NURBS-based thermo-elastic analyses of laminated and sandwich composite plates

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MS received 6 March 2018; revised 3 August 2018; accepted 26 November 2018; published online 21 March 2019

Abstract. Present research work is based on the unique properties of isogeometric analysis (IGA) like smarter, faster and cheaper analysis for the thermo-elastic bending of laminated and sandwich composite plates. IGA, based on isoparametric concept, is a breakthrough in the area of structural analysis, which employs non-uniform rational B-spline (NURBS) as a basis function to represent less erroneous geometry. Unlike finite-element method (FEM), increasing the polynomial order in IGA gives higher continuous basis functions naturally and easily with reduced computational cost. A procedure has been developed for thermo-elastic bending analysis of laminated composite plates and sandwich structures using IGA approach. The developed NURBS-based code is validated and computational efficacy of thermo-elastic analysis is investigated. A detailed parametric study has been carried out for the quadratic, cubic and quartic NURBS elements with respect to the variation of temperature. Different types of temperature profiles have been considered. Change of deflections, stresses and moment resultants are analysed with an aim to understand the thermo-elastic behaviour of laminated and sandwich composite plates. Several thermo-elastic numerical examples have been analysed extensively. Obtained numerical results are compared with available literature to show the advantage of current formulation.

Keywords. Thermo-elastic bending; isogeometric analysis (IGA); non-uniform rational B-splines (NURBS); laminated composite plates; sandwich composite plates; non-linear thermal loads.

1. Introduction

A combination of two or more materials, with different physical and chemical properties, in such a way that recognizable physical boundaries on macroscopic and microscopic level still exist after joining is called a composite material. Laminated and sandwich composites are widely used in aerospace, marine and wind turbine industries. Nowadays a large number of industrial products are made up of composite materials. The reasons for this are high strength-to-weight ratio, high stiffness, good dimensional stability after manufacturing and high impact, fatigue and corrosion resistance of composites. In addition to this, composites possess ability to follow complex mould shapes and to be specifically tailored through optimization of ply numbers and fibre orientations through the structure so that they can meet specific needs while minimizing weight [1].

Investigations on properties of laminate and sandwich structures have been addressed since long time [2, 3]. Composite structures are subjected to environmental conditions during the service life. Consequently, moisture and temperature have an adverse effect on the performance of composites. Stiffness and strength are reduced with the

increase in moisture concentration and temperature. The deformation and stress analyses of laminated and sandwich composite plates subjected to moisture and temperature have been the subject of research interest in recent years, but most of the researchers have studied the effect of temperature [4–10]. Wu and Taichert [4, 5] presented closed-form solutions for deflections and moments for symmetric and anti-symmetric laminates subjected to uniform change in temperature in addition to the external loading. Reddy and Hsu [6] applied the penalty finite element to the thermal stress analysis of laminates and compared the results to the closed-form solution. Effects of aspect ratio, side-to-thickness ratio and laminate construction are considered. Thangaratnam *et al* [7] used the semi-loof shell element for the thermal stress analysis of composite plates and shells. Results for deflections and moments are presented for the linearly varying temperature through the thickness and uniform temperature distribution over the surface. Whitney and Ashton [8] used the classical laminated plate theory to study the effect of environment on the stability, vibration and bending behaviour. To understand the properties of sandwich structures, Pagano [11] initially investigated the analytical three-dimensional (3D) elasticity method to predict the exact solution of simple static problems. Noor *et al* [10] have further developed a

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3D elasticity solution formulation for stress analysis of sandwich structures.

It is well known that an exact 3D approach is the most potential tool to obtain the true solution. However, it is not easy to solve practical problems in which complex geometric and boundary conditions are required. Alternatively, several plate theories can be utilized after reduction of the 3D full model to the two-dimensional (2D) model. Hitherto, many theories have been developed to analyse laminated and sandwich composite plates; among them, the equivalent single layer (ESL) theory is most widely used. The ESL theory provides sufficiently accurate description of the global response at low computational cost. The classical laminated plate theory (CLPT) [12] ignores the effect of transverse shear deformation and hence becomes inappropriate for thick plate analysis. The first-order shear deformation theory (FSDT) is simple to implement and gives better results than the CLPT, firstly because it requires only C^0 -continuity of generalized displacement fields and secondly, formulation includes transverse shear effects [2]. Also, the computational cost of using the FSDT is lesser than that of the higher-order shear deformation theory (HSDT). The FSDT can be applied for both moderately thick and thin plates. However, it requires the shear correction factors to take into account the non-linear distribution of shear stresses.

It is well known that because of the limitations of analytical approaches, various numerical methods have been developed such as finite-element method (FEM), BEM, smoothed FEM, mesh-free methods, etc. with its own advantages and disadvantages. Among different numerical techniques that seek approximate solutions, the FEM becomes a standard tool for treatment of structural analysis. Most existing finite elements and commercial codes use Lagrangian (C^0 inter-element continuity) and Hermitian (C^1 inter-element continuity) basis functions. The Lagrangian shape functions more commonly used in FEM are lower-order approximation of computer-aided design (CAD) basis function, which leads to less precise geometry. Hence, the requirement of complex geometry design and analysis with the demand of high precision and tighter integration paved the way for the development of new modelling-analysis process.

Ted Blacker from Sandia National Laboratory accounts that about 80% of overall analysis time is consumed for the modelling whereas 20% of overall time is actually devoted for the analysis [13]. This 80 / 20 ratio seems to be very common industrial experience and is therefore one of the major bottlenecks in CAD/computer-aided engineering (CAE)/computer-aided manufacturing (CAM) integration [14]. There is a great demand in industry for the integrated manufacturing process, which incorporates conceptual phase, design by means of CAD and analysis using CAE for the manufacturing done on CNC machines through CAM. CAD and CAM industries rely on the use of NURBS-based

geometry [15, 16] for the shape representation; thus, CAD/CAM integration is relatively straightforward. Specialized CAD/CAM/CAE systems have existed since the last 20–25 years (PTC Creo, CATIA V5, etc.). However, communication between CAD and CAE is tortuous; hence, it is necessary to build a new finite-element model in order to run the analysis.

Hence, for the execution of analysis on geometrical CAD model, Hughes *et al* [13] in 2005 introduced a new technique named isogeometric analysis (IGA) to bridge the gap between CAD and FEA. Instead of Lagrange or Hermite polynomial basis functions, the isogeometric FEM relies on non-uniform rational B-spline (NURBS) basis functions, the same as what almost every CAD or CAM packages do. Based on the isoparametric concept, the NURBS basis function from the CAD technology is employed for both the parameterization of the geometry and the approximation of the plate deformation. The Bezier extraction operator decomposes the NURBS-based elements to C^0 continuous Bezier elements, i.e., linear combinations of Bernstein polynomials that bear a close resemblance to the Lagrange elements. The construction of isogeometric Bezier elements using Bezier extraction operator provides an element structure for IGA [17]. This can be incorporated into existing finite-element codes as a basis for modelling and analysis. IGA has been applied to structural mechanics problem not only for the geometrical accuracy it provides but also for the high quality of stress fields resulting from the use of higher continuous basis function. Recently, several research papers have appeared that used the isogeometric approach for the composite plate and shell analysis [18–20], linear and non-linear elasticity and plasticity problem [21–23] and thermal buckling analysis [24]. The modelling using NURBS provides advantageous properties for structural vibrations problems [25–27], which offers more robust and accurate frequency spectra than those of typical higher-order FE p-methods.

After extensive literature survey, it is observed that the thermo-elastic bending analysis pertaining to IGA-FSDT approach on laminated and sandwich composite plates has not been reported in the literature. Also, the computational efficacy and the extent of accuracy of the isogeometric approach are not discussed in the literature for the present case. In the present investigation, an extensive thermo-elastic bending analysis is carried out using FSDT. NURBS-based codes are developed to study convergence, validation, comparison and computational efficacy of the present approach for thermal loading on laminated and sandwich composite plates. Thermo-elastic numerical procedure based on the framework of IGA is formulated for global stiffness matrix, thermal and mechanical load vector using quadratic, cubic and quartic NURBS elements. Obtained results are compared to other available data to demonstrate the accuracy and effectiveness of the proposed method.

2. Fundamentals of NURBS basis functions and surfaces

The quality of being able to exactly represent the geometry is one of the main reasons why NURBS is widely used in CAD at present. After decades of technology improvement, NURBS provides users with great control over the object shape in an intuitive way with low memory consumption, making them the most widespread technology for geometry representation [15, 16]. In this section, various fundamental components related to B-spline and NURBS like knot vectors, control points and basis function are discussed.

