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A Russell-based model for estimating overall and divisional efficiency in two-stage production systems with sets of convex hulls in intermediate products

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Abstract. Data Envelopment Analysis (DEA) research works have been recently examining production systems with a two-stage network structure. These studies consider system operations that are performed in two stages where intermediate products play dual roles: they are the outputs of the previous stage and inputs to the next stage. This dual role is incompatible with Pareto–Koopmans dominance in activity analysis. Also, disregarding intermediate products in assessing performance of two-stage systems compromises the models. The present work introduces a new production possibility set for two-stage network production systems by considering a convex hull for intermediate products. In addition, new models are introduced for evaluating overall efficiency and divisional efficiency of production units from the Pareto–Koopmans efficiency perspective. The proposed models are developed based on an enhanced Russell graph model for efficiency evaluation of two-stage network production structures with convex hulls for intermediate products. Numerical examples are further provided for illustration purposes.

Keywords. Data Envelopment Analysis (DEA); two-stage network; intermediate products; convex hull; overall efficiency.

1. Introduction

Data Envelopment Analysis (DEA) is a popular method for evaluating relative efficiency of a set of similar Decision-Making Units (DMUs). DEA methods are especially effective in evaluating efficiency in the presence of multi-input and multi-output variables. As an example, a model introduced by Charnes *et al* [1] has been extended to apply to different theoretical and practical contexts [2]. DEA has further been utilized in interpreting productivity of complex engineering–economic systems [3–13].

Early DEA methods were utilized to measure efficiency regardless of the internal structure of system operations. However, researchers in the last two decades have paid more attention to system operation in order to investigate inefficiency. The first study on a two-stage network structure using DEA was reported by Charnes *et al* [14] with an application to army recruitment system. The same two-stage network model was then used by many other researchers [15–17].

More recently, several models have been proposed for improving efficiency measurement in two-stage network systems: Wang *et al* [18] introduced a two-stage method utilizing Variable Returns to Scale (VRS), where each stage was characterized by independent variables and intermediate products. Rho and An [19] considered slack variables in a model that assessed DMUs with weak efficiency. Kao and Hwang [20] examined the possibility of decomposition in overall system efficiency by considering weights for intermediate products. Liang *et al* [21] presented a procedure to test the uniqueness of efficiency decomposition. Tone and Tsutsui [22] introduced models that used a production possibility set (PPS) based on a corresponding slack variable.

Chen *et al* [23] provided a new method for determining efficient projections for inefficient DMUs. They developed an additive efficiency decomposition approach by expressing the overall efficiency as a weighted sum of efficiencies of individual stages. This approach was applicable to both Constant Returns to Scale (CRS) and VRS situations [24]. Chen *et al* [25, 26] prove that the CRS version of the Chen and Zhu model is equivalent to the output-oriented Kao and Hwang approach [20]. Wang and Chin [27] proposed alternative two-stage network DEA models to show that the overall efficiency of two-stage process could also be formulated as weighted harmonic mean efficiencies of two individual stages.

Liu proposed a two-stage DEA model capable of calculating sub-process and overall process efficiencies

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simultaneously. In a previous work, Kao and Hwang [20] proposed a model where overall efficiency of the whole process had to be calculated prior to computing sub-process efficiency [28]. Du *et al* [29] proposed a Nash bargaining game model to measure the performance of DMUs with a two-stage structure. Li *et al* proposed two new two-stage DEA models, including a nonlinear centralized model and a non-cooperative model. Li *et al* [21] extended the Liang *et al* [21] model by assuming that input to second stage comprised the output from the first stage and additional exogenous input. Yu *et al* [30] recognized a deficiency in the Wang and Chin [27] model and proposed a novel method for transforming a two-stage process into a one-stage process consisting of two independent parallel sub-processes.

Lewis *et al* [31] used an iterative approach to solve a non-oriented two-stage DEA model using a classical radial objective approach that reduced input quantities and increased output quantities simultaneously. Yu and Shi [32] introduced a two-stage DEA model where intermediate products were freely distributed. Ahmadzadeh *et al* [33] evaluated the efficiency of a two-stage network structure by utilizing Charnes–Cooper transformation for linearization of fractional program.

