Approximate solution for high-order fractional integro-differential equations via trigonometric basic functions

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MS received 5 February 2018; revised 23 October 2018; accepted 24 October 2018; published online 14 March 2019

Abstract. An approach for searching an approximate solution of high-order integro-differential equations featuring fractional derivatives has been proposed in this paper. Initially, using trigonometric basic functions (TBFs), we determine the transformation functions in association to TBFs. Next, the approximate function is presented as a combination of TBFs and transform functions. The convergence of this approach is also presented. Using discrete derivatives of the solution to gain an approximate solution, we find the approximate solution, which satisfies the high-order integro-differential equations featuring fractional derivatives. An algorithm of this approach is applied for various examples, and one example is illustrated in detail.

Keywords. High-order fractional integro-differential equations; trigonometric transform; fractional derivative.

1. Introduction

Calculus and differential equations of non-integer order (or fractional differential equations (FDEs)) have many utilizations in the real world in different branches of sciences and topics of engineering. Some of these applications were offered by Sun and his associates in [1]. These topics may be included in sciences such as physics, biology, environmental science and topics of engineering such as control, signal processing, image processing, mechanics and dynamic systems.

We invite interested readers to check good books that have been written to get a better grasp of calculus with non-integer derivatives and non-integer integrals [2–4].

Consider fractional integro-differential equations (FIDEs) of the form

$$D^\gamma u(\tau) + \eta \int_0^\tau K(s, \tau) H[u(s)] ds = g(\tau), \quad (1)$$

$$M - 1 < \gamma \leq M, \quad 0 < \tau < b, \quad M \in \mathbb{Z}^+,$$

$$u(0) = u_0, \quad u^{(M)}(0) = \theta_M, \quad (2)$$

where $K(s, \tau)$ and $g(\tau)$ are given, $H[u(s)]$ is any linear function, u_0 and θ_M are real constants and D^γ signifies the Caputo derivative of non-integer order γ given by

$$D^\gamma u(\tau) = \frac{1}{\Gamma(m - \gamma)} \int_0^\tau (\tau - s)^{\gamma-1} u^{(M)}(s) ds, \quad (3)$$

$$M - 1 < \gamma \leq M, \quad M \in \mathbb{Z}^+.$$

We shown that by the trigonometric transforms method (TTM), it is possible to get an approximate solution for these FIDEs.

Some kinds of FIDEs are found in mathematical modelling of various real-life phenomena such as in

- **Physics**
 About these phenomena, a diversity of physical problems have been formulated as FIDEs. This kind of equation appears in the radiative equilibrium [5]. One of the different requirements for thermodynamic equilibrium is radiative equilibrium. Heat conduction in materials featuring memory is one example where these equations appear in the mathematical modelling of different physical phenomena [6].
- **Ecology**
 Population dynamics is a branch of ecology that studies the size and age composition of populations as dynamical systems. A population model is a type of mathematical model of FIDE that is utilized in the study of population dynamics [7, 8].
- **Control**
 The topic of optimal control is devoted to the problem of finding a control rule for the given system in such a way as to achieve a certain optimal criterion. The fractional optimal control problems (FOCPs) are optimal control problems (OCPs) where differential equations or implement indicator or both feature derivatives

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of non-integer order. A formulation to FOCPs with derivatives of non-integer order and a solution were introduced in [9]. Rabiei *et al* [10] presented a approach to solve the FOCP described by FIDEs.

Due to the difficulty and impossibility in obtaining precise solutions for a lot of FDEs and FIDEs, some researchers and scholars use numerical or approximate solution approaches to get the solution to these problems. Among the approximate approaches used by some researchers for FDEs and FIDEs are homotopic analysis (HAM) and optimum q -homotopic analysis method (Oq-HAM) [11, 12], optimal homotopic perturbation (OHPM) and homotopic perturbation method (HPM) [13, 14], Adomian's decomposition method (ADM) [15], variational iteration method (VIM) [13, 15] and collocation approach [16, 17], etc. [18, 19].

Perfilieva [20] was able to gain a solution for ordinary differential equations by the trigonometric transforms. In [21, 22] Khastan used the TTM to get an approximate solution for fuzzy differential equations of first order. Tomasiello [23] used the TTM to gain an approximate solution for delay differential equations.

Chen and Shen [24] have utilized an algorithm to get the approximate solutions of initial value problems (IVPs) of second order.