2.1 Knot vectors

A parametric space is partitioned into elements by a knot vector Ξ in each direction, which is a non-decreasing set of coordinates in one dimension [15, 16]. The knot vector is written as

$$\Xi = \{\xi_1, \xi_2, \dots, \xi_j, \dots, \xi_{n+p+1}\} \quad (1)$$

where the length of the knot vector is defined as $|\Xi| = n + p + 1$. Here ξ_j denotes the j^{th} knot, j is the knot index, n is the number of basis functions and p is the polynomial degree. The two main classes of knot vectors are periodic and open; further classification depends on the arrangement of knot values [16].

Open knot vectors are standard in the CAD literature. In one dimension, basis functions formed from open knot vectors are interpolatory at the ends of the parametric space interval, $[\xi_1, \xi_{n+p+1}]$, but they are not interpolatory at interior knots. This is a distinguishing feature between “knots” and “nodes” in finite-element analysis (FEA). With an open, non-uniform knot vector we can attain much richer behaviour in the characteristics of basis functions [28].

2.2 Basis functions

Given a knot vector, the B-spline basis functions $N_{i,p}^b(\xi)$ of degree $p = 0$ are defined as

$$N_{i,0}^b(\xi) = \begin{cases} 1 & \xi_i \leq \xi < \xi_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

The basis functions of degree $p > 0$ are defined by the following Cox–de Boor recursion formula [15]:

$$N_{i,p}^b(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}^b(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}^b(\xi) \quad (2)$$

which means that basis functions are in the parametric form, in contrast with FEA, i.e., the Lagrange polynomials

are explicit functions. Figure 1 illustrates a set of one-dimensional quadratic, cubic and quartic B-spline basis functions for open uniform knot vectors. A B-spline basis function is C^{p-1} continuous at a single knot. A knot value can appear more than once and is then called a multiple knot. At a knot of multiplicity $\hat{k}$, the continuity is reduced to $C^{p-\hat{k}}$.

2.3 NURBS surface

B-spline curves are defined as

$$C(\xi) = \sum_{i=1}^n N_{i,p}^b(\xi) P_i \quad (3)$$

where P_i are the control points and $N_{i,p}^b(\xi)$ is the p^{th} -degree B-spline basis function defined on the open knot vector. The control points can be dealt as analogues to the nodes in FEM, where deformations of the plate, i.e., field variables, are defined and can be thought as generalized coordinates.

B-spline surfaces are defined by the tensor product of B-spline curve in two parametric dimensions ξ and η with two knot vectors, $\Xi = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\}$ and $\mathcal{H} = \{\eta_1, \eta_2, \dots, \eta_{m+q+1}\}$, and expressed as

$$S(\xi, \eta) = \sum_{i=1}^n \sum_{j=1}^m N_{i,p}^b(\xi) M_{j,q}^b(\eta) P_{ij} \quad (4)$$

where P_{ij} is the bidirectional control net and $N_{i,p}^b(\xi)$ and $M_{j,q}^b(\eta)$ are the B-spline basis functions defined on the knot vectors Ξ and $\mathcal{H}$, respectively, over an $n \times m$ net of control points P_{ij} .

The logical coordinate (i, j) of B-spline surface is identically denoted as node “A” in context of FEM [19, 23] and Eq. (4) is rewritten as

$$S(\xi, \eta) = \sum_{A=1}^{n \times m} N_A^b(\xi, \eta) P_A \quad (5)$$

where $N_A^b(\xi, \eta) = N_{i,p}^b(\xi) M_{j,q}^b(\eta)$ is the shape function associated with a control point A. The superscript b indicates that N_A^b is a B-spline shape function.

NURBS curves and surfaces are the generalization of both B-splines and Bezier curves and surfaces. A NURBS entity in $\mathbb{R}^d$ Euclidean space is obtained by projecting a B-spline entity in $\mathbb{R}^{d+1}$, where d is the number of physical dimensions. Weights given to the control points make NURBS curves/surfaces rational, which are additional parameters demonstrating the projection from projective geometry. NURBS basis functions are obtained by augmenting every control point, P_A , in control mesh with the homogeneous coordinate w_A^s , which are scalars, also known as weights. The weighting function is constructed as follows:

Figure 1. Illustration of (a) quadratic, (b) cubic and (c) quartic B-spline basis functions.

$$w_g(\xi, \eta) = \sum_{A=1}^{n \times m} N_A^b(\xi, \eta) w_A^g \quad (6)$$

where $w_g(\xi, \eta)$ is the common denominator function. The NURBS surfaces are then defined by

$$S(\xi, \eta) = \frac{\sum_{A=1}^{n \times m} N_A^b(\xi, \eta) w_A^g P_A}{w_g(\xi, \eta)} = \sum_{A=1}^{n \times m} R_A(\xi, \eta) P_A \quad (7)$$

where $R_A(\xi, \eta) = N_A^b(\xi, \eta) w_A^g / w_g(\xi, \eta)$ is the NURBS basis function.

A null value of w_A^g will not contribute any influence of the polygon vertex on the shape of the corresponding curve. As w_A^g increases, the value of corresponding NURBS basis function increases; however, as a consequence, values of the other basis functions decrease due to the property of partition of unity [15, 16]. Note, in particular, that as w_A^g increases the curve is pulled closer to the corresponding control point. Hence, the homogeneous coordinates provide additional blending capability. The choice of the weight in the NURBS basis function depends on the CAD model considered for the analysis, which can be calculated using Eq. (7) [15, 16]. For rectangular geometry, B-splines basis functions can represent the geometry accurately by considering $w_A^g = 1$ [15, 16], which is a special case of NURBS basis functions.

The B-spline basis (or NURBS) can be enriched by three types of refinements – knot insertion, degree elevation (or

order elevation) and degree and continuity elevation. The first two are equivalent to h - and p -refinement, respectively, while the last one is k -refinement, which does not exist in standard FEM [13].

In IGA, the mesh is an exact encapsulation of the analysis-suitable geometry (ASG) and hence refinement takes place completely within the analysis framework [28]. A finite-element mesh is usually a piecewise polynomial approximation of the ASG. Hence, in FEA, mesh refinement requires interaction with an external description of the geometry (or physical domain) if the quality of the geometric approximation is to be improved simultaneously [28]. The other important feature of the NURBS-based IGA is the ability to provide higher-order and higher-continuity basis functions with less control points [14, 28]. The wide application of IGA technology in the industries is attributed to these smart features.

3. Isogeometric formulation for thermo-elastic bending

3.1 Displacement fields and strains

The basic configuration of the problems considered here is a laminated composite plate with dimensions ($a \times b \times h$) in Cartesian coordinate system ($X - Y - Z$) as shown in figure 2.

Figure 2. Schematic diagram of a laminated composite plate.

Table 1. Elastic moduli of graphite/epoxy lamina at different temperatures [30].

Elastic moduli (GPa)	Temperature (K)					
	300	325	350	375	400	425
E_1	130	130	130	130	130	130
E_2	9.5	8.5	8.0	7.5	7.0	6.75
G_{12}	6.0	6.0	5.5	5.0	4.75	4.5

The plate is made of isotropic or orthotropic laminae. The FSDT is used for the approximation of the displacement fields, and it can be stated in the following form:

$$\begin{aligned} u(x, y, z) &= u_o(x, y) + z\theta_x(x, y), \\ v(x, y, z) &= v_o(x, y) + z\theta_y(x, y), \\ w(x, y, z) &= w_o(x, y). \end{aligned} \quad (8)$$

The strain vector (ϵ) is written in terms of in-plane strain vector $\epsilon_p = [\epsilon_{xx} \ \epsilon_{yy} \ \gamma_{xy}]^T = \epsilon_m + z\epsilon_b$ and transverse shear strain vector $\epsilon_s = [\gamma_{yz} \ \gamma_{xz}]^T$ as

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_p \\ \epsilon_s \end{Bmatrix}, \quad \epsilon_p = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}, \quad \epsilon_s = \begin{Bmatrix} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{Bmatrix} \quad (9)$$

where

$$\epsilon_p = \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} + z \begin{Bmatrix} \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \end{Bmatrix}, \quad \epsilon_s = \begin{Bmatrix} \frac{\partial w_0}{\partial y} + \theta_y \\ \frac{\partial w_0}{\partial x} + \theta_x \end{Bmatrix}.$$

3.2 Constitutive relations

The constitutive relation for k^{th} orthotropic layer in global coordinates with initial thermal strain (ϵ_{th}) for plane stress problem is given by

$$\begin{aligned} \{\sigma\} &= [T_{trans}^{(k)}] [Q] [T_{trans}^{(k)}]^T \{\epsilon - \epsilon_{th}\} = [Q] \{\epsilon - \epsilon_{th}\} \\ \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}^{(k)} &= [T_{trans}^{(k)}] \\ \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0 \\ Q_{21} & Q_{22} & Q_{26} & 0 & 0 \\ Q_{61} & Q_{62} & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{54} & Q_{55} \end{bmatrix} & \begin{Bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^{(k)} - \begin{Bmatrix} \epsilon_{T_{xx}} \\ \epsilon_{T_{yy}} \\ \epsilon_{T_{xy}} \\ 0 \\ 0 \end{Bmatrix}^{(k)} \end{Bmatrix} \end{aligned} \quad (10)$$

where

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}$$

$$Q_{66} = G_{12}, \quad Q_{55} = G_{13}, \quad Q_{44} = G_{23}$$

E_1 and E_2 are Young's modulus; G_{12} , G_{13} and G_{23} are the shear modulus and ν_{12} and ν_{21} are Poisson's ratios; $\{\epsilon_{th}\} =$

Figure 3. Discretization of rectangular plate using IGA (control mesh).