Maghbouli *et al* [34] formulated a two-stage DEA model with undesirable intermediate measures. They proposed cooperative and non-cooperative game theories to assess the relative performance of operational units. Liu *et al* [35] investigated two-stage DEA models with undesirable input–intermediate output and used free-disposal axiom to construct a PPS. Jianfeng proposed a two-stage DEA model by considering the structure of input and intermediate products measures in efficiency evaluation and decomposition simultaneously [36].

Lim and Zhu [37] formulated equations to obtain frontier projections and divisional efficiency scores through a two-stage network DEA. Mahdiloo *et al* [38] proposed a multiple-criterion two-stage DEA model to find more realistic weights for input and output with better discriminating power than that of traditional two-stage DEA models. Guo *et al* [39] investigated factors involved in the overall efficiency variations of two-stage additive network DEA approach and examined the relationship of overall efficiency with stage efficiencies and weights. Wanke *et al* [40] proposed a new combined model based on directional distance functions, which improved the accuracy of efficiency scores under CRS production technologies in two-stage network DEAs.

The main contributions of the referenced models are their respective improvements to measuring efficiency in a two-stage network system. However, they all suffered deficiencies. For instance, Chen *et al* [41] reported limitations in efficiency measurements related to the different behaviours occurring due to intermediate products. Another researcher, Amir Teimoori [42], considered the presence of perfect and imperfect outputs in a two-stage decision process.

Ignoring intermediate products in assessing the performance of a two-stage network leads to implausible results, thus making it difficult for operators to determine the causes of inefficiencies. Furthermore, some envelopment models used for evaluating overall efficiency are nonlinear programming problems that make efficiency estimation more difficult and less accurate. Therefore, introduction of a method for linearizing models and then exploring a relation between overall (system) efficiency and divisional efficiency of their stages is an important challenge in using nonlinear envelopment models in two-stage production networks.

An important issue being treated in this study is the dissimilar and inconsistent roles of intermediate products in each stage of a two-stage network system. Intermediate products are outputs of a first stage and inputs to a second stage with free disposability assumptions for both stages. This dual-role behaviour imposes a unique structure on production sets in intermediate products. As an example, when being an output, associated production sets are bounded while as an input to the next stage, the same production sets are assumed to be unbounded. This is indicative of an axiomatic foundation of intermediate production sets in network production systems. There is clearly a need for fundamental modifications to the treatment of intermediate products.

It may be claimed that a different behaviour of intermediate products in many two-stage network models is one of the most significant and largely overlooked issues in this art. In addition, no compatibility exists between the two roles described (i.e., two different types of disposability) for a product in a system with the concept of dominance (Pareto–Koopmans efficiency). Thereby, it becomes necessary to revise the model constraints associated with intermediate products in both segments of two-stage network models. This is to comply with the principle of mathematical dominance, which is used to evaluate the system effectiveness.

A bounded production set for intermediate products is applied as per Soltanifar *et al* [43] using a convexity assumption. As a result, a new set of available intermediate products is introduced, which addresses the afore-mentioned issues.

The purpose of this study is to re-examine the disposability axioms of intermediate products to match management concepts. To this end, the scheme proposed by Ahmadzadeh *et al* [33] and Soltanifar *et al* [43] is utilized in envelopment-based network models. Thus, by focusing on the convexity axiom and the principle of mathematical dominance, a new PPS in two-stage systems is introduced that uses a convex hull for intermediate products. A new enhanced Russell graph measure to evaluate overall efficiency of units in a two-stage network structure is then proposed.

Furthermore, a method for linearizing the proposed model is provided based on Charnes–Cooper transformation. A unique approach for determining divisional efficiency in the new two-stage network set is also produced by formulating a mathematical relation between the efficiency of stages and their corresponding overall efficiency score. A numerical example is presented to demonstrate the proposed models.

