Terminal adaptive control for a class of uncertain underactuated systems

A KULKARNI^{1,*} and A KUMAR²

¹Department of Electronics Engineering, Medicaps University, Indore, India

²School of Electronics, Devi Ahilya Vishwavidyalaya, Indore, India

e-mail: a.kulkarni17@gmail.com

MS received 22 June 2018; revised 6 June 2019; accepted 13 July 2019

Abstract. This paper presents a backstepping-based terminal adaptive control solution for a class of uncertain underactuated systems with two degrees of freedom. The control scheme developed in this work is based on the finite time stability approach for the terminal convergence of system outputs. Due to the physical restrictions associated with underactuated systems, classical backstepping approach is not applicable to this class of non-linear systems and hence an approximate version is developed by designing a suitable error surface. Control term is developed in two steps: first, non-singular pseudo-control terms are developed for the dimensions of the configuration space and thereafter a feasible control term is designed to ensure the finite time convergence of error surface. The error surface is designed such that its finite time convergence in turn implies the terminal convergence of configuration variables. To ensure a well-defined control term over the entire state space, control components are embedded with corrective terms. Corrective terms are designed so as to avoid the singularity of the control term and to preserve the desired controller response up to certain extent. A fuzzy logic system is used for the approximation of unknown systems dynamics. Simulation results demonstrate the effectiveness of the control approach.

Keywords. Finite time stabilization; terminal backstepping control; underactuated systems; fuzzy logic system.

1. Introduction

Finite time stability control approaches are often characterized by their distinguished property of finite time error convergence. Besides this, these control approaches also provide higher robustness and better tracking precision as compared with other stability control schemes. Due to their ability to drive the tracking error to zero in finite time, several control schemes based on finite time stability are designed for linear and nonlinear control systems [1–3]. Terminal sliding mode control is a well-known finite-time-stable controller approach developed from classical sliding mode control. This approach not only preserves the robustness property of classical sliding mode control but also ensures the convergence of error terms in finite time. In terminal sliding mode control, instead of designing a linear hyperplane as the sliding manifold as in the case of conventional sliding mode control, a nonlinear sliding surface is designed by inducting the variables with fractional power. Sliding manifold so designed allows the finite time convergence of the system variables [4].

A modified version of terminal sliding mode control known as fast terminal sliding mode control has also been cited in the literature [5, 6]. This modification offers faster convergence rate than the conventional terminal sliding mode. However, terminal sliding mode control is associated with the problem of singularity, which means the system requires infinite control effort to maintain the desired terminal control. Non-singular terminal sliding mode control can be considered as one of the important modifications in terminal sliding mode control as it eliminates the singularity problem associated with terminal sliding mode control. Several research findings on the application of aforementioned control approach to various control systems have been cited in the literature [7–10]. Designing of a backstepping controller based on finite-time-stable approach is also cited in the literature. This approach integrates the features of backstepping methodology with finite time error convergence, resulting in terminal backstepping control [11]. Terminal control approaches have also been extended to the systems with uncertain dynamics in the form of adaptive terminal controllers. Several research findings addressing this issue have been cited in the literature [12–16].

*For correspondence
Published online: 19 August 2019

Underactuated systems are nonlinear mechanical systems in which the acceleration cannot be initiated in an arbitrary direction. As the acceleration is confined to a sub-manifold, these systems cannot be commanded to follow arbitrary trajectories in configuration space. A particular class of underactuated systems is one in which the dimension of the input space is less than the number of degrees of freedom to be controlled [17]. The dynamics of this class contains second-order nonholonomic constraints and reflects the presence of actuated and unactuated degrees of freedom with an inherent dynamic coupling among themselves [18]. Systems like twin rotor systems, underwater vehicles, robotic systems, cart pole and others belong to this category [17, 18]. Issues like controllability, existence and consequent designing of control terms for the concurrent control of all the degrees of freedom have been addressed by several researchers [17–32]. Several research findings emphasize on the development of control term using hierarchical methodology due to its flexible, effective and simplified approach. Hierarchical design utilizes the original state space representation of the underactuated system and does not require any nonlinear coordinate transformation [19–26]. Designing aspects of adaptive control terms for this class of nonlinear systems have also been cited in the literature [20–26]. Designing of semi- or fully generic control term relaxes the constraint of completely and accurately known system dynamics.

Backstepping is a Lyapunov-based controller designing methodology based on the concept of system decomposition and recursive consideration of system components for controller design. Backstepping can be viewed as a highly systematic and incremental designing approach that initiates the process of controller design by defining a positive-definite Lyapunov function for an elementary portion of the system and develops a pseudo-control term for the stabilization of the respective Lyapunov function. The process is further carried out by recursively considering the rest of the system components and hence redefining the Lyapunov function with suitable increment and the corresponding pseudo-control term to stabilize it. This process ultimately terminates with the development of actual control term, which ensures the stability of entire system [31–34]. As backstepping is a model-based approach, its classical approach demands completely and accurately known system dynamics. However, the classical version can be extended to adaptive version, which relaxes the aforementioned constraint on the system dynamics. These adaptive versions are developed by aggregating some adaptive approximation tools like fuzzy logic or neural network for the estimation of uncertain system dynamics [31–33]. Due to the physical restrictions associated with underactuated systems, classical structure of the backstepping methodology is not applicable to the concerned class of nonlinear systems without applying any nonlinear coordinate transformation. However, an approximate version of backstepping controller has been cited in the literature [31, 32]. This

approach defines an error term by linearly combining the velocity variables of the system. Stabilization of the Lyapunov function based on this error term in turn results in a feasible control term [32].

Due to the universal approximation property of fuzzy system, fuzzy-logic-based adaptive control has been developed for a wide variety of systems with uncertain dynamics. Adaptive schemes, developed by suitably aggregating the adaptable fuzzy logic system with classical feedback linearization schemes, utilize fuzzy logic system for the estimation of uncertain system dynamics [35]. As a consequence, control scheme ensures an accurate system performance in the presence of system uncertainties [21–26].

This paper presents an adaptive terminal control scheme for finite time convergence of a class of uncertain underactuated system. The controller scheme so developed integrates the approximate backstepping design with terminal backstepping concept. Initially, non-singular pseudo-control terms are developed for the dimensions of the configuration space and thereafter a positive-definite Lyapunov function is constructed by using an error term defined by aggregating the velocity variables. A feasible control term is thereafter designed to ensure the finite time convergence of the error surface. Finite time convergence of the error surface in turn implies the finite time regulation of configuration variables. Control components are designed to ensure a well-defined control term over the entire state space. Fuzzy logic system is used for the estimation of system uncertainties and controller gain parameters are set so as to ensure the finite time convergence of the system output in the presence of approximation error induced by the fuzzy system.