2. Preliminaries

In this section, two subsections have been considered. In the first subsection, discretization of the fractional derivative for different orders is presented. In the second subsection, we present the TBFs, lemmas and theorems to approximate the function of the solution, including convergence of approximate solution function with TBFs.

2.1 Discretization of the derivative of non-integer order

The discretization of derivative of non-integer order is introduced in this subsection. It is possible that by forward differences in finite difference, we gain the approximation of derivatives. Especially, for IVPs it is possible. The k -th order forward difference is given by

$$x^{(k)}(t) \approx \frac{1}{L^k} \sum_{p=0}^k \binom{k}{p} (-1)^p x((k-p)h + t), \quad k \in N,$$

in which L may be constant or variable pertaining to the application. We let $L = s_{j+1} - s_j$ and $s_j = a + jL$, $j = 0, 1, 2, 3, \dots$, for this paper.

Li *et al* [25] obtain the approximation concerning the Caputo derivative. For Eq. (3), regarding the approximation concerning the Caputo derivative and $0 < \gamma \leq 1$

$$D^\gamma u(s_{k+1}) \approx \frac{1}{L^\gamma \Gamma(2-\gamma)} \sum_{j=0}^k (u(s_{j+1}) - u(s_j)) \times \left((k-j+1)^{1-\gamma} - (k-j)^{1-\gamma} \right), \quad (4)$$

in which $u(s_0)$ is known and for $1 < \gamma \leq 2$

$$D^\gamma u(s_{k+1}) \approx \frac{1}{L^\gamma \Gamma(3-\gamma)} \sum_{j=0}^k (u(s_{j+1}) - 2u(s_j) + u(s_{j-1})) \times \left((k-j+1)^{2-\gamma} - (k-j)^{2-\gamma} \right), \quad (5)$$

in which $u(s_{-1}) = u(0) - L \frac{d}{dt} u(0)$, $u(s_0)$ and $\frac{d}{dt} u(s_0)$ are known and for $2 < \gamma \leq 3$

$$D^\gamma u(s_{k+1}) \approx \frac{1}{L^\gamma \Gamma(4-\gamma)} \sum_{j=0}^k (u(s_{j+2}) - 3u(s_{j+1}) + 3u(s_j) - u(s_{j-1})) \times \left((k-j+1)^{3-\gamma} - (k-j)^{3-\gamma} \right), \quad (6)$$

in which $u(s_1) = \frac{d}{dt} u(s_0) - L \frac{d^2}{dt^2} u(s_0)$, $u(s_0)$, $\frac{d}{dt} u(s_0)$ and $\frac{d^2}{dt^2} u(s_0)$ are known.

2.2 Trigonometric basic functions (TBFs)

We present the trigonometric basic functions (TBFs) and properties that are applied in the new approach to the approximate solution, in this subsection.

Definition 1 Consider nodes $a = s_0 < s_1 < \dots < s_{n-1} < s_n = b$ for $n \geq 1$. The basic functions (BFs)

$$S_0, S_1, \dots, S_n,$$

on interval $[a, b]$ are defined with trigonometric functions (TFs) $S_0, S_1(t), \dots, S_n(t)$, as follows, where $h_p = s_{p+1} - s_p$ and $p = 0, 1, 2, \dots, n-1$:

$$S_0(t) = \begin{cases} 0.5 \left(1 + \cos \frac{(t-s_0)}{h_0} \pi \right), & s_0 \leq t \leq s_1 \\ 0, & \text{otherwise,} \end{cases}$$

$$S_k(t) = \begin{cases} 0.5 \left(1 + \cos \frac{(t-s_k)}{h_{k-1}} \pi \right), & s_{k-1} \leq t \leq s_k, \\ 0.5 \left(1 + \cos \frac{(t-s_k)}{h_k} \pi \right), & s_k \leq t \leq s_{k+1}, \\ 0, & \text{otherwise,} \end{cases}$$

$$S_n(t) = \begin{cases} 0.5 \left(1 + \cos \frac{(t-s_n)}{h_{n-1}} \pi \right), & s_{n-1} \leq t \leq s_n \\ 0, & \text{otherwise,} \end{cases} \quad (7)$$

in which $k = 2, 3, 4, \dots, n-1$.