The presented work is organized in seven sections. Section 2 briefly reviews related literature on two-stage network DEA. A new approach of using convex hull set for intermediate products is explored in section 3 and a related technology set (T_{CHI}) is introduced. Section 4 discusses a novel Russell-based model to compute overall efficiency score in the proposed PPS. Section 5 is devoted to introducing a new approach for determining divisional efficiencies in T_{CHI}. An illustrative example is presented in section 6, and section 7 presents the conclusion as well as some suggestions for future research.

2. Two-stage production systems and intermediate products in DEA literature

Two different approaches are commonly used in evaluating efficiency of two-stage systems in conventional DEA models.

The first approach estimates efficiency of each division based on the relative efficiency ratio of units. Multiplier-based network DEA models are then derived as per this approach. Estimation of overall and divisional efficiency is generally done using multiplier-based network models. Researchers have presented a wide range of these types of models.

Kao and Hwang [20] calculated efficiency scores of stages one by one under a CRS assumption. Overall efficiency score was then obtained by considering a series relationship between stages as the product of stage efficiencies. It is noted that intermediate product weights are equal in both stages irrespective of the role of intermediate products being an output or an input. Kao and Hwang additionally provided a method for calculating overall efficiency scores of units under VRS [4]. Models were introduced in their studies for calculating divisional efficiency scores and overall efficiency.

The second method utilizes a PPS scheme to measure the efficiency of each division. Later, envelopment models are derived with a two-stage network structure.

Envelopment-based network models are used for determining projections on the efficiency frontier and for calculating overall and divisional efficiencies. Many models have been proposed in this category. Chen *et al* [24] calculated the overall efficiency in a two-stage system using specific weights in the objective function. Chen *et al* [25] also introduced a radial version of the envelopment-based network model to compute input-oriented CRS overall efficiency for the units. Chen *et al* [23] identified the DEA frontier for two-stage processes by imposing redundant constraints for intermediate products in the envelopment-based network model. Tone and Tsutsui [22] introduced slack-based network DEA models using PPS and explored several models based on intermediate products as fixed or free links.

How intermediate products in a two-stage system are treated and introduced within performance evaluation has a direct impact on results. Wang *et al* [18] presented separate models for evaluating efficiency of a two-stage network so that intermediate products at each stage are independent of the other stage. Chen and Zhu [26] proposed a two-stage model to maintain dependence between stages. To this end, they treated intermediate products as variables in evaluation so that the variables are identical in both stages. However, their model could not distinguish efficient units from other units. Rho and An [19] suggested a solution to this problem by considering slacks in the constraints of intermediate products.

Kao and Hwang [20] assumed the same weights for intermediate products in both stages for evaluating overall efficiency of a two-stage network. Tone and Tsutsui [22] examined intermediate products in two different fashions by considering the axiomatic foundation of a two-stage network. Tone and Tsutsui [22] referred to these products for evaluating performance as links in the form of fixed and free links. In general, most two-stage network envelopment models consider intermediate products as outputs of a first stage and inputs to the second stage with strong disposability.

Internal operation of the system is the strength in these models; however, models introduced in two-stage networks have considerable deficiencies. Some of these limitations may be due to discrepancy in the variations of utilized concepts of efficiency. Chen *et al* [41] described some of these limiting inadequacies in detail. As an example, a common limitation of two-stage network DEA models may be the employment of envelopment models for calculating divisional efficiency. It is known that the result of overall efficiency in envelopment models is equal to the efficiency of the first stage. Additionally, multiplier and envelopment models are not necessarily dual models in a two-stage network unlike conventional DEA models. Furthermore, these models demonstrate inadequacies in determining efficient frontier, projection of inefficient DMUs and divisional efficiencies.

Emerged issues could be due to the different nature of particular production sets. It must be noted that a PPS considers intermediate products in two different roles under free disposability assumption. This is incompatible with the concept of mathematical dominance as discussed earlier. Therefore, results of these models cannot directly be used for performance measurements of production units and need to be modified.

The next section introduces a novel approach for imposing a uniform behaviour on intermediate products and for estimating overall and divisional efficiencies.