This paper is organized as follows. Problem formulation is described in section 2, whereas section 3 details the designing of backstepping-based terminal controller for a class of underactuated systems with completely known system dynamics. Section 4 presents an extension of the controller developed in section 3 for uncertain underactuated systems. Results of the simulation performed for underactuated systems are illustrated in section 5, whereas conclusion is presented in section 6.

2. Problem formulation

Consider a class of underactuated system described by the following dynamics:

$$\begin{aligned}
 \dot{x}_1 &= x_2 \\
 \dot{x}_2 &= f_1(X) + g_1(\underline{X}_1)u \\
 \dot{x}_3 &= x_4 \\
 \dot{x}_4 &= f_2(X) + g_2(\underline{X}_1)u \\
 y_1 &= x_1 \\
 y_2 &= x_3
 \end{aligned} \tag{1}$$

Equation (1) describes the dynamics of a class of under-actuated system with two degrees of freedom and a single control term. As reflected by the dynamics (1), these systems are composed of two interconnected subsystems in Brunovsky canonical form. Several underactuated systems like cart pole, ball beam, furuta pendulum, two-link planar robotic systems like acrobat, pendubot and others belong to this class of underactuated systems [17, 18]. Elements of the vector $X = [x_1, x_2, x_3, x_4]^T \in \mathbb{R}^4$ represent the dimensions of the state space whereas the elements of the vector $\underline{X}_1 = [x_1, x_3]^T \in \mathbb{R}^2$ represent the dimensions of configuration space; $u \in \mathbb{R}$ is the control vector applied to the system whereas $y_i \in \mathbb{R} (i = 1, 2)$ is the output of i^{th} subsystem, $f_i(X) : \mathbb{R}^4 \rightarrow \mathbb{R}$ and $g_i(\underline{X}_1) : \mathbb{R}^2 \rightarrow \mathbb{R}; (i = 1, 2)$ are system nonlinearities abbreviated as f_i and g_i , respectively. Nonlinearities f_i are considered as smooth system uncertainties whereas the nonlinear function g_i is either strictly positive or negative and $g_{iq} \in L_\infty, \forall x \in \Omega_x, t \geq 0$. Here, $\Omega_x \subset \mathbb{R}^4$ is some compact set of allowable state trajectories.

The objective of this paper is to develop an adaptive terminal controller based on backstepping approach so as to ensure the finite time convergence of state variables of the dynamics described by (1). To streamline the consequent controller development, following assumptions have been considered.

Assumption 1 Desired trajectories for the system output $y_{id} \in \mathbb{R}$ are selected such that $y_{id}, \dot{y}_{id}, \ddot{y}_{id} \in L_\infty, i = 1, 2$.

Assumption 2 There exist a set of real valued elements such that

$$\begin{aligned} \Omega_\rho &= \{\rho | |\rho_1 g_{1\min}| > |\rho_2 g_{2\max}| \text{ or } |\rho_2 g_{2\min}| > |\rho_1 g_{1\max}|\}; \\ \Omega_\rho &\subset \mathbb{R}^2 \end{aligned} \quad (2)$$

where

$$\begin{aligned} g_{1\min} &= \min(|g_1|); g_{1\max} = \max(|g_1|) \\ g_{2\min} &= \min(|g_2|); g_{2\max} = \max(|g_2|). \end{aligned}$$

This assumption ensures that the term $\rho_1 g_1 + \rho_2 g_2$ is either strictly positive or strictly negative.

3. Backstepping-based terminal controller

This section presents a backstepping-based terminal controller for underactuated systems with dynamics (1) Control term developed in this section is based on the assumption that the system dynamics is completely and accurately known, and hence it can be treated as the baseline controller

for the further development. In accordance with the backstepping methodology, the control term is developed in two steps: first, pseudo-control terms are developed to ensure the finite time convergence of the error variables containing configuration variables whereas the control effort is developed in the second step. The control effort so developed ensures the finite time convergence of all the state variables [11].

Step 1 To facilitate the designing of pseudo-control terms, the system dynamics (1) is expressed as

$$\begin{aligned} \dot{\underline{X}}_1 &= \underline{X}_2 \\ \dot{\underline{X}}_2 &= f + gu \end{aligned} \quad (3)$$

where $\underline{X}_2 = [x_2, x_4]^T \in \mathbb{R}^2$ and the vector fields f and g are defined as

$$f = [f_1, f_2]^T; g = [g_1, g_2]^T.$$

For the given desired trajectories, tracking error vector can be defined as

$$e = [e_1, e_2]^T = \underline{X}_1 - y_d \quad (4)$$

$$\dot{e} = \underline{X}_2 - \dot{y}_d \quad (5)$$

where $y_d = [y_{1d}, y_{2d}]^T$ is the vector of desired trajectories.

Using the error dynamics (4), a Lyapunov function can be defined as [34]

$$V_1 = \frac{1}{2} e^T e. \quad (6)$$

Differentiation of function V_1 (6) results in

$$\begin{aligned} \dot{V}_1 &= e^T \dot{e} \\ &= e^T (\underline{X}_2 - \dot{y}_d) \end{aligned} \quad (7)$$

For the terminal convergence of the error dynamics (4), the pseudo-term is defined as [11]

$$\underline{X}_{2d} = -Pe - K \left[|e_1|^{\frac{p-1}{q}} \text{sign}(e_1), |e_2|^{\frac{p-1}{q}} \text{sign}(e_2) \right]^T + \dot{y}_d \quad (8)$$

where $P = \text{diag}[P_{11}, P_{11}]; P_{11} > 0$ and $K = \text{diag}[k_1, k_1]; k_1 > 0$ are gain matrices; p and q are positive integers satisfying the condition $1 < \frac{p}{q} < 2$. Also $\text{sign}(x)$ is the sign function defined as

$$\begin{cases} \text{sign}(x) = -1; & \text{if } x < 0 \\ \text{sign}(x) = 0; & \text{if } x = 0. \\ \text{sign}(x) = 1; & \text{if } x > 0 \end{cases} \quad (9)$$