Consider the TFs in Definition 1. They are TBFs whenever the following properties are satisfied:

- 1) Continuity for S_k of $[a, b]$ to $[0, 1]$,
- 2) $\sum_{k=0}^n S_k(t) = 1$, $t \in [a, b]$,
- 3) $S_k(s_k) = 1$, for $k = 0, 1, 2, 3, \dots, n$,
- 4) $S_k(t) = 0$, if $t \notin (s_{k-1}, s_{k+1})$, for $k = 1, 2, \dots, n-1$,
- 5) $S_0(t) = 0$, if $t \notin (s_0, s_1)$,
- 6) $S_n(t) = 0$, if $t \notin (s_{n-1}, s_n)$,
- 7) on interval $[s_{k-1}, s_{k+1}]$, $S_k(t)$, increases function on $[s_{k-1}, s_k]$, $k = 1, 2, 3, \dots, n-1$,
- 8) on interval $[s_{k-1}, s_{k+1}]$, $S_k(t)$, decreases function on $[s_k, s_{k+1}]$, $k = 1, 2, 3, \dots, n-1$,
- 9) $S_k(t + s_k) = S_k(s_k - t)$, $t \in [0, h]$, $k = 1, 2, \dots, n-1$,
- 10) $S_k(t) = S_{k-1}(t - h)$, $S_{k+1}(t) = S_k(t - h)$, $k = 1, 2, \dots, n-1$, $t \in [s_k, s_{k+1}]$.

BFs are named uniform functions with regard to $s_{k+1} - s_k = h = \frac{b-a}{n}$ and properties (9) and (10) coincide.

Lemma 1 [20] Consider $n \geq 2$, $S_0, S_1, \dots, S_n$, are TBFs defined on $[a, b]$. Then we have

$$\int_{s_{k-1}}^{s_{k+1}} S_k(t) dt = h, \quad k = 1, 2, 3, \dots, n-1. \quad (8)$$

The interval between two neighbouring nodes is assumed be h , and

$$\int_{s_0}^{s_1} S_1(t) dt = \int_{s_{n-1}}^{s_n} S_n(t) dt = \frac{h}{2}. \quad (9)$$

Lemma 2 Suppose sinusoidal membership functions for $n \geq 2$, $S_0, S_1, S_2, \dots, S_n$, be the TBFs defined on $[a, b]$. Thus for $k = 0, 1, 2, 3, \dots, n$

$$\int_0^{s_k} S_j(t) dt = \begin{cases} h, & k \geq j+1, \\ 0, & k \leq j-1, \\ \frac{h}{2}, & k = j. \end{cases}$$

Proof When $t \notin (s_{j-1}, s_{j+1})$, from Definition 1, $S_j(t) = 0$. When $k \geq j+1$, from Definition 1 and Lemma 1, the result will be

$$\int_0^{s_k} S_j(t) dt = \int_0^{s_{k-1}} S_j(t) dt + \int_{s_{k-1}}^{s_{k+1}} S_j(t) dt + \int_{s_{k+1}}^{s_k} S_j(t) dt = h.$$

When $k \leq j-1$, $t \leq s_k$; hence, $S_j(t) = 0$ and we get $\int_0^{s_k} S_j(t) dt = 0$.

When $k = j$, from Lemma 2 and Definition 1

$$\int_0^{s_k} S_j(t) dt = \int_0^{s_{k-1}} S_j(t) dt + \int_{s_{k-1}}^{s_k} S_j(t) dt = \frac{h}{2}.$$

Definition 2 Consider intervals $H = C([a, b])$, $I = [a, b]$. Let $f \in H$ and S_j , $j = 0, 1, \dots, n$, be the TBFs defined on

I . We assume that P_j is the transform of function p on I ; considering BFs S_j , $j = 0, 1, 2, 3, \dots, n$, we get

$$P_j = \frac{\int_a^b f(t) S_j(t) dt}{\int_a^b S_j(t) dt}. \quad (10)$$

Definition 3 Consider intervals $H = C([a, b])$, $I = [a, b]$. Let $f \in H$ and S_i , $i = 0, 1, \dots, n$, be the TBFs defined on I and P_k be the transform of function p on I considering BFs S_k . Then

$$P_n(t) = \sum_{k=0}^n P_k S_k(t)$$

is an approximation of function p on I , considering TBFs. For convergence, we have the following theorem.