Table 2. Central deflection $\bar{w}$ of isotropic plate for different aspect ratios (a/b) subject to simply supported boundary condition (SSSS-1) under thermal loading.

b/h	Loading	Solution	$a/b = 1$	$a/b = 1.5$	$a/b = 2$	$a/b = 2.5$	$a/b = 3$
10	Case: B	Closed form [29]	0.6586	0.9119	1.0537	1.1355	1.1855
		Present ($p = 2$) †	0.6585	0.9118	1.0537	1.1354	1.1854
		Present ($p = 3$) †	0.6585	0.9118	1.0537	1.1355	1.1854
		Present ($p = 4$) †	0.6585	0.9118	1.0537	1.1355	1.1854
	Case: C	Closed form [29]	0.9575	1.3097	1.4798	1.5582	1.5938
		Present ($p = 2$) †	0.9577	1.3100	1.4803	1.5589	1.5948
		Present ($p = 3$) †	0.9577	1.3100	1.4803	1.5589	1.5948
		Present ($p = 4$) †	0.9588	1.3100	1.4803	1.5589	1.5948

†NURBS-based solution; $\nu = 0.3$ is used for given isotropic material.

$\{\epsilon_{T_{xx}}, \epsilon_{T_{yy}}, \epsilon_{T_{xy}}\}^T$ are the initial thermal strains of a layer, which are expressed as

$$\begin{Bmatrix} \epsilon_{T_{xx}} \\ \epsilon_{T_{yy}} \\ \epsilon_{T_{xy}} \end{Bmatrix}^{(k)} = \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ \alpha_{xy} \end{Bmatrix}^{(k)} \Delta T = \left[\mathcal{T}_{trans}^{(k)} \right]^{-T} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{Bmatrix} (T(x, y, z) - T_{ref}) \quad (11)$$

where α_1 and α_2 are the thermal expansion coefficients of a layer in the material coordinate system. $T(x, y, z)$ is the

realistic temperature distributions for the specified thermal boundary conditions, which becomes part of the input to the present thermal stress analysis. T_{ref} is the reference temperature. $\mathcal{T}_{trans}$ used in Eqs. (10) and (11) is the transformation matrix [29].

In-plane force and moment resultants are expressed as

$$\begin{aligned} \{N_{xx} \ N_{yy} \ N_{xy}\}^T &= \int_{-h/2}^{h/2} \{\sigma_{xx} \ \sigma_{yy} \ \tau_{xy}\} dz, \\ \{M_{xx} \ M_{yy} \ M_{xy}\}^T &= \int_{-h/2}^{h/2} \{\sigma_{xx} \ \sigma_{yy} \ \tau_{xy}\} z dz. \end{aligned} \quad (12)$$

Table 3. Central deflection $\bar{w}$ of orthotropic plate for different aspect ratios (a/b) and side-to-thickness ratios (b/h) subject to simply supported boundary condition (SSSS-1) under thermal load with Material-I.

Loading	b/h	Solution	$a/b = 1$	$a/b = 1.5$	$a/b = 2$	$a/b = 2.5$	$a/b = 3$
Case: B	10	Closed form [29]	1.0440	2.1129	3.0623	3.6394	3.8883
		Present ($p = 2$) †	1.04395	2.11287	3.06229	3.63931	3.88825
		Present ($p = 3$) †	1.04397	2.11291	3.06236	3.63939	3.88833
		Present ($p = 4$) †	1.04397	2.11290	3.06234	3.63937	3.88832
	20	Closed form [29]	1.0346	2.1128	3.0758	3.6560	3.9002
		Present ($p = 2$) †	1.03457	2.11272	3.07571	3.65594	3.90015
		Present ($p = 3$) †	1.03459	2.11277	3.07578	3.65602	3.90024
		Present ($p = 4$) †	1.03458	2.11276	3.07577	3.65601	3.90022
	100	Closed form [29]	1.0312	2.1127	3.0806	3.6619	3.9044
		Present ($p = 2$) †	1.03128	2.11263	3.08022	3.66144	3.90396
		Present ($p = 3$) †	1.03131	2.11272	3.08039	3.66168	3.90424
		Present ($p = 4$) †	1.03131	2.11271	3.08038	3.66167	3.90423
Case: C	10	Closed form [29]	1.4603	3.1321	4.5966	5.4269	5.6987
		Present ($p = 2$) †	1.46063	3.13259	4.59725	5.42775	5.69971
		Present ($p = 3$) †	1.46060	3.13254	4.59721	5.42769	5.69963
		Present ($p = 4$) †	1.46061	3.13255	4.59722	5.42770	5.69965
	20	Closed form [29]	1.4409	3.1339	4.6243	5.4609	5.7239
		Present ($p = 2$) †	1.44121	3.13434	4.62496	5.46175	5.72499
		Present ($p = 3$) †	1.44118	3.13429	4.62491	5.46169	5.72492
		Present ($p = 4$) †	1.44119	3.13430	4.62492	5.46170	5.72493
	100	Closed form [29]	1.4334	3.1343	4.6342	5.4729	5.7327
		Present ($p = 2$) †	1.43379	3.13491	4.63497	5.47384	5.73372
		Present ($p = 3$) †	1.43374	3.13478	4.63478	5.47366	5.73362
		Present ($p = 4$) †	1.43376	3.13478	4.63477	5.47366	5.73364

†NURBS-based solution.

Table 4. Central deflection $\bar{w}$ of anti-symmetric ($0^\circ/90^\circ$) laminated plate for different aspect ratios (a/b) and side-to-thickness ratios (b/h) subject to simply supported boundary condition (SSSS-1) under thermal load with Material-I.

Loading	b/h	Solution	$a/b = 1$	$a/b = 1.5$	$a/b = 2$	$a/b = 2.5$	$a/b = 3$
Case: B	10	Closed form [29]	1.1504	1.4673	1.5186	1.5122	1.4984
		Present ($p = 2$) †	1.15033	1.46725	1.51860	1.51216	1.49834
		Present ($p = 3$) †	1.15036	1.46728	1.51863	1.51219	1.49837
		Present ($p = 4$) †	1.15035	1.46727	1.51863	1.51219	1.49836
	20	Closed form [29]	1.1504	1.4613	1.5091	1.5026	1.4898
		Present ($p = 2$) †	1.15033	1.46124	1.50904	1.50259	1.48976
		Present ($p = 3$) †	1.15036	1.46127	1.50908	1.50263	1.48980
		Present ($p = 4$) †	1.15035	1.46127	1.50907	1.50262	1.48979
	100	Closed form [29]	1.1504	1.4592	1.5058	1.4994	1.4869
		Present ($p = 2$) †	1.15031	1.45912	1.50577	1.49936	1.48688
		Present ($p = 3$) †	1.15036	1.45918	1.50584	1.49943	1.48695
		Present ($p = 4$) †	1.15035	1.45918	1.50583	1.49942	1.48695
Case: C	10	Closed form [29]	1.7213	2.1446	2.1100	1.9862	1.8796
		Present ($p = 2$) †	1.72149	2.14495	2.11065	1.98704	1.88079
		Present ($p = 3$) †	1.72147	2.14492	2.11061	1.98701	1.88080
		Present ($p = 4$) †	1.72148	2.14493	2.11062	1.98703	1.88081
	20	Closed form [29]	1.7269	2.1394	2.0965	1.9703	1.8649
		Present ($p = 2$) †	1.72709	2.13980	2.09705	1.97114	1.86600
		Present ($p = 3$) †	1.72707	2.13977	2.09701	1.97112	1.86604
		Present ($p = 4$) †	1.72708	2.13977	2.09702	1.97114	1.86605
	100	Closed form [29]	1.7293	2.1377	2.0918	1.9649	1.8600
		Present ($p = 2$) †	1.72954	2.13811	2.09236	1.96541	1.86004
		Present ($p = 3$) †	1.72950	2.13805	2.09236	1.96570	1.86099
		Present ($p = 4$) †	1.72950	2.13806	2.09238	1.96576	1.86113

† NURBS-based solution.