Substituting the pseudo-control term (8) in (7) and then utilizing the fact $|b_1|^r + |b_2|^r \geq (b_1^2 + b_2^2)^{r/2}$ where b_i and $1 < r < 2$ are real numbers [10]

$$\begin{aligned}
\dot{V}_1 &= e^T(\underline{X}_2 - \underline{X}_{2d} + \underline{X}_{2d} - \dot{y}_d) \\
&= e^T(-Pe - K[e_1^{\frac{p}{q}-1} \text{sign}(e_1), |e_2|^{\frac{p}{q}-1} \text{sign}(e_2)]^T + \gamma) \\
&= (-e^T Pe - e^T K[e_1^{\frac{p}{q}-1} \text{sign}(e_1), |e_2|^{\frac{p}{q}-1} \text{sign}(e_2)]^T + e^T \gamma) \\
&= (-\lambda_{\min}(P)\|e\|^2 - \lambda_{\min}(K)[|e_1|^{\frac{p}{q}} + |e_2|^{\frac{p}{q}}] + e^T \gamma) \\
&\leq (-\lambda_{\min}(P)\|e\|^2 - \lambda_{\min}(K)[\|e\|^2]^{\frac{p}{2q}} + e^T \gamma) \\
&\leq (-2\lambda_{\min}(P)V_1 - 2^{\frac{p}{2q}}\lambda_{\min}(K)V_1^{\frac{p}{2q}} + e^T \gamma) \\
&\leq (-2^{\frac{p}{2q}}\lambda_{\min}(K)V_1^{\frac{p}{2q}} + e^T \gamma) \quad (10)
\end{aligned}$$

Here the error vector γ is defined as

$$\gamma = [\gamma_1, \gamma_2]^T = \underline{X}_2 - \underline{X}_{2d} \quad (11)$$

whereas $\lambda_{\min}(\cdot)$ represents the minimum eigenvalue of respective matrices. It is apparent from (10) that finite time convergence of the elements of the vector e (4) indeed requires the terminal convergence of the elements of γ (11). The next step describes the development of control input so as to ensure the finite time convergence of all the error variables.

Step 2 Due to the physical constraints associated with underactuated systems, a feasible control term cannot be deduced by applying the conventional backstepping approach. In this work, an approximate version of the backstepping is used to develop a feasible control law. This approach relies on the development of an error term by suitably aggregating the elements of the error vector γ [25, 32].

Consider the error vector γ and define an error term of the form

$$S = \rho^T \gamma = \rho_1 \gamma_1 + \rho_2 \gamma_2. \quad (12)$$

Here, $\rho = [\rho_1, \rho_2]^T$; $\rho \in \Omega_\rho$ is the vector of coupling parameters. To ensure the positive definiteness of the Lyapunov function constructed using the error term S , the following algorithm is used for the construction of error term. The error term S can be constructed using either of the following cases [19].

Case a $\rho_1 \in \mathbb{R}^+$; $\rho_2 = \rho_{20} \text{sign}(\gamma_1 \gamma_2)$; $\rho_{20} \in \mathbb{R}^+$. Error term (12) resulting from this parameter selection is suitable for $|\rho_1 g_{1\min}| > |\rho_2 g_{2\max}|$.

Case b $\rho_1 \in \mathbb{R}^-$; $\rho_2 = -\rho_{20} \text{sign}(\gamma_1 \gamma_2)$; $\rho_{20} \in \mathbb{R}^+$. Error term (12) resulting from this parameter selection is suitable for $|\rho_1 g_{1\min}| > |\rho_2 g_{2\max}|$.

Case c $\rho_1 = \rho_{10} \text{sign}(\gamma_1 \gamma_2)$; $\rho_{10} \in \mathbb{R}^+$; $\rho_2 \in \mathbb{R}^+$. Error term (12) resulting from this parameter selection is suitable for $|\rho_2 g_{2\min}| > |\rho_1 g_{1\max}|$.

Case d $\rho_1 = -\rho_{10} \text{sign}(\gamma_1 \gamma_2)$; $\rho_{10} \in \mathbb{R}^+$; $\rho_2 \in \mathbb{R}^-$. Error term (12) resulting from this parameter selection is suitable for $|\rho_2 g_{2\min}| > |\rho_1 g_{1\max}|$.

The Lyapunov function constructed using error term S is positive definite in nature and can be used for the development of terminal control term [10].

To deduce the control term, define a Lyapunov function of the form

$$V_2 = \frac{1}{2} S^2. \quad (13)$$

Differentiation of V_2 (13) results in

$$\begin{aligned}
\dot{V}_2 &= S\dot{S} \\
&= S(\rho^T \dot{\gamma}) = S(\rho^T (\dot{\underline{X}}_2 - \dot{\underline{X}}_{2d})) \\
&= S(\rho^T (f + gu - \dot{\underline{X}}_{2d})) \\
&= S(\rho^T f + \rho^T gu - \rho^T \dot{\underline{X}}_{2d}) \quad (14)
\end{aligned}$$

Define the control input as

$$u = \frac{1}{\rho^T g} (-\rho^T f + \rho^T \dot{\underline{X}}_{2d} - K_1 S - K_2 |S|^{\frac{p}{q}-1} \text{sign}(S)) \quad (15)$$

where $K_1 > 0$; $K_2 > 0$ are gain terms.

Substitution of control term (15) in (14) results in

$$\begin{aligned}
\dot{V}_2 &= S(-K_1 S - K_2 |S|^{\frac{p}{q}-1} \text{sign}(S)) \\
&= (-K_1 S^2 - K_2 |S|^{\frac{p}{q}}) \\
&\leq (-K_1 S^2 - K_2 (|S|^2)^{\frac{p}{2q}}) \\
&\leq (-2K_1 V_2 - 2^{\frac{p}{2q}} K_2 V_2^{\frac{p}{2q}}) \\
&\leq -2^{\frac{p}{2q}} K_2 V_2^{\frac{p}{2q}} \quad (16)
\end{aligned}$$

Thus V_2 satisfies the following inequality:

$$V_2^{1-\frac{p}{2q}}(t) \leq V_2^{1-\frac{p}{2q}}(t_0) - 2^{\frac{p}{2q}} K_2 (1 - \frac{p}{2q})(t - t_0); t_0 \leq t \leq t_1. \quad (17)$$

Hence, it is apparent that $V_2(t) \equiv 0 \forall t \geq t_1$ where

$$t_1 = t_0 + \frac{1}{2^{\frac{p}{2q}} K_2 (1 - \frac{p}{2q})} V_2^{1-\frac{p}{2q}}(t_0). \quad (18)$$

Here, t_0 is the initial time [10].

It is apparent from the inequality (16) that the Lyapunov function V_2 (13) will converge to origin in finite time and this in turn implies the terminal convergence of error term S in (12) with $S = 0, \forall t \geq t_1$.