Theorem 1 Suppose h is a monotonic continuous function on $I = [a, b]$. For all $\epsilon > 0$, there exists n_ϵ such that for any $n \geq n_\epsilon$

$$|h(t) - h_{n_\epsilon}(t)| < \epsilon. \quad (11)$$

Proof For $0 < \delta < \epsilon$, we have $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon)$, $|x - t| < \delta \Rightarrow |h(x) - h(t)| < \epsilon$, because h is a monotonic continuous function on I .

When $\epsilon > 0$

$$\begin{aligned} |h(t) - h_n(t)| &= \left| \sum_{i=0}^n S_i(t) h(t) - \sum_{i=0}^n P_i S_i(t) \right| \\ &\leq \sum_{i=0}^n S_i(t) |h(t) - P_i| < \epsilon. \end{aligned}$$

This is sufficient to show that $|h(t) - P_i| < \epsilon$. Assume $t, x \in [x_{i-1}, x_{i+1}]$, $i = 1, 2, \dots, n-1$. Then

$$\begin{aligned} |h(x) - P_i| &= \left| h(x) - \frac{\int_a^b h(t) S_i(t) dt}{\int_a^b S_i(t) dt} \right| \\ &\leq \frac{\int_{x_{i-1}}^{x_{i+1}} S_i(t) |h(x) - h(t)| dt}{\int_{x_{i-1}}^{x_{i+1}} S_i(t) dt} < \epsilon \\ &\text{iff } \delta < 2h < \epsilon. \end{aligned}$$

Thus, $n_\epsilon > \frac{2(b-a)}{\epsilon}$ is suitable; assume $h = \frac{b-a}{n}$.

Proposition 1 Replace $p_n(t) = \sum_{k=0}^n P_k S_k(t)$ in Eqs. (4)–(6); then for $k = 0, 1, 2, 3, \dots, n-1$

$$\begin{aligned} D^\gamma P_n(s_{k+1}) &\approx \frac{1}{L^\gamma \Gamma(2-\gamma)} \sum_{j=0}^k (P_{j+1} - P_j) \\ &\quad \times \left((k-j+1)^{1-\gamma} - (k-j)^{1-\gamma} \right), \quad (12) \\ 0 &< \gamma \leq 1. \end{aligned}$$

$$D^\gamma p_n(s_{k+1}) \approx \frac{1}{L^\gamma \Gamma(3-\gamma)} \sum_{j=0}^k (P_{j+1} - 2P_j + P_{j-1}) \times \left((k-j+1)^{2-\gamma} - (k-j)^{2-\gamma} \right), \quad (13)$$

$$1 < \gamma \leq 2.$$

$$D^\gamma p_n(s_{k+1}) \approx \frac{1}{L^\gamma \Gamma(4-\gamma)} \sum_{j=0}^k (P_{j+2} - 3P_{j+1} + 3P_j - P_{j-1}) \times \left((k-j+1)^{3-\gamma} - (k-j)^{3-\gamma} \right), \quad (14)$$

$$2 < \gamma \leq 3.$$

3. Description of the new approach

Assume the solution of Eq. (1) to be continuous on $I_1 = [0, b]$. To obtain an approximate solution of $u(x)$, divide I_1 to n equal partitions involving step length L , for $i = 0, 1, \dots, n$:

$$s_0 = 0, \quad s_i = s_0 + iL, \quad L = \frac{b}{n}. \quad (15)$$

Considering the TFs with Definition 1 on I_1 and Definition 3, we get an approximation of function $u(x)$ as $u_n(x) = \sum_{k=0}^n U_k S_k(t)$. Thus, to calculate $u_n(t)$, we should calculate U_k , $k = 0, 1, \dots, n$, when $t \in [0, b]$. To get an approximate solution of Eq. (1), $u_n(t)$ for points $s_0, s_1, \dots, s_n$ must satisfy Eq. (1). Due to the boundary conditions Eq. (2), $u_n(s_0) := u(s_0) = u_0$ and for other points $s_1, s_2, \dots, s_n$, $m-1 < \gamma \leq m$, we have

$$D_t^\gamma u_n(s_{k+1}) + \eta \int_0^{s_{k+1}} K(s_{k+1}, s) u_n(s) ds = g(s_{k+1}), \quad (16)$$

$$k = 0, 1, 2, 3, \dots, n-1,$$

where $m \in \mathbb{Z}^+$.