3.3 NURBS-based discretization

In the present isogeometric approach, the discretization is based on NURBS basis functions. The displacement fields of plate for quadratic element are approximated as

$$\mathbf{u} = \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_x \\ \theta_y \end{Bmatrix} = \sum_{I=1}^{(p+1) \times (q+1)} \begin{bmatrix} R_I & 0 & 0 & 0 & 0 \\ 0 & R_I & 0 & 0 & 0 \\ 0 & 0 & R_I & 0 & 0 \\ 0 & 0 & 0 & R_I & 0 \\ 0 & 0 & 0 & 0 & R_I \end{bmatrix} \begin{Bmatrix} u_{0I} \\ v_{0I} \\ w_{0I} \\ \theta_{xI} \\ \theta_{yI} \end{Bmatrix}$$

where $(p+1) \times (q+1)$ is the number of basis functions; $R_I(\xi, \eta)$ and $\mathbf{q}_I = [u_{0I} \ v_{0I} \ w_{0I} \ \theta_{xI} \ \theta_{yI}]$ are the NURBS basis functions and the degrees of freedom (DOFs) associated with control point I , respectively.

The in-plane and shear strains for degree $(p, q) = 2$ are written as

$$\begin{aligned} \epsilon_m &= \sum_{I=1}^{3 \times 3} \mathbf{B}_I^m \mathbf{q}_I = \mathbf{B}^m \mathbf{q} \\ \epsilon_b &= \sum_{I=1}^{3 \times 3} \mathbf{B}_I^b \mathbf{q}_I = \mathbf{B}^b \mathbf{q} \\ \epsilon_s &= \sum_{I=1}^{3 \times 3} \mathbf{B}_I^s \mathbf{q}_I = \mathbf{B}^s \mathbf{q} \end{aligned} \quad (13)$$

where $\mathbf{B}$ is a strain-displacement matrix written in terms of NURBS basis function and its first derivatives:

$$\begin{aligned} \mathbf{B}_I^m &= \begin{bmatrix} R_{I,x} & 0 & 0 & 0 & 0 \\ 0 & R_{I,y} & 0 & 0 & 0 \\ R_{I,y} & R_{I,x} & 0 & 0 & 0 \end{bmatrix}, \\ \mathbf{B}_I^s &= \begin{bmatrix} 0 & 0 & R_{I,y} & 0 & R_I \\ 0 & 0 & R_{I,x} & R_I & 0 \end{bmatrix}, \\ \mathbf{B}_I^b &= \begin{bmatrix} 0 & 0 & 0 & R_{I,x} & 0 \\ 0 & 0 & 0 & 0 & R_{I,y} \\ 0 & 0 & 0 & R_{I,y} & R_{I,x} \end{bmatrix}. \end{aligned}$$

First derivatives of NURBS basis functions can be calculated in terms of B-spline basis functions as shown:

$$\frac{dR_i(\xi)}{d\xi} = w_i \left(\frac{1}{w^g} \frac{dN_i^b(\xi)}{d\xi} - \frac{w_{,\xi}^g}{(w^g)^2} N_i^b(\xi) \right)$$

in which

$$w^g = \sum_{j=1}^n w_j N_j^b(\xi), \quad w_{,\xi}^g = \sum_{j=1}^n w_j \frac{dN_j^b(\xi)}{d\xi}$$

where k^{th} derivative of B-spline basis function is expressed as

$$\begin{aligned} \frac{d^k}{d\xi^k} N_{i,p}^b(\xi) &= \frac{p!}{(p-k)!} \sum_{j=0}^k \beta_{k,j} N_{i+j,p-k}^b(\xi) \\ \beta_{0,0} &= 1, \quad k=0, j=0 \\ \beta_{k,0} &= \frac{\beta_{k-1,j}}{\xi_{i+p+j-k+1} - \xi_{i+j}}, \quad j=0 \\ \beta_{k,k} &= \frac{-\beta_{k-1,j-1}}{\xi_{i+p+j-k+1} - \xi_{i+j}}, \quad j=k \\ \beta_{k,j} &= \frac{\beta_{k-1,j} - \beta_{k-1,j-1}}{\xi_{i+p+j-k+1} - \xi_{i+j}}, \quad j=1, \dots, k-1. \end{aligned}$$

3.4 Virtual work principle

In this subsection, equation of equilibrium for thermo-elastic bending problem is derived in the framework of

NURBS-based isogeometric approach. A weak-form of governing equation for composite plate can be obtained by employing the principle of virtual work and is stated as

$$\int_{\Omega} \{\delta \epsilon\}^T \{\sigma\} d\Omega = \int_A \delta w^T P dA \quad (14)$$

where δ is the variation operator and P is the transverse load.

Setting the generalized Hook's law from Eq. (10), which includes the effect of initial strain in constitutive relation, Eq. (14) can be rewritten as

$$\int_{\Omega} \{\delta \epsilon\}^T [\bar{Q}] (\{\epsilon\} - \{\epsilon_{th}\}) d\Omega = \int_A \delta w^T P dA. \quad (15)$$

Substituting Eqs. (9) and (11) in Eq. (15) and integrating along the transverse direction, the weak-form equation becomes

$$\begin{aligned} \int \left\{ \begin{matrix} \delta \epsilon_m \\ \delta \epsilon_b \end{matrix} \right\}^T \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \left\{ \begin{matrix} \epsilon_m \\ \epsilon_b \end{matrix} \right\} dA + \int \{\delta \epsilon_s\}^T [\mathbf{S}] \{\epsilon_s\} dA \\ - \int \left\{ \begin{matrix} \delta \epsilon_m \\ \delta \epsilon_b \end{matrix} \right\}^T \left\{ \begin{matrix} \mathbf{A}^{th} \\ \mathbf{B}^{th} \end{matrix} \right\} dA = \int \delta w^T P dA \end{aligned} \quad (16)$$

in which

Table 5. Central deflection $\bar{w}$ of symmetric ($0^\circ/90^\circ/90^\circ/0^\circ$) laminated plate for different aspect ratios (a/b) and side-to-thickness ratios (b/h) subject to simply supported boundary condition (SSSS-1) under thermal load with Material-I.

Loading	b/h	Solution	$a/b = 1$	$a/b = 1.5$	$a/b = 2$	$a/b = 2.5$	$a/b = 3$
Case: B	10	Closed form [29]	1.0421	1.7130	1.9680	1.9807	1.9227
		Present ($p = 2$) †	1.04204	1.71294	1.96795	1.98069	1.92262
		Present ($p = 3$) †	1.04206	1.71298	1.96799	1.98074	1.92266
		Present ($p = 4$) †	1.04206	1.71297	1.96799	1.98073	1.92266
	20	Closed form [29]	1.0343	1.7339	1.9858	1.9854	1.9193
		Present ($p = 2$) †	1.03426	1.73389	1.98572	1.98539	1.91924
		Present ($p = 3$) †	1.03428	1.73392	1.98576	1.98543	1.91928
		Present ($p = 4$) †	1.03528	1.73392	1.98576	1.98543	1.91927
	100	Closed form [29]	1.0313	1.7419	1.9923	1.9871	1.9181
		Present ($p = 2$) †	1.03127	1.74179	1.99217	1.98700	1.91801
		Present ($p = 3$) †	1.03130	1.74186	1.99226	1.98710	1.91811
		Present ($p = 4$) †	1.03130	1.74185	1.99226	1.98709	1.91810
Case: C	10	Closed form [29]	1.5452	2.5733	2.8961	2.8045	2.5921
		Present ($p = 2$) †	1.54540	2.57361	2.89654	2.80522	2.59296
		Present ($p = 3$) †	1.54538	2.57359	2.89650	2.80517	2.59290
		Present ($p = 4$) †	1.54538	2.57359	2.89651	2.80518	2.59292
	20	Closed form [29]	1.5357	2.6169	2.9352	2.8191	2.5877
		Present ($p = 2$) †	1.53593	2.61723	2.93565	2.81975	2.58850
		Present ($p = 3$) †	1.53590	2.61721	2.93561	2.81970	2.58849
		Present ($p = 4$) †	1.53591	2.61721	2.93562	2.81971	2.58851
	100	Closed form [29]	1.5318	2.6343	2.9500	2.8243	2.5859
		Present ($p = 2$) †	1.53202	2.63462	2.95052	2.82491	2.58676
		Present ($p = 3$) †	1.53197	2.63457	2.95047	2.82488	2.58677
		Present ($p = 4$) †	1.53197	2.63457	2.95048	2.82489	2.58680

†NURBS-based solution.