To verify the terminal convergence of the elements of the error vector γ , consider the error dynamics S in (12) constructed using either of the aforementioned cases (a)–(d). Squaring of S results in

$$S^2 = \rho_1^2 \gamma_1^2 + \rho_2^2 \gamma_2^2 + |\rho_1 \rho_2 \gamma_1 \gamma_2|, \quad (19)$$

$$S^2 \geq \rho_1^2 \gamma_1^2 + \rho_2^2 \gamma_2^2. \quad (20)$$

As the error term S undergoes terminal convergence, it is clear from the inequality (18) that the error terms γ_1 and γ_2 also converge in finite time with $\gamma_1 = 0$ and

$\gamma_2 = 0 \forall t \geq t_1$. With the convergence of error terms γ_1 and γ_2 , dynamics (10) becomes

$$\begin{aligned} \dot{V}_1 &\leq -2^{\frac{p}{2q}} \lambda_{\min}(K) V_1^{\frac{p}{2q}} \\ V_1^{1-\frac{p}{2q}}(t) &\leq V_1^{1-\frac{p}{2q}}(t_1) - 2^{\frac{p}{2q}} \lambda_{\min}(K) (1 - \frac{p}{2q})(t - t_1); t_1 \leq t \leq t_2 \end{aligned} \quad (21)$$

Thus V_1 (6) also undergoes finite time convergence with $V_1(t) \equiv 0$ for $t \geq t_2$ where

$$t_2 = t_1 + \frac{1}{2^{\frac{p}{2q}} \lambda_{\min}(K) (1 - \frac{p}{2q})} V_1^{1-\frac{p}{2q}}(t_1). \quad (22)$$

Finite time convergence of V_1 in turn implies the terminal convergence of the error terms e_1 and e_2 . Thus, $e_1 = 0$ and $e_2 = 0 \forall t \geq t_2$.

This development implies the finite time convergence of the error variables of the underactuated systems with dynamics (1). However the aforementioned development is applicable for the systems with completely and accurately known system dynamics. The next section presents an adaptive version of the control law (15) for underactuated systems associated with uncertainties. Also there are certain points in the state space where the system dynamics $\dot{\underline{X}}_{2d}$, $\dot{S}$ and $\dot{V}_2$ show impulsive nature. The impulsive nature shown by the system dynamics may result in singularity in the control term (15) or ambiguous nature of $\dot{V}_2$. The next part of this section deals with these issues.

Case I: The term $\dot{\underline{X}}_{2d}$ present in the control term (15) is defined as

$$\dot{\underline{X}}_{2d} = -P\dot{e} - K \left[|e_1|^{\frac{p}{q}-2} \dot{e}_1, |e_2|^{\frac{p}{q}-2} \dot{e}_2 \right]^T + \ddot{y}_d. \quad (23)$$

At the points on the trajectories of x_1 and x_3 where the conditions $e_1 = 0, \dot{e}_1 \neq 0$ and/or $e_2 = 0, \dot{e}_2 \neq 0$ are satisfied, component $\dot{\underline{X}}_{2d}$ (22) and the control term u (14) cease to be analytic since $\frac{p}{q} - 2 < 0$.

To avoid the singularity at these points, the term $\dot{\underline{X}}_{2d}$ is modified as [33]

$$\dot{\underline{X}}_{2d} = -P\dot{e} - K \left[\frac{\dot{e}_1}{|e_1|^{2-\frac{p}{q}+\eta_1 d_1}}, \frac{\dot{e}_2}{|e_2|^{2-\frac{p}{q}+\eta_2 d_2}} \right]^T + \ddot{y}_d. \quad (24)$$

The term $\eta_i (i = 1, 2)$ is defined as

$$\begin{cases} \eta_i = 0; & \text{if } |e_i| \geq e_{ia} \\ \eta_i = \frac{e_{ia} - |e_i|}{e_{ia}}; & \text{else} \end{cases}. \quad (25)$$

Here $e_{ia} (i = 1, 2)$ is the value of the error variable e_i close to origin and $d_i (i = 1, 2) \in [0, 1]$ is the design parameter. This modification will restrict the term $\dot{\underline{X}}_{2d}$ and hence u (15) from getting singular at the points with $(e_1 = 0, \dot{e}_1 \neq 0)$ and/or $(e_2 = 0, \dot{e}_2 \neq 0)$. Although this modification will

restrict the control effort u (14) to some finite value, the control effort may deviate from the actual control effort required to maintain the terminal convergence. However, due to its natural characteristic, the system will not remain in this state for a long period of time. Since, at these points, $e_i = 0 (i = 1, 2)$ and $\dot{e}_i \neq 0$, trajectories of the error variable e_i will cross $e_i = 0$ at the next time instant and the $\dot{\underline{X}}_{2d}$ will resume the dynamics (23) within a short span of time. Thus, due to the short lasting nature of this phenomenon and continuous nature of system variables, boundedness of variables and system stability is assured [11]. Also, parameter d_i can be used to control the accuracy of the control term (15); by lowering the value of d_i , control effort can be kept close to the control effort required for terminal convergence; however the resulting control term may contain high-frequency localized but bounded components at these points.

Case II: Due to the $\text{sign}(\cdot)$ used in the designing of the error term S (12), derivative of S (12) and hence $\dot{V}_2$ (14) will become impulsive at the points $(\gamma_1 = 0, \dot{\gamma}_1 \neq 0, \gamma_2 \neq 0)$; for cases a and b) and $(\gamma_2 = 0, \dot{\gamma}_2 \neq 0, \gamma_1 \neq 0)$; for cases c and d). Depending on the sign of $\dot{\gamma}_1$ (for cases a and b) and $\dot{\gamma}_2$ (for cases c and d), these impulses are either positive or negative; hence, in accordance to the existence of positive or negative impulses, $\dot{V}_2$ becomes positive or negative at these points. Similar to case I, this is also short lasting phenomenon. At these singularities, the condition $\gamma_i = 0$ and $\dot{\gamma}_i \neq 0 (i = 1, 2)$ is satisfied and the trajectories of γ_i will cross $\gamma_i = 0$ at the next time instant; hence, the singular part of $\dot{S}$ will vanish within a short span of time. Thus, under the condition of momentary existence of singular part in $\dot{S}$ and continuous nature of state variables, boundedness of the error term S and hence the system stability is still assured [11].

The next section presents an adaptive version of the control term (15) for a class of underactuated systems with dynamics (1) associated with uncertainties $f_i (i = 1, 2)$. In this work, fuzzy logic system is used for the estimation of nonlinear term $\rho^T f$, which is induced by system uncertainties. State variables $(X \in \mathbb{R}^4)$ are input to the fuzzy approximator.