Regarding Caputo's derivative, using Eqs. (12)–(14), Eq. (16) is converted to the following form for $k = 0, 1, 2, \dots, n-1$:

$$\frac{1}{L^\gamma \Gamma(2-\gamma)} \sum_{j=0}^k (U_{j+1} - U_j) \times \left((k-j+1)^{1-\gamma} - (k-j)^{1-\gamma} \right) + \eta \int_0^{s_{k+1}} K(s_{k+1}, s) u_n(s) ds = g(s_{k+1}), \quad (17)$$

$$0 < \gamma \leq 1,$$

$$\frac{1}{L^\gamma \Gamma(3-\gamma)} \sum_{j=0}^k (U_{j+1} - 2U_j + U_{j-1}) \times \left((k-j+1)^{2-\gamma} - (k-j)^{2-\gamma} \right) \quad (18)$$

$$+ \eta \int_0^{s_{k+1}} K(s_{k+1}, s) u_n(s) ds = g(s_{k+1}),$$

$$0 < \gamma \leq 1,$$

$$\frac{1}{L^\gamma \Gamma(4-\gamma)} \sum_{j=0}^k (U_{j+2} - 3U_{j+1} + 3U_j - U_{j-1}) \left((k-j+1)^{3-\gamma} - (k-j)^{3-\gamma} \right) \quad (19)$$

$$+ \eta \int_0^{s_{k+1}} K(s_{k+1}, s) u_n(s) ds = g(s_{k+1}),$$

$$2 < \gamma \leq 3,$$

where $U_0 := u(0)$, $U_1 := \frac{d}{dt}u(0) - L \frac{d^2}{dt^2}u(0)$ and $U_{-1} := u(0) - \frac{d}{dt}u(0)$ are known initial conditions.

Otherwise, for $\int_0^{s_{k+1}} K(s_{k+1}, s) u_n(s) ds$, we have

$$\int_0^{s_{k+1}} K(s_{k+1}, s) u_n(s) ds = \int_0^{s_{k+1}} K(s_{k+1}, s) \sum_{i=0}^n U_i S_i(s) ds$$

$$= \sum_{i=0}^n U_i \int_0^{s_{k+1}} K(s_{k+1}, s) S_i(s) ds.$$

Let

$$W_i = \int_0^{s_{k+1}} K(s_{k+1}, s) S_i(s) ds, \quad i = 0, 1, \dots, n. \quad (20)$$

Hence $\int_0^{s_{k+1}} K(s_{k+1}, s) u_n(s) ds = \sum_{i=0}^{k+1} U_i W_i$, ($W_i = 0$, for $i > k+1$). Now, using the boundary condition, we may calculate $U_1, U_2, \dots, U_n$ by the obtained recursive equations (17)–(19) and thus for Eq. (1), we may achieve the approximation $u(t) = u_n(t)$.

An algorithm for approximation of FIDEs by TTM is presented as Algorithm 1.

Algorithm 1. An algorithm to approximate solution of high-order FIDEs via TTM.

- Stage 1.** Input $g(t)$, $K(t, s)$, n and b .
Stage 2. Put $L \leftarrow \frac{b}{n}$.
Stage 3. For $k = 0, 1, 2, 3, \dots, n$, set $s_k \leftarrow kL$.
Stage 4. For $k = 0, 1, 2, 3, \dots, n$, select TBFs $S_k(t)$.
Stage 5. For $k = 0, 1, 2, 3, \dots, n-1$ and $i = 0, 1, 2, 3, \dots, k+1$, assume

$$W_i = \int_0^{s_{k+1}} K(s_{k+1}, s) S_i(s) ds. \quad (21)$$

Stage 6. Put
(a) for $m = 1$:

$$\frac{1}{L^\gamma \Gamma(2-\gamma)} \sum_{j=0}^k (U_{j+1} - U_j) \\ \left((k-j+1)^{1-\gamma} - (k-j)^{1-\gamma} \right) \\ + \eta \sum_{i=0}^{k+1} U_i W_i = g(s_{k+1}),$$

where $U_0 = u(0)$.