Figure 4. Transverse thermal deflection ($\bar{w}$) with respect to various aspect ratios (a/b) at a given side-to-thickness ratio ($b/h = 100$) of simply supported (SSSS-1) plate subjected to linearly varying temperature field for degree $p = 2$ along $y = 0.5b$.

Table 6. Non-dimensionalized central deflection $\bar{w}$ of cross-ply square plate subjected to thermal loading Case: C for various boundary conditions with Material-I.

Lamination	b/h	Solution	SSSS	SSSC	SSCC	SSFF	SSFS	CCCC
(0°)	5	Closed form [36]	1.0721	0.7613	0.3915	2.2894	1.5859	–
		Present ($p = 2$) †	1.0720	0.7612	0.3914	2.2893	1.5858	0.3171
	10	Closed form [36]	1.0440	0.5677	0.2912	2.2928	1.5952	–
		Present ($p = 2$) †	1.0439	0.5677	0.2912	2.2928	1.5952	0.2783
$(0^\circ/90^\circ)$	5	Closed form [36]	1.1504	0.8547	0.6231	1.2784	1.2170	–
		Present ($p = 2$) †	1.1503	0.8547	0.6230	1.2784	1.2169	0.3189
	10	Closed form [36]	1.1504	0.7703	0.5307	1.2736	1.2176	–
		Present ($p = 2$) †	1.1503	0.7703	0.5306	1.2735	1.2176	0.3174
$(0^\circ/90^\circ/0^\circ)$	5	Closed form [36]	1.0763	0.8155	0.4578	1.6597	1.3698	–
		Present ($p = 2$) †	1.0762	0.8154	0.4577	1.6596	1.3697	0.3100
	10	Closed form [36]	1.0460	0.6037	0.3211	1.6640	1.3737	–
		Present ($p = 2$) †	1.046	0.6037	0.3210	1.6639	1.3736	0.2813
$(0^\circ/90^\circ)_{10}$	5	Closed form [36]	1.0331	0.8191	0.5847	1.0736	1.0549	–
		Present ($p = 2$) †	1.0331	0.8191	0.5847	1.0735	1.0548	0.2594
	10	Closed form [36]	1.0331	0.7157	0.4949	1.0722	1.0558	–
		Present ($p = 2$) †	1.0331	0.7156	0.4949	1.0721	1.0557	0.2591

†NURBS-based solution.

Table 7. Non-dimensionalized thermal stresses of cross-ply square laminated plate subjected to thermal loading Case: C for various boundary conditions with Material-I.

Stresses	Laminate	b/h	Solution	Mesh	SSSS	SSSC	SSCC	SSFF	SSFS	SSFC	CCCC
$\bar{\sigma}_x$	$(0^\circ/90^\circ/0^\circ)$	5	Closed form [36]	–	0.4072	6.8460	15.6783	1.0220	0.7501	4.5545	–
			Present ($p = 2$) †	16×16	0.4075	6.8463	15.6790	1.0223	0.7504	4.5549	16.187
			Present ($p = 3$) †	14×14	0.4071	6.8459	15.6780	1.0218	0.7499	4.5544	16.187
			Present ($p = 4$) †	6×6	0.4072	6.8460	15.6780	1.0220	0.7500	4.5545	16.187
		10	Closed form [36]	–	0.0847	4.7320	7.7020	0.6103	0.3782	2.8579	–
			Present ($p = 2$) †	16×16	0.0848	4.7321	7.7021	0.6106	0.3784	2.8582	7.9259
			Present ($p = 3$) †	14×14	0.0846	4.7319	7.7019	0.6103	0.3781	2.8579	7.9257
			Present ($p = 4$) †	6×6	0.0846	4.7320	7.7020	0.6103	0.3781	2.8579	7.9258
$\bar{\sigma}_y$	$(0^\circ/90^\circ)$	5	Closed form [36]	–	– 0.6148	1.7502	3.6323	– 1.7733	– 1.2190	1.7814	–
			Present ($p = 2$) †	16×16	– 0.6144	1.7506	3.6327	– 1.7728	– 1.2186	1.7817	15.171
			Present ($p = 3$) †	14×14	– 0.6148	1.7501	3.6322	– 1.7734	– 1.2191	1.7813	15.171
			Present ($p = 4$) †	6×6	– 0.6147	1.7502	3.6323	– 1.7733	– 1.2190	1.7814	15.171
		10	Closed form [36]	–	– 0.3074	1.8991	3.3902	– 1.0399	– 0.7075	1.8168	–
			Present ($p = 2$) †	16×16	– 0.3072	1.8993	3.3904	– 1.0394	– 0.7072	1.8170	7.5364
			Present ($p = 3$) †	14×14	– 0.3074	1.8990	3.3901	– 1.0399	– 0.7075	1.8168	7.5362
			Present ($p = 4$) †	6×6	– 0.3073	1.8991	3.3902	– 1.0399	– 0.7075	1.8168	7.5363

†NURBS-based solution.

$$[A \quad B \quad D] = \int_{-h/2}^{h/2} [1 \quad z \quad z^2] \mathcal{Q} dz, \quad i, j = 1, 2, 6,$$

$$[S] = \int_{-h/2}^{h/2} \kappa \mathcal{Q} dz, \quad i, j = 4, 5,$$

$$[A^{th} \quad B^{th}] = \int_{-h/2}^{h/2} [1 \quad z] \mathcal{Q} \epsilon_{th} dz, \quad i, j = 1, 2, 6.$$

(17)

Now, Eq. (16) is discretized into an isogeometric system, utilizing NURBS as the basis function and substituting strain-displacement relation from Eq. (13). The obtained expression is as follows:

$$\{\delta q\}^T [K] \{q\} - \{\delta q\}^T \{F_{TH}\} = \{\delta q\}^T \{F_M\}.$$

Eliminating the virtual displacement δq , elemental system of equilibrium equations are obtained and can be written in the matrix form as

$$[K] \{q\} - \{F_{TH}\} = \{F_M\} \quad (18)$$

The scalar κ here is the shear correction factor with value equal to $5/6$.

Table 8. Normalized deflection and moment resultants of anti-symmetric ($0^\circ/90^\circ/0^\circ/90^\circ$) square laminated plate at temperature, $T = 400$ K for simply supported (SSSS-1) boundary condition with Material-II and $b/h = 100$.

Parameters	Solution	Point			
		A	B	C	D
$w(mm)$	Closed form [5]	0	0.0085	0.0267	0.0337
	FEM* [30]	0	0.0085	0.0266	0.0337
	Present ($p = 2$) †	0	0.00852	0.02665	0.03369
	Present ($p = 3$) †	0	0.00851	0.02664	0.03369
	Present ($p = 4$) †	0	0.00851	0.02664	0.03369
$M_{xx}(Nmm)$	Closed form [5]	-2.753	-2.518	-1.869	-0.966
	FEM* [30]	-2.796	-2.553	-1.880	-0.970
	Present ($p = 2$) †	-2.7531	-2.5196	-1.8743	-0.9733
	Present ($p = 3$) †	-2.7537	-2.5206	-1.8751	-0.9732
	Present ($p = 4$) †	-2.7537	-2.5206	-1.8752	-0.9734
$M_{yy}(Nmm)$	Closed form [5]	2.753	2.752	2.657	2.237
	FEM* [30]	2.796	2.787	2.690	2.285
	Present ($p = 2$) †	2.7552	2.7471	2.6435	2.2370
	Present ($p = 3$) †	2.7537	2.7459	2.6430	2.2365
	Present ($p = 4$) †	2.7537	2.7460	2.6431	2.2363

*Finite-element solution.

†NURBS-based solution.

where $[K]$ is the stiffness matrix and $\{q\}$ is the displacement vector; $\{F_{TH}\}$ and $\{F_M\}$ are thermal and mechanical load vector, respectively.

The first and second terms of Eq. (16) form an elemental stiffness matrix $[K]$ and can be written as

$$K = \int \left\{ \begin{matrix} B^m \\ B^b \end{matrix} \right\}^T \begin{bmatrix} A & B \\ B & D \end{bmatrix} \left\{ \begin{matrix} B^m \\ B^b \end{matrix} \right\} dA + \int \{B^s\}^T [S] \{B^s\} dA$$

where B^m , B^b and B^s are strain-displacement matrix; A , B , D and S are material rigidity matrix expressed in Eq. (17).