4. Adaptive terminal controller

Due to their universal approximation property, fuzzy logic systems are used for the approximation system uncertainties. Fuzzy approximators are composed of set IF-THEN rules, which are used for mapping the input $(X \in \mathbb{R}^4)$ to output $(y_f \in \mathbb{R})$. The fuzzy rule is expressed as [15, 25, 26, 35]

Rule - j: if x_1 is $A_1^j(x_1)$ and $\dots$ and if x_4 is $A_4^j(x_4)$, .
then y_f is θ^j for $j = 1, 2, \dots, N$

Here A_i^j is a fuzzy set with membership function $\mu_{A_i^j}(x_i)$, θ^j is the fuzzy singleton corresponding to fuzzy sets and N is the number of fuzzy rules. Then the output of the fuzzy system can be defined as

$$y_f = \frac{\sum_{j=1}^N \theta^j \prod_{i=1}^4 \mu_{A_i^j}(x_i)}{\sum_{j=1}^N \left(\prod_{i=1}^4 \mu_{A_i^j}(x_i) \right)}. \quad (26)$$

This equation can be expressed in the following form:

$$y_f = \theta^T \varphi(X) \quad (27)$$

where $\theta = [\theta^1, \theta^2, \dots, \theta^N]^T$ is the vector of adjustable parameters and $\varphi = [\varphi^1, \varphi^2, \dots, \varphi^N]^T$ is the regression vector with

$$\varphi^j = \frac{\prod_{i=1}^4 \mu_{A_i^j}(x_i)}{\sum_{j=1}^N \left(\prod_{i=1}^4 \mu_{A_i^j}(x_i) \right)}; j = 1, 2, \dots, N. \quad (28)$$

Hence the fuzzy-logic-based approximation of the non-linear term $\rho^T f$ can be expressed as

$$\hat{F} = \theta^T \varphi(X). \quad (29)$$

Under the condition of optimal approximation

$$\rho^T f = \theta^{*T} \varphi(X) + \varepsilon. \quad (30)$$

Here, θ^* is the optimal estimate of θ and is defined as

$$\theta^* = \arg \min_{\theta} \left(\sup_{X \in \Omega_X} |\rho^T f - \theta^T \varphi(X)| \right) \quad (31)$$

Ω_X is the compact set of allowable state trajectories whereas ε is approximation error such that $|\varepsilon| \leq \varepsilon_F \forall X \in \Omega_X$; ε_F is a positive constant.

Assumption 3 Optimal parameter vector θ^* is norm bounded so that

$$\|\theta^*\| \leq A_\theta. \quad (32)$$

Here, A_θ is a positive constant [10]. With the fuzzy logic approximation, under the condition of system uncertainties $f_i (i = 1, 2)$, control term $u()$ is defined as

$$u = \frac{1}{\rho^T g} \left(-\hat{F} + \rho^T \ddot{X}_{2d} - K_1 S - K_2 |S|^{\frac{p}{q}-1} \text{sign}(S) - \text{asign}(S) - u_s \right) \quad (33)$$

where $\hat{F}$ is the fuzzy approximation of the nonlinear uncertainty $\rho^T f$ and $a > 0$. Additional compensation term u_s incorporated in the control law is defined as

$$u_s = \begin{cases} \frac{K_2(\|\theta\| + A_\theta)^{\frac{p}{q}} S}{S^2}; & \text{if } S \neq 0 \\ 0; & \text{if } S = 0 \end{cases} \quad (34)$$

Architecture of the control scheme is shown in figure 1. To analyse the terminal convergence under the condition of system uncertainty and to deduce the online update laws for adjustable parameters, consider the following Lyapunov function [36]:

$$V_3 = V_2 + \frac{1}{2} \tilde{\theta}^T \tilde{\theta} = \frac{1}{2} S^2 + \frac{1}{2} \tilde{\theta}^T \tilde{\theta} \quad (35)$$

where $\tilde{\theta}$ is the estimation error of adjustable parameters, defined as

$$\tilde{\theta} = \theta^* - \theta. \quad (36)$$

Differentiation of V_3 (35) results in

$$\begin{aligned} \dot{V}_3 &= S\dot{S} + \tilde{\theta}^T \dot{\tilde{\theta}} \\ \dot{V}_3 &= S(\rho^T(\dot{X}_2 - \dot{X}_{2d})) + \tilde{\theta}^T \dot{\tilde{\theta}} \\ &= S(\rho^T(f + gu - \dot{X}_{2d})) + \tilde{\theta}^T \dot{\tilde{\theta}} \\ &= S(\rho^T f + \rho^T gu - \rho^T \dot{X}_{2d}) + \tilde{\theta}^T \dot{\tilde{\theta}} \end{aligned} \quad (37)$$

Substituting the control term u (33) in (37)

$$\begin{aligned} \dot{V}_3 &= S \left(\rho^T f - \hat{F} - K_1 S - K_2 |S|^{\frac{p}{q}-1} \text{sign}(S) - \text{asign}(S) - \frac{K_2(\|\theta\| + A_\theta)^{\frac{p}{q}} S}{S^2} \right) + \tilde{\theta}^T \dot{\tilde{\theta}} \\ &= S \left(\tilde{F} - K_1 S - K_2 |S|^{\frac{p}{q}-1} \text{sign}(S) - \text{asign}(S) - \frac{K_2(\|\theta\| + A_\theta)^{\frac{p}{q}} S}{S^2} \right) + \tilde{\theta}^T \dot{\tilde{\theta}} \end{aligned} \quad (38)$$

Figure 1. Closed loop system architecture.

where $\tilde{F}$ is the function estimation error, defined as

$$\begin{aligned}\tilde{F} &= \rho^T f - \hat{F} \\ &= \theta^{*T} \varphi - \theta^T \varphi + \varepsilon \\ &= \tilde{\theta}^T \varphi + \varepsilon\end{aligned}\quad (39)$$

$$\begin{aligned}\dot{V}_3 &= S \left(\tilde{\theta}^T \varphi + \varepsilon - K_1 S - K_2 |S|^{\frac{p}{q}-1} \text{sign}(S) - \frac{a \text{sign}(S) - \frac{K_2(\|\theta\| + A_\theta)^{\frac{p}{q}} S}{S^2}}{S^2} \right) + \tilde{\theta}^T \dot{\tilde{\theta}} \\ &\leq \left(S \tilde{\theta}^T \varphi + S \varepsilon_F - K_1 S^2 - K_2 |S|^{\frac{p}{q}} - \frac{a |S| - K_2(\|\theta\| + A_\theta)^{\frac{p}{q}}}{S} \right) + \tilde{\theta}^T \dot{\tilde{\theta}}\end{aligned}\quad (40)$$