(b) For $m = 2$:

$$\frac{1}{L^\gamma \Gamma(3-\gamma)} \sum_{j=0}^k (U_{j+1} - 2U_j + U_{j-1}) \\ \left((k-j+1)^{2-\gamma} - (k-j)^{2-\gamma} \right) \\ + \eta \sum_{i=0}^{k+1} U_i W_i = g(s_{k+1})$$

where $U_0 = u(0)$ and $U_{-1} = u(0) - L \frac{du(0)}{dt}$.

c) For $m = 3$:

$$\frac{1}{L^\gamma \Gamma(4-\gamma)} \sum_{j=0}^k (U_{j+2} - 3U_{j+1} + 3U_j - U_{j-1}) \\ \times \left((k-j+1)^{3-\gamma} - (k-j)^{3-\gamma} \right) \\ + \eta \sum_{i=0}^{k+1} U_i W_i = g(s_{k+1}),$$

where $U_0 = u(0)$, $U_1 = \frac{d}{dt}u(0) - L \frac{d^2u(0)}{dt^2}$ and $U_{-1} = u(0) - L \frac{du(0)}{dt}$.

Stage 7. For $p = 1, 2, 3, \dots, n$, from an equation of degree one, calculate every U_p .

Stage 8. The approximate solution for the FIDEs is

$$u_n(t) = \sum_{k=0}^n U_k S_k(t).$$

4. Examples

Various examples in this section are now provided to get familiar with the TTM for high-order FDIEs. The software *Mathematica11* has been utilized in these examples for computations and graphs.

Example 1 We offer the Volterra integro-differential equation for the first example:

$$D^\gamma u(t) - \int_0^t (t+s)u(s)ds = 1 - \frac{(2\gamma+3)t^{\gamma+2}}{\Gamma(\gamma+3)}, \quad (22) \\ 0 \leq t \leq 1, \quad 0 \leq \gamma \leq 1,$$

with the exact solution $u(t) = \frac{t^\gamma}{\Gamma(\gamma+1)}$ and the initial condition

$$u(0) = 0. \quad (23)$$

Conforming to what was mentioned in section 3, utilizing TTM for Eq. (22), we may calculate $U_1, U_2, \dots, U_n$ and then gain the approximate solution $u_n(t)$ for Eq. (22).

The approximate solutions for $\gamma = 0.9$ and $n = 50$ obtained via various values of t in table 1 are illustrated. We may see the exact and approximate solution via $\gamma = 0.9$ and $n = 50$, in figure 1.

Test example 1 with details is illustrated in Appendix I.

Example 2 We offer the Volterra integro-differential equation for the second example:

$$D^\gamma u(t) - \int_0^t u(t)dt = \frac{4^\gamma t^\gamma}{\sqrt{\pi}} - \frac{t^{2\gamma+1}}{2\Gamma(\gamma+\frac{3}{2})}, \quad (24) \\ 0 \leq t \leq 1, \quad 1 \leq \gamma \leq 2,$$

via the exact solution $u(t) = \frac{t^{2\gamma}}{\Gamma(\gamma+\frac{1}{2})}$ and the initial conditions

$$u(0) = 0, \quad \frac{du(0)}{dt} = 0. \quad (25)$$

Table 1. Comparative outcomes of Example 1.

t	TTM	Exact	Absolute error
0.0	0.0	0.0	0.0
0.2	0.242498	0.244263	3.85405×10^{-3}
0.4	0.44815	0.455811	7.66069×10^{-3}
0.6	0.642711	0.656548	13.837×10^{-3}
0.8	0.844837	0.850573	5.73563×10^{-3}
1.0	1.07128	1.03975	31.5248×10^{-3}

Figure 1. Agreement by TTM for Eq. (22) and exact solution.

Figure 2. Agreement by TTM for Eq. (24) and exact solution.

Table 2. Comparative outcomes of Example 2.

t	TTM	Exact	Absolute error
0.0	3.0	3.0	0.0
0.2	2.99983	2.99983	3.04308×10^{-6}
0.4	2.99299	2.99295	35.2995×10^{-6}
0.6	2.94082	2.94063	191.503×10^{-6}
0.8	2.79051	2.79012	386.906×10^{-6}
1.0	2.99898	3.0	1.02379×10^{-6}

The unknown coefficients U_i , $i = 1, 2, \dots, n$ for Eq. (24) are determined by the TTM based on section 3.

In figure 2 and in table 2, the exact and approximate solutions featuring $\gamma = 0.8$ and $n = 500$ through applying TTM can be seen. With the knowledge that $\gamma = 0.8$ and $n = 500$, the approximate solution obtained by the TTM matches the exact solution.