3. Convex hull set for intermediate products in two-stage production systems

Separate axioms are proposed for each stage in a two-stage network structure in this section. A new PPS is formed for a two-stage network DEA with convex hull set in the intermediate products with employment of these axioms. Relevant properties are then described.

Assume a set of DMUs containing a two-stage internal system where all outputs from the first stage are the only inputs to the second stage. Figure 1 illustrates a two-stage network structure for members of a set of n DMUs [26].

The Kao and Hwang [20] model is applied to explain the main concepts of two-stage DEA models. For each DMU_j $(j = 1, 2, \ldots, n)$ in the first stage, x_{ij} $(i = 1, 2, \ldots, m)$ inputs are used to produce a set of D intermediate products $z_{dj}(d = 1, 2, \ldots, D)$. In the second stage, all outputs of the first stage, namely z_{dj} $(d = 1, 2, \ldots, D)$, are used to produce the final outputs y_{rj} $(r = 1, 2, \ldots, s)$ where $\boldsymbol{x} = (x_1, \ldots, x_m) \in R^m_+, \boldsymbol{z} = (z_1, \ldots, z_D) \in R^D_+$ and $\boldsymbol{y} = (y_1, \ldots, y_S) \in R^S_+$ represent the input vector, intermediate products vector and output vector, respectively. It follows that axioms of "observations inclusion", "convexity" and "strong disposability of inputs" for a set of *DMUs* in the first stage exist. Thus, the PPS that satisfies these axioms is

$$T_1 = \left\{(\boldsymbol{x}, \boldsymbol{z}) : \sum_j \lambda_j \boldsymbol{x}_j \le \boldsymbol{x}, \sum_j \lambda_j \boldsymbol{z}_j = \boldsymbol{z}, \sum_j \lambda_j = 1, \lambda_j \ge 0; j = 1, \ldots, n\right\}, \tag{1}$$

where variable $\lambda \in R^n$ is the vector of intensity variables of the first stage.

Theorem 1 *Technology set of the first stage* T_1*, defined by Eq. (1), is the minimal set that contains all observations and satisfies the axioms of convexity and strong disposability of inputs.*

Proof of Theorem 1 is provided in the appendix.

Axioms of "observations inclusion", "convexity" and "strong disposability of outputs" are postulated for the PPS of the second stage. Hence, the associated production set T_2 is defined as follows:

$$T_2 = \left\{(\boldsymbol{z}, \boldsymbol{y}) : \sum_j \mu_j \boldsymbol{z}_j = \boldsymbol{z}, \sum_j \mu_j \boldsymbol{y}_j \ge \boldsymbol{y}, \sum_j \mu_j = 1, \mu_j \ge 0; j = 1, \ldots, n\right\}, \tag{2}$$

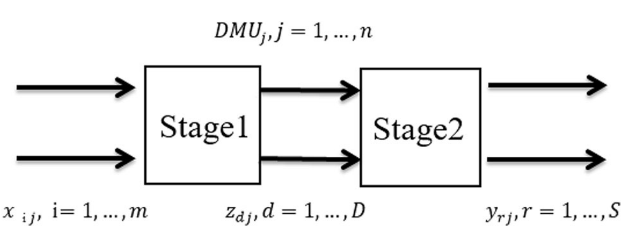

Figure 1. A two-stage production process where each DMU_j (j = 1, 2, …, n) has m inputs, S outputs and D intermediate measures.

where the variable $\mu \in R^n$ denotes the intensity levels of DMUs for the second stage.

Theorem 2 *The second-stage technology set* T_2*, defined by (2), is a minimal set that contains all observations and satisfies the axioms of convexity and strong disposability of outputs.*

Proof of Theorem 2 is identical to the proof of Theorem 1; hence it is not repeated.