The third term of Eq. (16) leads to the elemental thermal load vector $\{F_{TH}\}$ and can be written as

$$\{F_{TH}\} = \int \left\{ \begin{matrix} B^m \\ B^b \end{matrix} \right\}^T \left\{ \begin{matrix} A^{th} \\ B^{th} \end{matrix} \right\} dA.$$

The material coefficients A^{th} and B^{th} , expressed in Eq. (17), depend upon material properties of lamina, thermal expansion coefficients α_1 and α_2 , ply orientation and temperature difference $\Delta T(x, y, z)$.

Finally, the fourth term of Eq. (16) leads to elemental load vector $\{F_M\}$ due to the transverse mechanical load P and can be written as

$$\{F_M\} = \int \{W\}^T P dA, \\ \{W\} = \{W_1 \ W_2 \ W_3 \ W_4 \ W_5 \ W_6 \ W_7 \ W_8 \ W_9\}, \\ W_I = [0 \ 0 \ R_I \ 0 \ 0], \quad I = 1, \dots, 9.$$

Figure 5. Deflection along x -axis at different temperatures subject to simply supported (SSSS-1) boundary condition of anti-symmetric ($0^\circ/90^\circ/0^\circ/90^\circ$) square laminated plate for side-to-thickness ratio $b/h = 100$.

Global system of equilibrium equations can be obtained by direct assembly of elemental equations (18), which can be written as

$$[K^{global}] \{q^{global}\} = \{F_M^{global}\} + \{F_{TH}^{global}\}. \quad (19)$$

(a) M_{xx} along x-axis. (b) M_{yy} along x-axis.

Figure 6. Moment resultants M_{xx} and M_{yy} along x-axis at different temperatures subject to simply supported (SSSS-1) boundary condition of anti-symmetric $(0^\circ/90^\circ/0^\circ/90^\circ)$ square laminated plate for side-to-thickness ratio $b/h = 100$.

4. Results and discussion

In this section, we present the thermo-elastic bending analyses of laminated and sandwich composite plates using NURBS-based isogeometric approach.

4.1 Material properties

Following sets of material properties are used in this section.

Figure 7. The total simulation time plotted against the total number of elements of anti-symmetric $(0^\circ/90^\circ/0^\circ/90^\circ)$ square laminated plate at temperature, $T = 400$ K for simply supported (SSSS-1) boundary condition with Material-II and $b/h = 100$.

Figure 8. Variation of moment resultants, M_{xy} with fibre orientations at different temperatures of anti-symmetric $(\theta/-\theta/\theta/-\theta)$ square laminated plate under clamped (CCCC) boundary condition for side-to-thickness ratio $b/h = 100$.

- Material-I: (Orthotropic) [29] $E_1 = 25E_2$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$, $\nu_{12} = 0.25$, $\alpha_2 = 3\alpha_1$, $\alpha_1 = 10^{-6}/\text{K}$.
- Material-II: (Temperature-dependent elastic properties) The material properties [30] at the elevated temperatures, shown in table 1, are used for the

Table 9. Moment resultants variation at different temperatures for anti-symmetric ($0^\circ/90^\circ/0^\circ/90^\circ$) square laminated plate with simply supported (SSSS-1) and clamped boundary conditions, Material-II and $b/h = 100$.

Moment (N mm)	Solution	Temperature T (K)					
		300	325	350	375	400	425
M_{xx}	FEM* [30]	0	−0.323	−0.615	−0.876	−1.106	−1.344
	Present ($p = 2$) †	0	−0.3229	−0.6146	−0.8757	−1.1057	−1.3435
M_{yy}	FEM* [30]	0	0.323	0.615	0.876	1.106	1.344
	Present ($p = 2$) †	0	0.323	0.615	0.876	1.106	1.344

*Finite-element solution.

†NURBS-based solution.

Table 10. Non-dimensionalized displacements and stresses for orthotropic, two-layer anti-symmetric and three-layer symmetric cross-ply square laminated plates subjected to non-linear thermal load Case: D, for side-to-thickness ratio $b/h = 10$ with Material-I.

Plate	Solution	$\bar{u}$	$\bar{v}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$
(0°)	Closed form [32]	0.2860	0.3032	1.8520	−1.6323	1.4859	0.9259
	Present ($p = 2$) †	0.28598	0.30323	1.8520	−1.6321	1.486	0.92557
$(0^\circ/90^\circ)$	Closed form [32]	0.2926	0.3325	1.9899	−2.1765	2.1765	0.9820
	Present ($p = 2$) †	0.29260	0.3325	1.98985	−2.1762	2.1762	0.98198
$(0^\circ/90^\circ/0^\circ)$	Closed form [32]	0.2857	0.3001	1.8583	−1.6103	1.4960	0.9203
	Present ($p = 2$) †	0.28576	0.30012	1.85841	−1.6123	1.4959	0.92033

†NURBS-based solution.

analysis of anti-symmetric laminated plate, where $G_{12} = G_{13}$, $G_{23} = 0.5G_{12}$, $\nu_{12} = 0.3$, $\alpha_1 = -0.3 \times 10^{-6}/K$, $\alpha_2 = 28.1 \times 10^{-6}/K$.

- **Material-III:** (Sandwich) [31] Face sheets: $E_1 = 200$ GPa, $E_2 = 1$ GPa, $G_{12} = G_{13} = 5$ GPa, $G_{23} = 2.2$ GPa, $\nu_{12} = 0.25$, $\alpha_1 = -2 \times 10^{-6}/K$, $\alpha_2 = 50 \times 10^{-6}/K$. Core: $E_1 = E_2 = 1$ GPa, $G_{12} = 3.7$ GPa, $G_{13} = G_{23} = 0.8$ GPa, $\nu_{12} = 0.35$, $\alpha_1 = \alpha_2 = 30 \times 10^{-6}/K$.

$$\Delta T(x, y, z) = zT_1 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right). \quad (22)$$

- **Case: D** – Non-linearly distributed temperature across the thickness [32]

$$\Delta T(x, y, z) = zT_1 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) + \left(\frac{1}{\pi}\right) \sin\left(\frac{\pi z}{h}\right) T_2 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right). \quad (23)$$

4.2 Thermal load distributions

Followings are the temperature distribution (ΔT) pattern enlisted as different cases for the analysis:

- **Case: A** – Uniformly distributed temperature [30]

$$\Delta T = \text{constant}. \quad (20)$$

- **Case: B** – Linearly distributed temperature across the thickness and uniformly distributed over the planform [29]

$$\Delta T(x, y, z) = zT_1. \quad (21)$$

- **Case: C** – Linearly distributed temperature across the thickness and sinusoidally distributed over the planform [29]

4.3 Boundary conditions

As the present formulation is based on the displacement approach, it is required to satisfy only the kinematics boundary conditions ($u_0, v_0, w_0, \theta_x, \theta_y$). Different types of boundary conditions that most commonly occur in practice are considered for isogeometric thermo-elastic bending analysis of laminated and sandwich composite plates to assess the efficacy of the present approach.

- Simply supported

1. For cross-ply SSSS-1: $v_0 = w_0 = \theta_y = 0$ at $x = 0, a$ and $u_0 = w_0 = \theta_x = 0$ at $y = 0, b$.
2. For angle-ply SSSS-2: $u_0 = w_0 = \theta_y = 0$ at $x = 0, a$ and $v_0 = w_0 = \theta_x = 0$ at $y = 0, b$.

Figure 9. Variation of in-plane stress $\bar{\sigma}_{xx}$ across the thickness of (a) orthotropic plate, (b) two-layer laminated plate (0°/90°) and (c) three-layer laminated plate (0°/90°/0°) under non-linear thermal loading for aspect ratio $a/b = 10$ and side-to-thickness ratio $b/h = 10$ with Material-I.

- Clamped

1. CCCC: $u_0 = v_0 = w_0 = \theta_x = \theta_y = 0$ at $x = 0, a$ and $y = 0, b$.

4.4 Numerical examples and discussion

This subsection deals with some numerical investigations using NURBS-based elements on the thermo-elastic behaviour of composite plates. It includes convergence, validation and comparison of the present results to analytical results as well as available numerical results. Also, the computational efficacy of isogeometric approach is assessed in reference to the standard finite-element approach. It has been assumed that the thickness and the material for all the layers are the same, unless stated otherwise.