Example 3 We offer the Volterra integro-differential equation for the third example:

$$D^\gamma u(t) + \int_0^t t u(t) dt = \frac{t^{3\gamma+2}}{3\gamma+1} - \frac{t^{6\gamma+2}}{6\gamma+1} - \frac{3\gamma\Gamma(3\gamma)t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{6\gamma\Gamma(6\gamma)t^{5\gamma}}{\Gamma(5\gamma+1)} - 3t^2, \quad (26)$$

$$0 \leq t \leq 1, \quad 1 \leq \gamma \leq 2,$$

including the following initial conditions and the exact solution $u(t) = 3 - t^{3\gamma} + t^{6\gamma}$:

$$u(0) = 3, \quad \frac{du(0)}{dt} = 0. \quad (27)$$

In figure 3 and in table 3, the exact and approximate answers featuring $\gamma = 1.8$ and $n = 500$ through applying TTM, can be seen. For $\gamma = 1.8$ and $n = 500$, the solution that we have gained is in match with the exact solution.

Example 4 We offer the Volterra integro-differential equation, for the fourth example:

Figure 3. Agreement by TTM for Eq. (26) and exact solution.

Table 3. Comparative outcomes of Example 3.

t	TTM	Exact	Absolute error
0.0	3.0	3.0	0.0
0.2	2.99983	2.99983	3.04308×10^{-6}
0.4	2.99299	2.99295	35.2995×10^{-6}
0.6	2.94082	2.94063	191.503×10^{-6}
0.8	2.79051	2.79012	386.906×10^{-6}
1.0	2.99898	3.0	1.02379×10^{-3}

$$D^\gamma u(t) - \int_0^t u(t) dt = \frac{3t(t^3)^\gamma}{6\gamma+2} - \frac{2t^{\frac{3\gamma}{2}+1}}{3\gamma+2} + \frac{3\Gamma(\frac{3\gamma}{2})t^{\gamma/2}}{\Gamma(\frac{\gamma}{2})} - \frac{9\Gamma(3\gamma)t^{2\gamma}}{4\Gamma(2\gamma)} - t, \quad (28)$$

$$0 \leq t \leq 1, \quad 2 \leq \gamma \leq 3,$$

although the initial conditions:

$$u(0) = 1, \quad \frac{du(0)}{dt} = 0, \quad \frac{d^2u(0)}{dt^2} = 0. \quad (29)$$

The unknown coefficients U_i , $i = 1, 2, \dots, n$ with the TTM, matching to section 3 for Eq. (28) are determined. The approximate solutions for $\gamma = 2.3$ and $n = 50$ obtained via various values of t in table 4 are illustrated.

In figure 4, we can confirm the correspondence between the exact answer and the approximate solution with $\gamma = 2.3$.

For $\gamma = 2.3$, the exact solution with $u(t) = 1 - \frac{3t^{3\gamma}}{2} + t^{\frac{3\gamma}{2}}$ and the obtained approximate solution are consistent.

Example 5 We offer the Volterra integro-differential equation for the fifth example:

$$D^\gamma u(t) - \int_0^t u(s) ds = 18t - \frac{t^2}{2} + t^3 - \frac{3t^4}{4} - 6, \quad (30)$$

Table 4. Comparative outcomes of Example 4.

t	TTM	$Exact$	$Absolute\ error$
0.0	1.0	1.0	0.0
0.2	1.00382	1.00385	36.0925×10^{-6}
0.4	1.0395	1.03968	185.174×10^{-6}
0.6	1.12715	1.12745	298.121×10^{-6}
0.8	1.14206	1.14141	641.586×10^{-6}
1.0	0.504706	0.5	4.70572×10^{-3}

Figure 4. Agreement by TTM for Eq. (28) and exact solution.

$$1 \leq \gamma \leq 2, \quad 0 \leq t \leq 1, \quad (31)$$

with the following initial conditions and the exact solution $u(t) = 3t^3 - 3t^2 + t$:

$$u(0) = -6, \quad \frac{du(0)}{dt} = 1. \quad (32)$$

Table 5 shows the approximate solution determined for various values of γ , $n = 500$ and various values of t .

In figure 5, we confirm the correspondence between the exact answer and the approximate solution with $\gamma = 1.1, 1.5, 1.9$ by TTM for $n = 500$.

Table 5. Comparative outcomes of Example 5 with various values of γ and $n = 500$.