According to proposed axioms for each stage, and under VRS assumption, the overall PPS for a two-stage network with convex hull in intermediate products may be stated as follows:

$$T_{CHI} = \left\{(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y}) : \sum_j \lambda_j \boldsymbol{x}_j \le \boldsymbol{x}, \sum_j \lambda_j \boldsymbol{z}_j = \boldsymbol{z}, \sum_j \mu_j \boldsymbol{z}_j = \boldsymbol{z}, \sum_j \mu_j \boldsymbol{y}_j \ge \boldsymbol{y}, \sum_j \mu_j = 1, \sum_j \lambda_j = 1, \lambda_j \ge 0, \mu_j \ge 0, j = 1, \ldots, n\right\}, \tag{3}$$

where *CHI* represents a set of convex hulls for the intermediate products.

It should be noted that the intermediate products in T_{CHI} are examined by two separate sets of intensity weights $\boldsymbol{\lambda} \in R^n$ and $\boldsymbol{\mu} \in R^n$. Thus, λ determines the relationship between inputs and intermediate products as μ determines the relation between intermediate products and final outputs. The main difference between the technology expressed in (3) and conventional technology of a two-stage network is that the former allows inequality constraints for the intermediate products while the latter approach suggests equality constraints.

In the described technology, produced output ratio in the first stage is equal to consumed input ratio in the second stage. Therefore, access to resources is restricted and bounded. Thereby, the PPS generated by technology set (3) becomes a subset of the conventional two-stage network PPS.

An illustrative example is presented to compare the overall efficiency and frontier of each stage of a typical two-stage production system using both the proposed and conventional approaches under the conditions of CRS and VRS.

Example 1. Consider a system with four DMUs. Each DMU has one input, one output and one intermediate measure. Table 1 presents the data set.

The overall production technology and PPS for each stage may be clearly seen in figures 2–4. Bold lines in figure 2 represent the production frontier of stage 1 (T_1). Note that the efficiency frontier has been expanded by the convex hull of the observations and strong disposability in inputs. Bold lines in figure 3 represent the production frontier of stage 2 (T_2), where frontier points are obtained with convex hull of the observations and strong disposability axiom in outputs. The thin black lines and dotted lines in figures 2 and 3 represent the conventional

Table 1. Data set of example 1.

dmu	x	z	y
a	1	2	4
b	2	1	1
c	5	4	2
d	2	2	2

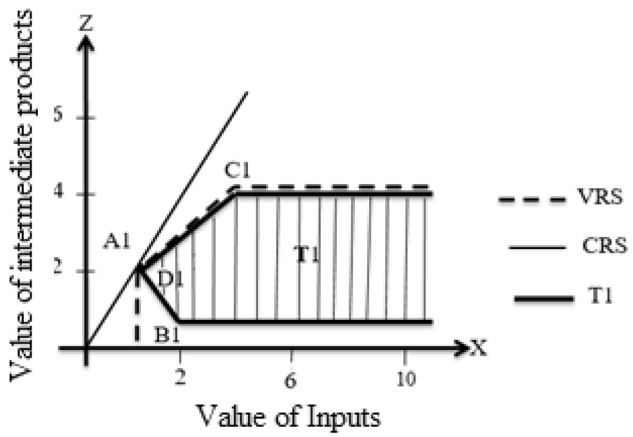

Figure 2. Histogram showing efficient frontier of first stage generated by a two-stage system with the convex hull in intermediate products (T_1). Conventional efficiency frontier is also displayed under the assumptions of CRS and VRS in the first stage.

efficiency frontier, under the assumptions of CRS and VRS, respectively. Clearly, the PPS with the convex hull in the intermediate products is a subset of the PPS under both CRS and VRS assumptions. Note that units $A1$–$C1$ in T_1 and units $A2$–$C2$ in T_2 belong to boundary of PPS.

