute ratio of the difference of change in temperature to the initial one is much smaller than unity, that is $|\frac{\theta - \theta_0}{\theta_0}| \ll 1$.

Basically, the heat conduction models of thick circular plates require examining the thermal behaviour when the plate comes in contact with an internal or external heat source. The mathematical model presented in this article examines the thermal properties of a thick circular plate subjected to an axisymmetric ramp-type heat flux. The finite speed of thermal waves has been achieved due to the application of one relaxation time $\tau_1 > 0$ in association with Caputo time fractional derivative of order $\alpha \in (0, 1)$ to the classical heat conduction equation. The presented model can be investigated further for dual- and triple-phase-lag heat conduction problems related to science and engineering. Referring to Cotterell and Perks [30], realistically the presented heat conduction model can be applied to solve aerospace engineering problems.

The results found in this article reproduce fractional-order generalized thermoelasticity. Scientifically, one can conclude that the model presented in this manuscript can be closely related to recent aerodynamics engineering problems related to thick circular plates of various materials. It appears that fractional-order α can serve as a new indicator to classify various materials as per their ability of heat conduction.

Appendix I.

Laplace transform is an appreciably well known mathematical tool to solve many real world engineering problems mathematically. Moreover these realistic problems are converted to various boundary value problems involving partial differential equations in time and space domain, which are simply reduced to corresponding ordinary differential equations using Laplace transforms. Naturally, to arrive at the final solution of these problems, one has to achieve the inverse Laplace transform of the involved parameters. Due to the nonlinear nature of transformed equations in Laplace domain, analytical inversions are unavailable; rather, it is good to approach numerical inversion. Authors applied the Gaver–Stehfest algorithm described in sections 3.4 and 3.5, in preference over so many available methods, due to computational simplicity. One can alternatively use the Post–Widder method to

achieve the inversion; a brief description of the Post–Widder method is described here.

Post–Widder method

Following Cohen [39], if the integral

$$\bar{f}(s) = \int_0^\infty e^{-su} f(u) du \text{ converges } \forall s > \gamma, \text{ then}$$

$$f(t) = \lim_{n \rightarrow \infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \bar{f}^{(n)}\left(\frac{n}{t}\right)$$

$\forall t > 0$ of continuity of $f(t)$.

To compare the iteration schemes, authors reiterated the inversions of the results shown in Eqs. (44)–(50) for a particular value of time $t = 0.1$ s for $b = 1.5$ m, $r_b = 0.8$ m, $h = 0.6$ m. Results obtained by PostWidder algorithm are illustrated by figures 11, 12, 13, 14, 15, 16, 17, 18, 19 and 20 given below:

Moreover, it is observed that the numerically inverted dimensionless values of temperature, displacement and thermal stresses obtained by the Gaver–Stehfest algorithm and Post–Widder method for two sets of dimensions are different in terms of magnitude; however, they are quite similar graphically except for a few parameters.

Figure 11. Distribution of temperature $\Theta(r, t)$ in radial direction.

Figure 12. Distribution of temperature $\Theta(r, t)$ in axial direction.

Figure 13. Distribution of displacement $u(r, t)$ in radial direction.

Figure 16. Distribution of radial stress $\sigma_{rr}(r, t)$ in axial direction.

Figure 14. Distribution of displacement $u(z, t)$ in axial direction.

Figure 17. Distribution of axial stress $\sigma_{zz}(r, t)$ in radial direction.

Figure 15. Distribution of radial stress $\sigma_{rr}(r, t)$ in radial direction.

Figure 18. Distribution of axial stress $\sigma_{zz}(z, t)$ in axial direction.

List of symbols

α	fractional order
α_t	coefficient of linear thermal expansion
γ	material constant
ν	reciprocal of thermal diffusivity
ϵ	dimensionless coupling constant
κ	thermal conductivity
λ, μ	Lamé constants
ρ	density

$\sigma_{rr}, \sigma_{zz}, \sigma_{rz}$	thermal stresses
τ_1, τ_2	relaxation times
C_e	specific heat capacity
θ	absolute temperature
Θ	dimensionless absolute temperature
θ_0	reference temperature
a	speed of iso-thermal elastic wave
e	cubical dilatation
t_0	ramping time parameter
q_i	heat flux vector

Figure 19. Distribution of shear stress $\sigma_{rz}(r, t)$ in radial direction.

Figure 20. Distribution of shear stress $\sigma_{rz}(r, t)$ in axial direction.

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