TTM				
t	$\gamma = 1.50$	$\gamma = 1.70$	$\gamma = 1.90$	Exact
0.0	0.00	0.00	0.00	0.00
0.2	0.156927	0.241038	0.319699	0.32
0.4	0.567237	0.58592	0.639287	0.64
0.6	1.60812	1.37563	1.19871	1.2
0.8	3.52252	2.87066	2.23804	2.24
1.0	6.51626	5.30314	3.99825	4.0

Figure 5. Comparative outcomes via TTM of Example 5 for various values of t , $n = 500$ and γ .

5. Conclusion

We have prosperously utilized TTM to acquire approximate solutions of the FIDEs. In this paper, initially various real-life phenomena related to FIDEs are introduced. Following this, discretization of the fractional derivative for different orders and the TBFs, lemmas and theorems are presented in section 2, to approximate the function of the solution, including convergence of approximate solution function with TBFs. In section 3, via discretization of the fractional derivative, the approximate solution of FIDEs is determined from a linear combination of BF's with trigonometric transforms.

The results demonstrate that a few iterations of TTM will result in useful approximate solutions.

The suggested technique can be utilized for solving fractional integral equations *FIEs*, fractional integro-partial differential equations *FIPDEs*, *FDEs*, fractional partial differential equations *FPDEs*, fractional differential system equations *FDSEs* and fractional partial differential system equations *FPDSEs*.

Appendix I. Illustration of Example 1 with details

Consider the fractional Volterra integro-differential equation of example 1:

$$D^\gamma u(t) - \int_0^t (t+s)u(s)ds = 1 - \frac{(2\gamma+3)t^{\gamma+2}}{\Gamma(\gamma+3)}, \quad (A1)$$

$$0 \leq t \leq 1, \quad 0 \leq \gamma \leq 1, \quad (A2)$$

with the exact solution $u(t) = \frac{t^\gamma}{\Gamma(\gamma+1)}$ and the initial condition

$$u(0) = 0. \quad (A3)$$

Algorithm 1 is implemented as follows.

Stage 1. Input $g(t) = 1 - \frac{(2\gamma+3)t^{\gamma+2}}{\Gamma(\gamma+3)}$, $K(t, s) = t + s$, $n = 50$ and $b = 1$.

Stage 2. Put $L \leftarrow \frac{1}{50}$.

Stage 3. Set $s_k \leftarrow \frac{k}{50}$, $k = 0, 1, 2, 3, \dots, 50$.

Stage 4. From Eq. (7)

$$S_0(t) = \begin{cases} 0.5(1 + \cos(50\pi t)), & 0 \leq t \leq \frac{1}{50}, \\ 0, & \text{otherwise,} \end{cases}$$

$$S_k(t) = \begin{cases} 0.5(1 + \cos(50\pi(t - s_k))), & s_{k-1} \leq t \leq s_k, \\ 0.5(1 + \cos(50\pi(t - s_k))), & s_k \leq t \leq s_{k+1}, \\ 0, & \text{otherwise,} \end{cases}$$

$$S_{50}(t) = \begin{cases} 0.5(1 + \cos(50\pi(t - 1))), & \frac{1}{50} \leq t \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $k = 2, 3, 4, \dots, 49$.

Stage 5. Consider

$$W_i = \int_0^{s_{k+1}} (s_{k+1} + s) S_i(s) ds,$$

for $k = 0, 1, \dots, 49$ and $i = 0, 1, \dots, k + 1$.

Stage 6. Construct recursive equations

$$\frac{1}{(\frac{1}{50})^\gamma \Gamma(2 - \gamma)} \sum_{j=0}^k (U_{j+1} - U_j) \times \left((k - j + 1)^{1-\gamma} - (k - j)^{1-\gamma} \right) + \eta \sum_{i=0}^{k+1} U_i W_i = 1 - \frac{(2\gamma + 3)(s_{k+1})^{\gamma+2}}{\Gamma(\gamma + 3)}.$$

where $U_0 = u(0) = 0$.

Stage 7. For $p = 0$, we have

$$\frac{1}{(\frac{1}{50})^\gamma \Gamma(2 - \gamma)} (U_1 - U_0) + \eta U_0 W_0 = 1 - \frac{(2\gamma + 3)(s_1)^{\gamma+2}}{\Gamma(\gamma + 3)},$$

Hence we may calculate U_1 . For $p = 2, \dots, n$, we may calculate every U_k by solving an equation of degree one.

Stage 8. The approximate solution is

$$u_{50}(t) = U_0 S_0(t) + U_1 S_1(t) + \dots + U_{50} S_{50}(t).$$

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