By selecting the gain term a to be large enough, the following condition can be satisfied:

$$a \geq \varepsilon_F. \quad (41)$$

With the value of a satisfying the inequality (41), (40) becomes

$$\dot{V}_3 \leq \left(S \tilde{\theta}^T \varphi - K_1 S^2 - K_2 |S|^{\frac{p}{q}} - K_2(\|\theta\| + A_\theta)^{\frac{p}{q}} \right) + \tilde{\theta}^T \dot{\tilde{\theta}}. \quad (42)$$

Selecting the following adaptation law and substituting in (42) gives

$$\dot{\tilde{\theta}} = -\dot{\theta} = -S\varphi, \quad (43)$$

$$\dot{V}_3 \leq \left(-K_1 S^2 - K_2 |S|^{\frac{p}{q}} - K_2(\|\theta\| + A_\theta)^{\frac{p}{q}} \right). \quad (44)$$

With assumption 3, this inequality becomes

$$\dot{V}_3 \leq \left(-K_1 S^2 - K_2 |S|^{\frac{p}{q}} - K_2(\|\theta^*\| + \|\theta\|)^{\frac{p}{q}} \right). \quad (45)$$

By triangle inequality, this expression becomes

$$\begin{aligned}\dot{V}_3 &\leq \left(-K_1 S^2 - K_2 \left(|S|^2 \right)^{\frac{p}{2q}} - K_2(\|\theta^* - \theta\|)^{\frac{p}{q}} \right) \\ &\leq \left(-K_1 S^2 - K_2 \left(|S|^2 \right)^{\frac{p}{2q}} - K_2 \left(\|\tilde{\theta}\|^2 \right)^{\frac{p}{2q}} \right) \\ &\leq -K_2 \left(|S|^2 + \|\tilde{\theta}\|^2 \right)^{\frac{p}{2q}} \\ &\leq -2^{\frac{p}{2q}} K_2 V_3^{\frac{p}{2q}}\end{aligned}\quad (46)$$

Thus, V_3 satisfies the condition of terminal convergence with

$$V_3^{1-\frac{p}{2q}}(t) \leq V_3^{1-\frac{p}{2q}}(t_0) - 2^{\frac{p}{2q}} K_2 \left(1 - \frac{p}{2q} \right) (t - t_0); t_0 \leq t \leq t_3. \quad (47)$$

This inequality indicates that $V_3(t) \equiv 0$, for $t \geq t_3$ where

$$t_3 = t_0 + \frac{1}{2^{\frac{p}{2q}} K_2 \left(1 - \frac{p}{2q} \right)} V_3^{1-\frac{p}{2q}}(t_0). \quad (48)$$

Here, t_0 is the initial time.

Thus, the Lyapunov function $V_3(t)$ undergoes finite time convergence, which in turn implies the terminal convergence of error term S (12) with $S = 0, \forall t \geq t_3$. Due to the error dynamics S (12), elements of the error vector γ (11) converge in finite time. With $\gamma_1 = 0; \gamma_2 = 0, \forall t \geq t_3$, Lyapunov function V_1 satisfies Eq. (21) and hence the function V_1 and the elements of the error vector e (4) converge in finite time t_4 given by

$$t_4 = t_3 + \frac{1}{2^{\frac{p}{2q}} \lambda_{\min}(K) \left(1 - \frac{p}{2q} \right)} V_1^{1-\frac{p}{2q}}(t_3). \quad (49)$$

The following theorem summarizes the convergence characteristics of the control term developed in this work.

Theorem For a class of underactuated systems governed by the dynamics (1) and associated with the aforementioned uncertainties, under the condition of assumptions 1–3, the control term (33) with additional component (34) and update laws (36) ensures the finite time convergence of the configuration and other state variables.

5. Simulation results

In order to illustrate effectiveness of the controller approach developed in this work (33), simulations carried out with a cart–pole system and a pendubot are presented in this section. Both these systems are underactuated systems with dynamics governed by (1).

5.1 Cart–pole system

Cart–pole system is a typical case of underactuated systems comprising a cart and a pole, which are coupled using a revolute joint. Dynamics of the system is described in a two-dimensional plane. Angular orientation of the pole with respect to vertical axis and position of the cart along the horizontal axis with respect to origin are two degrees of freedom whereas the driving force applied to the cart is the only actuation available. Simulation is carried out to steer the pole to upright position and cart to origin in finite time from any arbitrary initial position under the action of adaptive control term (33).

With reference to dynamics (1), the system nonlinearities along with system variables and other parameters are

$$\begin{aligned}
 f_1 &= \frac{mg \sin x_3 \cos x_3 - mLx_4^2 \sin x_3}{m \cos^2 x_3 - (M + m)}, \\
 g_1 &= \frac{-1}{m \cos^2 x_3 - (M + m)} \\
 f_2 &= \frac{-(M + m)g \sin x_3 + mLx_4^2 \sin x_3 \cos x_3}{mL \cos^2 x_3 - (M + m)L} \\
 g_2 &= \frac{\cos x_3}{mL \cos^2 x_3 - (M + m)L}
 \end{aligned} \quad (50)$$

where x_1 denotes the position of the cart with respect to origin, x_3 is pole angle with respect to vertical axis, x_2 and x_4 are respective velocity variables and u is the force applied to drive the cart. Parameters appearing in the system dynamics are m , which is the mass of the bob, M , the cart mass, L , the pendulum length and g , the acceleration due to gravity. Nonlinear terms f_1 and f_2 are assumed to be unknown and the nonlinear component appearing in the error dynamics due to these nonlinearities is approximated using the fuzzy logic system. For simulation, system parameters, initial settings and controller parameters are taken as

$$m = 0.15 \text{ kg}, M = 1 \text{ kg}, L = 0.5 \text{ m} \quad \text{and} \quad g = 9.8 \text{ m/s}^2$$

$$x(0) = [0.3, 0, -\pi/7, 0]^T$$

$$P = \text{diag}[1.8, 1.2]; \quad K = \text{diag}[1.4, 1.4];$$

$$\rho_0 = [1.5, 1]^T \quad K_1 = 8, K_2 = 2.51$$

$$p = 5, q = 3, A_\theta = 15, a = 5;$$

$$e_{ia} = 0.01(i - 1, 2); d_i = 0.01; (i = 1, 2)$$

A fuzzy system with arguments (x_1, x_2, x_3, x_4) is constructed for the approximation of uncertain nonlinearities. For each argument, Gaussian functions are selected as the membership functions. An online update of the adjustable parameters is carried out using adaptation laws (36) with initial conditions set to zero for all the parameters.