Figure 4 illustrates the 3-D network technology of a convex hull in intermediate products. Note that the

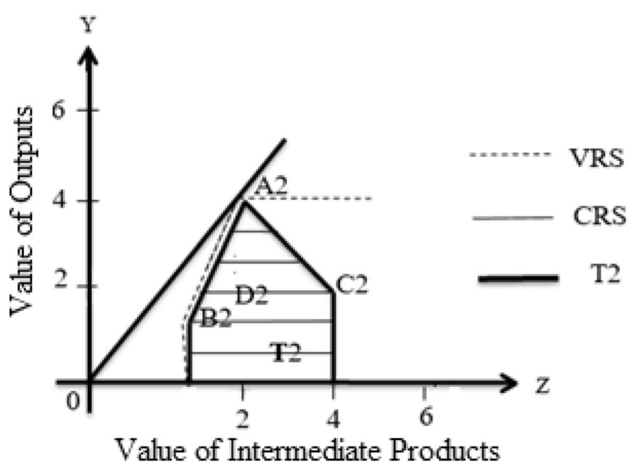

Figure 3. Histogram showing efficient frontier of second stage generated by a two-stage system with convex hull in intermediate products (T_2). Conventional efficiency frontier is also shown under the assumptions of CRS and VRS in the first stage.

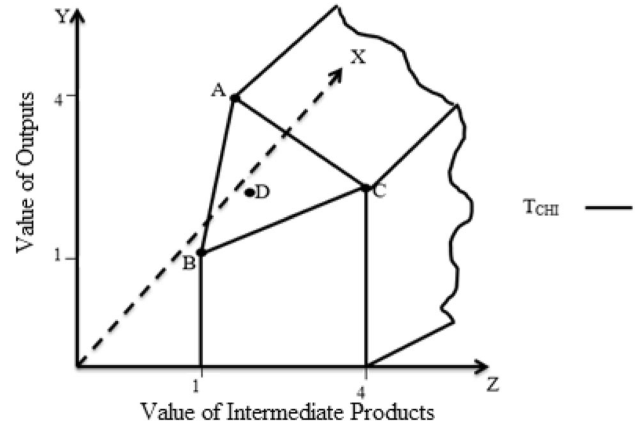

Figure 4. Histogram showing efficient frontier generated by T_{CHI}.

intermediate products in T_{CHI} are bounded, and units A–C are located on the boundary of T_{CHI}.

Selecting a suitable model for efficiency evaluation would possibly further improve the units on the boundary. More details are presented in the next section.

4. Determining overall efficiency in T_{CHI}

A new model for evaluating the overall efficiency of production units in T_{CHI} is introduced in this section that utilizes a modified version of the enhanced Russell graph model. A new suggested network model for evaluating overall efficiency of DMU_O in T_{CHI} is given as follows:

$$E_o = \min\left(\hat{w}_1 E_o^1 + \hat{w}_2 E_o^2\right) = \min\left(\hat{w}_1 \frac{\frac{1}{m}\sum_{i=1}^{m}\theta_i}{\frac{1}{D}\sum_{d=1}^{D}\varphi_d} + \hat{w}_2 \frac{\frac{1}{D}\sum_{d=1}^{D}\bar{\varphi}_d}{\frac{1}{s}\sum_{r=1}^{s}\beta_r}\right) \quad (4)$$

$$\begin{aligned}
\text{s.t.} \quad & \textstyle\sum_{j=1}^{n}\lambda_j x_{ij} \le \theta_i x_{io}, && i = 1,2,\ldots,m \\
& \textstyle\sum_{j=1}^{n}\lambda_j z_{dj} = \varphi_d z_{do}, && d = 1,2,\ldots,D \\
& \textstyle\sum_{j=1}^{n}\mu_j z_{dj} = \bar{\varphi}_d z_{do}, && d = 1,2,\ldots,D \\
& \textstyle\sum_{j=1}^{n}\mu_j y_{rj} \ge \beta_r y_{ro}, && r = 1,2,\ldots,S \\
& \textstyle\sum_{j=1}^{n}\mu_j = \sum_{j=1}^{n}\lambda_j = 1, && \\
& \lambda_j \ge 0, \mu_j \ge 0, && j = 1,2,\ldots,n \\
& \theta_i \le 1, && i = 1,2,\ldots,m \\
& \varphi_d \ge 1, && d = 1,2,\ldots,D \\
& \beta_r \ge 1, && r = 1,2,\ldots,S \\
& \bar{\varphi}_d \le 1, && d = 1,2,\ldots,D
\end{aligned}$$

where $\theta_i, \varphi_d, \bar{\varphi}_d, \beta_r, \lambda_j, \mu_j$ are decision variables and $\hat{w}_1 > 0$ and $\hat{w}_2 > 0$ are weights such that $\hat{w}_1 + \hat{w}_2 = 1$, and they may be defined as a constant or a variable; θ_i denotes contraction coefficients of inputs and β_r represents the expansion coefficients of the final outputs.