In IGA, each knot span is a physical element where actual integration is carried out. Also, each control point is associated with the NURBS basis function, which makes it shared within the knot spans (elements). The NURBS basis functions ordered from quadratic to quartic were employed. The Gauss–Legendre quadrature rule of integration has been employed in all the analyses with selective integration scheme as in FEM [33–35] to avoid shear locking behaviour. The order of Gauss points $(p + 1) \times (q + 1)$ for bending and $p \times q$ for transverse shear part have been used, where p and q are the polynomial degrees of the NURBS basis functions in x and y directions, respectively. The integration is carried out at each element and assembling is done at the control points, as IGA uses isoparametric mapping.

An 8×8 mesh of NURBS element is used for the present study unless stated otherwise. The discretization detail of the rectangular plate using mesh-size of 5×5 quadratic elements is shown in figure 3. The formulation and accuracy of the present IGA are verified with the closed-form solution and with the available FEM solution.

A number of examples are shown in the subsequent sub-subsections, which include problems of cross-ply and angle-ply laminated and sandwich composite plates. Thermal deflections, stresses and moment resultants are investigated for various side-to-thickness ratios, aspect ratios, boundary conditions, fibre orientations and material properties with the variation of temperature.

4.4a Simply supported isotropic homogeneous plate under thermal load: In order to verify the accuracy of the present technique, uniform and sinusoidal distribution of thermal load [29] over the planform and a linear variation across the thickness have been considered for an isotropic plate as shown in Case: B and Case: C.

An isotropic rectangular plate with $b/h = 10$, under SSSS-1 boundary condition, subjected to different temperature gradients is analysed. The transverse deflection is non-dimensionalized as $\bar{w} = w(10h)/(\alpha_1 T_1 b^2)$. Table 2 presents the effects of aspect ratio (a/b) on steady-state thermo-elastic bending of isotropic rectangular plates. It may be observed in table 2 that with the increase of plate aspect ratio (a/b), transverse deflection increases. The formulation and accuracy of the present IGA approach are verified with the closed-form solution [29] for all aspect ratios and thermal load distributions.

4.4b Simply supported orthotropic plate under thermal load: Steady-state thermo-elastic bending of an orthotropic simply supported rectangular plate is analysed for thermal loading Case: B and Case: C, as elucidated in the previous example in 4.4a. Table 3 presents the effects of aspect ratios (a/b) and side-to-thickness ratios (b/h) on the non-dimensionalized transverse deflection for the orthotropic rectangular plate of Material-I. The present isogeometric results show excellent agreement with the available closed-form solutions [29]. The results also show that for the present plate model, an increase of aspect ratio (a/b) has a greater influence on thermo-elastic response, compared with side-to-thickness ratio (b/h) as shown in table 3. With the increase of aspect ratio (a/b), transverse deflection increases for both types of thermal loading.

4.4c Simply supported anti-symmetric cross-ply laminated plate under thermal load: Steady-state thermo-elastic bending of an anti-symmetric cross-ply ($0^\circ/90^\circ$) simply supported rectangular plate subjected to thermal loading Case: B and Case: C is analysed. Table 4 presents the effects of aspect ratios (a/b) and side-to-thickness ratios (b/h) on non-dimensionalized transverse deflection subjected to thermal load. Unlike the previous example in 4.4b, the increase of transverse deflection with the increase of aspect ratio (a/b) is not monotonic. For instance, in table 4, after initial increase, transverse deflection shows reduction at $a/b \geq 2$ for uniformly loaded plates, while for sinusoidally loaded plates this reduction starts at $a/b > 2$. The accuracy of the present model is validated with the available closed-form solution [29].

4.4d Simply supported symmetric cross-ply laminated plate under thermal load: Steady-state thermo-elastic bending of a simply supported rectangular cross-ply ($0^\circ/90^\circ/90^\circ/0^\circ$) plate subjected to thermal loading Case: B and Case: C is analysed. Table 5 presents the effects of aspect ratios (a/b) and side-to-thickness ratios (b/h) on non-dimensionalized transverse deflection due to thermal load. The same pattern of transverse displacement is observed with the increase of aspect ratio (a/b), i.e., it is not monotonic. For instance in table 5, after an initial increase, transverse deflection shows a reduction at $a/b \geq 2.5$ for uniformly loaded plates (Case: B), while for sinusoidally loaded plates (Case: C) this reduction starts at $a/b > 2.5$. The present model shows excellent agreement with the available closed-form solution [29].

Figure 4 shows the variation of non-dimensionalized transverse thermal deflection for degree $p = 2$, with respect to various aspect ratios (a/b) at a given side-to-thickness ratio ($b/h = 100$) subject to Case: B and Case: C for symmetric and anti-symmetric laminated plates.

4.4e Effect of various boundary conditions on cross-ply square laminated plates: For this example, Material-I has been used for the analysis and the following non-dimensional deflections and stresses have been used throughout the calculations:

$$\bar{w} = w(a/2, b/2) \frac{10}{\alpha_1 T_1 b^2},$$

$$\bar{\sigma}_x = \sigma_x(a/2, b/2, -h/2) \frac{10}{b \alpha_1 T_1 E_2},$$

$$\bar{\sigma}_y = -\sigma_y(a/2, b/2, h/2) \frac{10}{b \alpha_1 T_1 E_2}.$$

The notation SCFS represents that the edge $y = 0$ is simply supported, $x = a$ is clamped, $y = b$ is free and $x = 0$ is simply supported. The non-dimensionalized centre deflections and stresses have been evaluated for thermal load Case: C under various boundary conditions and side-to-thickness ratios (b/h) and results are tabulated in tables 6 and 7. For moderately thick plates, the results predicted by IGA for deflections and axial stresses including all lamination schemes are in excellent agreement with available closed-form results [36].

4.4f Effect of change in temperature and corresponding variation in material properties on anti-symmetric square cross-ply laminate: In this example, the effect of uniform change in temperature (Case: A) with corresponding material properties (Material-II, shown in table 1) on the anti-symmetric square cross-ply laminated plate is analysed, where $\Delta T = T - 300 = \text{constant}$.

For the cross-ply laminated plate with SSSS-1 boundary condition at elevated temperature $T = 400$ K, the solution is verified with closed-form solution [5] and finite-element solution [30]. Table 8 shows normalized transverse deflection and moment resultants (M_{xx} and M_{yy}) at given points A(0.5a, 0.5b), B(0.625a, 0.5b), C(0.75a, 0.5b) and D(0.875a, 0.5b), where 'a' and 'b' are side-lengths as mentioned in figure 2. The obtained isogeometric solutions for normalized deflections and moment resultants are closer to closed-form solution than the available FEM results.

For SSSS-1 and CCCC boundary conditions, the variations of M_{xx} and M_{yy} at different elevated temperatures T are shown in table 9. The obtained isogeometric solutions for moment resultants have been compared to available FEM results [30] and are in good agreement with the same. Monotonically increasing patterns of M_{xx} and M_{yy} are observed with increase in elevated temperature [30].

The plots of deflection w using IGA approach along the x -axis ($0.5a \leq x \leq a$) and y -axis ($y = 0.5b$) with the variation of temperature for SSSS-1 boundary condition are shown in figure 5. The deflection is maximum near the supported edge, i.e., $x = 0.875a$. The plots of moment resultants (M_{xx} and M_{yy}) versus selected points (A–D) using IGA along the x -axis for SSSS-1 boundary condition are shown in figure 6. Both M_{xx} and M_{yy} are maximum at the centre of the plate $x = 0.5a$, having the same value but opposite in sign (figure 6). The plate edges are not found to be free from moments.

To assess the computational efficacy, the variation of total computational time (meshing, assembly and solution time) with respect to total number of elements for both FEA

Figure 10. Schematic diagram of a sandwich plate with face sheets and core.

and IGA is studied as shown in figure 7. Due to the recursive calculation of basis functions, its derivatives and weights, the assembly procedure in the IGA approach is much more time consuming than in the FEA approach. This requirement makes FEA approach initially faster. However, after assembly, IGA approach is faster than FEA because at

the same mesh size IGA requires less control points/DOFs and hence solution time reduces drastically. A comparative study of total computational time versus total number of elements reveals that the IGA approach is more efficient. As total number of elements increases, the total simulation time for the IGA is less than that for the FEA, which is

Table 11. Comparison and convergence of non-dimensionalized deflection of a square sandwich plate subjected to thermal load Case: C for various side-to-thickness (b/h) and core-to-thickness ratios (CTR) using Material-III for simply supported boundary condition (SSSS-1).