Simulation results are shown in figures 2 and 3. As revealed from figure 2, trajectories of the error variables γ_1, γ_2 and S approach zero within $t_3 = 2.3$ sec, whereas the trajectories of the configuration variables x_1 and x_3 approach zero into $t_4 = 2.77$ sec. Figure 2 depicts the control input, which is bounded in nature.

5.2 Pendubot

Pendubot is a two-link underactuated planar robot with one active and one passive joint. These links are coupled using revolute joints in serial link mechanism, with the first link connected to the base and angular orientations of these links are two degrees of freedom whereas the torque applied at the first (lower) joint is the only actuation available. Simulation is carried out to steer the system from

Figure 2. (a) Trajectories followed by error and state variables. (b) Expanded view of error and state variables.

any arbitrary initial condition to the equilibrium position $X^e = (\pi/2, 0, -\pi, 0)$ in finite time using the control term (33). The Euler–Lagrange equation for system dynamics is [19]

$$M(\underline{X}_1)\ddot{\underline{X}}_2 + C(\underline{X}_1, \underline{X}_2) + H(\underline{X}_1) = \Gamma u \quad (51)$$

where

$$M(\underline{X}_1) = \begin{bmatrix} a_1 + a_2 + 2a_3 \cos x_3 & a_2 + a_3 \cos x_3 \\ a_2 + a_3 \cos x_3 & a_2 \end{bmatrix},$$

Figure 3. Control effort applied to the system.

$$C(\underline{X}_1, \underline{X}_2) = a_3 \sin x_3 \begin{bmatrix} -2x_2x_4 - x_4^2 \\ x_2^2 \end{bmatrix},$$

$$H(\underline{X}_1) = \begin{bmatrix} a_4 g \cos x_1 + a_5 g \cos(x_1 + x_3) \\ a_5 g \cos(x_1 + x_3) \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Here, $M(\underline{X}_1) \in \mathbb{R}^{2 \times 2}$, $C(\underline{X}_1, \underline{X}_2) \in \mathbb{R}^{2 \times 1}$ and $H(\underline{X}_1) \in \mathbb{R}^{2 \times 1}$ are inertia matrix, Coriolis matrix and gravity matrix, respectively.

Element of the vector $\underline{X}_1 = [x_1, x_3]^T \in \mathbb{R}^{2 \times 1}$ denotes the link angles. Variable x_1 denotes the angular position of the first link with respect to horizontal axis whereas x_2 denotes the angular positions of second (upper) link with respect to axis passing through the centre of mass of the preceding link. Vector $\underline{X}_2 = [x_2, x_4]^T \in \mathbb{R}^{2 \times 1}$ contains corresponding velocity variables. $u \in \mathbb{R}$ is the torque applied to the system. Parameters appearing in the system dynamics are [19]

$$\begin{aligned} a_1 &= m_1 c_1^2 + m_2 l_1^2 + I_1; a_2 = m_2 c_2^2 + I_2. \\ a_3 &= m_2 l_1 c_2; a_4 = m_1 c_1 + m_2 l_1; a_5 = m_2 c_2 \end{aligned} \quad (52)$$

System parameters m_i, l_i, c_i and $I_i (i = 1, 2)$ are link parameters and correspondingly designate the mass, length, distance from i^{th} joint to centre of mass and moment of inertia about the centre of mass for i^{th} link and g is acceleration due to gravity. System dynamics (51) can be transformed into the state space representation (1). Control scheme (33) is applied to the system dynamics with the consideration of nonlinearities $f_i (i = 1, 2)$ as system uncertainties.

For simulation, system parameters, initial settings and controller parameters are taken as

$$\begin{aligned} a_1 &= 0.0308; a_2 = 0.0106; a_3 = 0.0095 \\ a_4 &= 0.2086; a_5 = 0.0630 \end{aligned}$$

$$x(0) = [pi/6, 0, -pi/5, 0]^T$$

$$P = \text{diag}[1, 1]; \quad K = \text{diag}[1.2, 1.2];$$

$$\rho_0 = [0.5, 1]^T \quad K_1 = 7.6, K_2 = 2.51$$

Figure 4. (a) Trajectories followed by error variable γ_1 and lower link angle x_1 . (b) Expanded view of trajectories.

$$\begin{aligned} p &= 5, q = 3, A_\theta = 25, a = 8; \\ e_{ia} &= 0.01(i - 1, 2); d_i = 0.01; (i = 1, 2) \end{aligned}$$

Similar to cart-pole system, a fuzzy logic system is constructed for the approximation of uncertain nonlinearities with Gaussian functions as the membership functions. Online update of the adjustable parameters is carried out using adaptation laws (38) with initial conditions set to zero for all the parameters.

Simulation results are shown in figures 4, 5, 6 and 7. As revealed from figures 4, 5 and 6, trajectories of the error variables γ_1, γ_2 and S approach zero within $t_3 = 9.1$ sec whereas the trajectories of the configuration variables x_1 and x_3 approach zero in $t_4 = 10.8$ sec. Figure 6 depicts the

Figure 5. (a) Trajectories followed by error variable γ_2 and upper link angle x_3 . (b) Extended view of trajectories.

control input; although some high-frequency localized components are present in the control input, the control term is well defined and bounded in nature.

6. Conclusion

In this work, a backstepping-based terminal adaptive controller is presented for a class of uncertain underactuated systems with two degrees of freedom. The control scheme developed in this work not only ensures the

Figure 6. Trajectory followed by error variable S .

Figure 7. Control effort applied to the system.

stability of the system under consideration but also finite time convergence of the system variables to their desired trajectories under the presence of system uncertainties. The control scheme developed in this work utilizes fuzzy logic system for the approximation of system uncertainties, and update laws are developed for the online tuning of adjustable parameters of the fuzzy system. The control term is embedded with corrective terms to deal with impulses that are supposed to appear at some points in the state space. Simulation study reveals that the control term is bounded; the quality of the control effort is improved by the induction corrective terms and the trajectories followed by the configuration variables are almost smooth.

References

- [1] Bhat S and Bernstein D 2000 Finite time stability of continuous autonomous systems. *SAIM J. Control Optim.* 38: 751–766
- [2] Hong Y 2002 Finite time stabilization and stabilizability of a class of controllable systems. *Syst. Control Lett.* 46: 231–236
- [3] Gao T G, Chen Z Q, Chen G R and Yuan Z Z 2006 Finite time control of chaotic systems with nonlinear inputs. *Chin. Phys.* 15: 1190–1195
- [4] Yu X and Man Z 1998 Multi-input uncertain linear systems with terminal sliding mode control. *Automatica* 34: 389–392

- [5] Yu S, Yu X, Shirinzadeh B and Man Z 2005 Continuous finite time control for robotic manipulators with terminal sliding mode. *Automatica* 41: 1957–1964
- [6] Zhao D, Li S and Gao F 2009 A new terminal sliding mode control for robotic manipulators. *Int. J. Control* 82: 1804–1813
- [7] Yu X, Zhihong M, Feng Y and Guan Z 2002 Nonsingular terminal sliding mode code of a class of nonlinear dynamical systems. In: *Proceedings of the 15th Triennial World Congress*, pp. 1–5
- [8] Feng Y, Yu X and Man Z 2002 Nonsingular terminal sliding mode control of rigid manipulators. *Automatica* 38: 2159–2167
- [9] Jin M, Lee J, Chang P H and Choi C 2009 Practical nonsingular terminal sliding mode control of robot manipulators for high accuracy tracking control. *IEEE Trans. Ind. Electron.* 56: 3593–3601
- [10] Aghababa M P and Feizi H 2012 Nonsingular terminal sliding mode approach applied to synchronize chaotic systems with unknown parameters and nonlinear inputs. *Chin. Phys.* 21: 060506-1–060506-10
- [11] Lin Y and Cheng C 2015 Design of terminal block backstepping controllers for perturbed systems in semi-strict feedback form. *Int. J. Control* 88: 2107–2116
- [12] Tao C W, Taur J S and Chan M L 2004 Adaptive fuzzy terminal sliding mode controller for linear systems with mismatched time varying uncertainties. *IEEE Trans. Syst. Man Cybern. Part B: Cybern.* 34: 255–262
- [13] Lin C K 2006 Nonsingular terminal sliding mode control of robot manipulators using fuzzy wavelet networks. *IEEE Trans. Fuzzy Syst.* 14: 849–859
- [14] Wang L, Chai T and Zhai L 2009 Neural network based terminal sliding mode control of robotic manipulators including actuator dynamics. *IEEE Trans. Ind. Electron.* 9: 3296–3304
- [15] Nekoukar V and Erfanian A 2011 Adaptive fuzzy terminal sliding mode control for a class of MIMO uncertain nonlinear systems. *Fuzzy Sets Syst.* 179: 34–49
- [16] Fan L and Yu Y 2011 Adaptive nonsingular terminal sliding mode control for DC–DC converters. *Adv. Electr. Comput. Eng.* 11: 119–122
- [17] Spong M W 1997 Underactuated mechanical systems. *Control Prob. Robot. Automat.* 230: 135–150
- [18] Olfati-Saber R 2002 Normal forms for underactuated mechanical systems with symmetry. *IEEE Trans. Automat. Control* 47: 305–308
- [19] Wang W, Yi J, Zhao D and Liu D 2004 Design of a stable sliding-mode controller for a class of second-order underactuated systems. *IEE Proc.-Control Theory Appl.* 151: 683–690
- [20] Chyau H and Feng C Y 2010 Adaptive control for a class of underactuated systems with mismatched uncertainties. In: *Proceedings of the 29th Chinese Control Conference*, pp. 2053–2059
- [21] Hung L C, Lin H P and Chung H Y 2007 Design of self-tuning fuzzy sliding mode control for TORA system. *Expert Syst. Appl.* 32: 201–212
- [22] Qian D and Yi J 2010 Fuzzy aggregated hierarchical sliding mode control for underactuated systems. In: *Proceedings of the IEEE Conference on Mechatronics and Automation*, pp. 196–201
- [23] Qian D, Yi J and Zhao D 2011 Control of overhead crane systems by combining sliding mode with fuzzy regulator. In: *Proceedings of the 18th IFAC World Congress*, pp. 9320–9325
- [24] Chiang C C and Yah Y W 2014 Hierarchical fuzzy sliding mode control for uncertain nonlinear underactuated systems. In: *Proceedings of the IEEE Conference on Fuzzy Systems*, pp. 662–669
- [25] Hwang C L, Chiang C C and Yah Y W 2014 Adaptive fuzzy hierarchical sliding mode control for the trajectory tracking of uncertain underactuated nonlinear dynamic systems. *IEEE Trans. Fuzzy Syst.* 22: 286–299
- [26] Su S, Hsueh Y, Tseng C, Chen S and Lin Y 2015 Direct adaptive fuzzy sliding mode control for underactuated uncertain systems. *Int. J. Fuzzy Logic Intell. Syst.* 15: 240–250
- [27] Marton L, Hodel A S, Lantos B and Hung J 2008 Underactuated robot control: comparing LQR, subspace stabilization and combined error matrix approaches. *IEEE Trans. Ind. Electron.* 55: 3724–3730
- [28] Choukchou-Braham A, Cherki B, Djemaï M and Busawon K 2016 *Analysis and control of underactuated mechanical systems*. Berlin: Springer
- [29] Moreno-Valenzuela J and Aguilar-Avelar C 2018 *Motion control of underactuated mechanical systems*. Berlin: Springer
- [30] Marton L and Lantos B 2006 Stable adaptive ball and beam control. In: *Proceedings of the IEEE Conference on Mechatronics*, pp. 507–512
- [31] Zhixiang T, Hongtao W U and Chun F 2010 Hierarchical adaptive backstepping sliding mode control for underactuated space robot. In: *Proceedings of IACICAR*, pp. 500–503
- [32] Kulkarni A and Kumar A 2015 Backstepping based adaptive control for underactuated systems. *Int. J. Syst. Control Inf. Process.* 1: 340–352
- [33] Hsu C F, Lin C M and Lee T T 2006 Wavelet adaptive backstepping control for a class of nonlinear systems. *IEEE Trans. Neural Netw.* 17: 1175–1183
- [34] Khalil H K 2002 *Nonlinear systems*. Upper Saddle River, NJ: Prentice Hall
- [35] Wang L X 1993 Stable adaptive fuzzy control of nonlinear systems. *IEEE Trans. Fuzzy Syst.* 1: 146–155
- [36] Astrom K J and Wittenmark B 1995 *Adaptive control*. New York: Addison Wesley