CTR (h_c/h)	Solution	Mesh	No. of control points/nodes	Deflection ($\bar{w}$)		
				$b/h = 8$	$b/h = 12$	$b/h = 20$
0.6	FEM*	16×16	1089	11.2165	16.0564	29.7470
	Present ($p = 2$)†	16×16	324	11.2162	16.056	29.7464
	Present ($p = 3$) †	14×14	289	11.2164	c 16.0564	29.7471
	Present ($p = 4$) †	6×6	100	11.2163	16.0562	29.7468
0.8	FEM*	16×16	1089	12.5713	19.4638	39.7818
	Present ($p = 2$) †	16×16	324	12.5710	19.4634	39.7810
	Present ($p = 3$) †	14×14	289	12.5712	19.4638	39.7819
	Present ($p = 4$) †	6×6	100	12.5711	19.4636	39.7815

* Finite-element solution.

†NURBS-based solution.

Table 12. Comparison and convergence of non-dimensionalized deflection of a square sandwich plate subjected to thermal load Case: C for various side-to-thickness (b/h) and core-to-thickness ratios (CTR) using Material-III for clamped boundary condition.

CTR (h_c/h)	Solution	Mesh	No. of control points/nodes	Deflection ($\bar{w}$)		
				$b/h = 8$	$b/h = 12$	$b/h = 20$
0.6	FEM*	16×16	1089	8.78647	11.8111	16.4233
	Present ($p = 2$) †	16×16	324	8.78598	11.8106	16.4230
	Present ($p = 3$) †	14×14	289	8.78619	11.8106	16.4229
	Present ($p = 4$) †	6×6	100	8.78573	11.8095	16.4183
0.8	FEM*	16×16	1089	9.2466	12.7838	19.1504
	Present ($p = 2$) †	16×16	324	9.2461	12.7833	19.1499
	Present ($p = 3$) †	14×14	289	9.2464	12.7834	19.1500
	Present ($p = 4$)†	6×6	100	9.2457	12.7819	19.1449

*Finite-element solution.

†NURBS-based solution.

Figure 11. IGA convergence analysis and comparison of non-dimensionalized centre deflection ($\bar{w}$) of a square sandwich plate for simply supported boundary condition (SSSS-1) using $b/h = 8$ and CTR (h_c/h) = 0.8.

quite evident from figure 7. This important aspect of IGA has a far-reaching impact on the complex real world problem over FEM, where a large number of elements are required. Considering the overall time, to analyse a complicated structure, IGA involves the creation of ASG that exactly represents the features of interest for the calculation [13, 28]. The IGA approach provides considerable time saving, by providing refinement completely within the analysis framework, whereas in FEA, mesh refinement requires interaction with an external description of the geometry. Hence, the present study ascertains the fact that the IGA is faster than the FEA approach.

4.4g Effect of change in temperature and corresponding variation in material properties on anti-symmetric square angle-ply laminate: In this example, the effects of uniform change in temperature (Case: A) with corresponding material properties (Material-II, shown in table 1) are considered for the analysis of anti-symmetric angle-ply ($\theta/ - \theta/\theta/ - \theta$) laminated plate as in the previous example in 4.4f. In this case, zero transverse deflection is observed throughout the plate due to uniform change in temperature. The evaluated values of M_{xy} using IGA are plotted as a function of the fibre orientations θ as shown in figure 8 for CCCC boundary condition, which are in good agreement with the available result [30]. M_{xy} is maximum when the fibre orientation is at 45° . Exactly same result is obtained for M_{xy} under simply supported (SSSS-2) boundary condition.

4.4h Square orthotropic, anti-symmetric and symmetric laminated plates subjected to non-linear thermal load: In this example, displacements and in-plane stresses are determined for orthotropic, anti-symmetric ($0^\circ/90^\circ$) and symmetric ($0^\circ/90^\circ/0^\circ$) square laminated plates using Material-I subjected to non-linear thermal load of Case: D,

where $T_1 = T_2 = 1$ [32]. Results are presented in the following non-dimensionalized forms for the discussion:

$$\bar{u} = u(0, b/2, -h/2) \times \frac{1}{\alpha_1 T_1 a^2},$$

$$\bar{v} = v(a/2, 0, -h/2) \times \frac{1}{\alpha_1 T_1 a^2},$$

$$\bar{w} = w(a/2, b/2, 0) \times \frac{10h}{\alpha_1 T_1 b^2},$$

$$\bar{\sigma}_x = \sigma_x(a/2, b/2, -h/2) \times \frac{1}{\alpha_1 T_1 E_2 a^2},$$

$$\bar{\tau}_{xy} = \tau_{xy}(0, 0, -h/2) \times \frac{1}{\alpha_1 T_1 E_2 a^2},$$

$$\bar{\sigma}_y = \sigma_y(a/2, b/2, -h/2) \times \frac{1}{\alpha_1 T_1 E_2 a^2}, \text{ orthotropic \& 3 layer,}$$

$$\bar{\sigma}_y = \sigma_y(a/2, b/2, h/2) \times \frac{1}{\alpha_1 T_1 E_2 a^2}, \text{ 2 layer.}$$

Non-dimensionalized displacements and stresses for orthotropic, two-layer antisymmetric and three-layer symmetric cross-ply square laminated plates are shown in table 10. The through thickness variation of normal stress $\bar{\sigma}_x$ as shown in figure 9 shows that there is a substantial inter-laminar jump in the value for two-layer anti-symmetric laminate compared to the three-layer symmetric laminate for the side-to-thickness ratio, $b/h = 10$. Deflections and in-plane stresses obtained by present formulation show good agreement with closed-form solution [32].

4.4i Square sandwich plate under sinusoidal linear thermal load: In this example, a square sandwich plate ($0^\circ/\text{Core}/0^\circ$) is taken up for the analysis [31], which is subjected to thermal loading of temperature gradient shown in Case: C. The face sheets (i.e., layers 1 and 3, see figure 10) are assumed to be orthotropic whereas the core material (layer 2) is transversely isotropic (Material-III).

Central deflection subjected to thermal loading has been evaluated using present NURBS-based formulations; a finite-element solution is also obtained for comparison using the method prescribed in literature [37]. Using these formulations, a parametric study has been conducted for both clamped (CCCC) and simply supported (SSSS-1) boundary conditions taking core thickness (h_c) to be $0.6h$ or $0.8h$ where h is the plate thickness. A variation of side-to-thickness ratios, $b/h = 8, 12$ and 20 , has also been considered and presented in tables 11 and 12. The dimensionless deflections are calculated as $(\bar{w}) = w(a/2, b/2)/(\alpha_0 T_0 h^2)$, where $\alpha_0 = 1 \times 10^{-6}/\text{K}$. The present method is capable of predicting the thermo-elastic behaviour of sandwich plate, and deflections obtained by both approaches are found to be in good agreement.

A convergence study based on present NURBS-based approach reveals that quartic element converges faster than the cubic and quadratic elements as shown in figure 11.

Also, achieving the converged results with the increase in NURBS polynomial order from quadratic to quartic requires mesh size reduction (see tables 11 and 12), and hence the control points reduce from 324 to 100. Hence, quartic element converges faster with less control points. This behaviour can be attributed to the fact that quartic elements are less prone to shear locking and have a more stable stiffness matrix than the other two lower-order elements. Also, from tables 11 and 12, it can be observed that for quadratic elements of the same mesh size (16×16), FEM requires 1089 nodes whereas IGA needs only 324 control points for nearly same accuracy. Hence, for the same mesh size, IGA requires less control points than FEA, and therefore less DOFs. Less DOFs means less memory consumption and less memory storage and, hence, this convergence study substantiates the fact that the IGA is cheaper in terms of DOFs for thermo-elastic analysis.

5. Conclusion

A flexible and efficient NURBS-based solution for the thermo-elastic analysis of laminated plates and sandwich structures has been proposed. The variations of central deflection, stresses and moment resultants are investigated. The study investigates convergence, computational efficiency and cost on total DOFs basis for IGA approach, which have been presented exclusively.

To show the efficiency and wider applicability of present approach, several types of examples are carried out. Obtained isogeometric results are closer to closed-form solution in comparison with the finite-element solutions with less number of DOFs. This observation emphatically ascertains that on the total DOFs basis, computational cost is reduced and accuracy is enhanced using the present isogeometric approach.

Several novel results have been presented for the isogeometric thermo-elastic bending analysis of laminated and sandwich composite plates. As no results have been reported in the literature using the present methodology, the gap is rightly filled with the standard solution to provide a reference for the further analysis.

This paper finds IGA to be a very promising alternative to the FEM in the analysis domain of thermo-elasticity. The present thermo-elastic study of laminated and sandwich plate opens up a plethora of scopes, such as analysing stiffened plate incorporating HSSTs in the framework of NURBS based on IGA.

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