Model (4) uses two different patterns for evaluating the overall efficiency in a two-stage system in intermediate products: it estimates intermediate products individually according to their roles in each stage. In other words, intermediate products in the first stage have an output role while they have an input role to the second stage. Therefore, in order to evaluate overall efficiency in Model (4), these products are initially scaled with expansion coefficients φ_d and in the second stage with contraction coefficients $\bar{\varphi}_d$. The weights $\hat{w}_1$ and $\hat{w}_2$ may be determined by employing the model and according to the importance of each division or during optimization. Model (4), which is developed upon Model (3), is an envelopment DEA network model under the assumption of VRS. Since the pattern of Russell's model is compatible with the concept of Pareto–Koopmans efficiency, it is not necessary to consider the slacks in the proposed models. Model (4) is a nonlinear programming model and management priorities may be considered for linearization.

Model (4) can be converted into a linear program (LP) using Charnes–Cooper transformation [33]. In this model, E_o^1 and E_o^2 rely on individual efficiencies of the first and second stages, respectively, to evaluate DMU_o. Assuming that both stages possess the same importance in evaluation ($\hat{w}_1 = \hat{w}_2 = \frac{1}{2}$, Model (4) may be rewritten by allowing $t_1 = \frac{1}{\frac{1}{D}\sum_{d=1}^{D}\varphi_d}$ and $t_2 = \frac{1}{\frac{1}{s}\sum_{r=1}^{s}\beta_r}$ and multiplying the constraints by t_1 and t_2 separately.

Allowing $\theta_i^1 = t_1\theta_i$, $\theta_i^2 = t_2\theta_i$, $\varphi_d^1 = t_1\varphi_d$, $\varphi_d^2 = t_2\varphi_d$, $\bar{\varphi}_d^1 = t_1\bar{\varphi}_d$, $\bar{\varphi}_d^2 = t_2\bar{\varphi}_d$, $\beta_r^1 = t_1\beta_r$, $\beta_r^2 = t_2\beta_r$, $\lambda_j^1 = t_1\lambda_j$, $\lambda_j^2 = t_2\lambda_j$, $\mu_j^1 = t_1\mu_j$, $\mu_j^2 = t_2\mu_j$, Model (4) transforms into the following LP Model (5):

$$E_o = \min\frac{1}{2}\left(\frac{1}{m}\sum_{i=1}^{m}\theta_i^1 + \frac{1}{D}\sum_{d=1}^{D}\bar{\varphi}_d^2\right) \tag{5}$$

$$\begin{array}{llll}
\text{s.t.} & \sum_{j=1}^{n}\lambda_j^1 x_{ij} \le \theta_i^1 x_{io}; & \sum_{j=1}^{n}\lambda_j^2 x_{ij} \le \theta_i^2 x_{io}, & i = 1,2,\ldots,m \\
& \sum_{j=1}^{n}\lambda_j^1 z_{dj} = \varphi_d^1 z_{do}; & \sum_{j=1}^{n}\lambda_j^2 z_{dj} = \varphi_d^2 z_{do}, & d = 1,2,\ldots,D \\
& \sum_{j=1}^{n}\mu_j^1 z_{dj} = \bar{\varphi}_d^1 z_{do}; & \sum_{j=1}^{n}\mu_j^2 z_{dj} = \bar{\varphi}_d^2 z_{do}, & d = 1,2,\ldots,D \\
& \sum_{j=1}^{n}\mu_j^1 y_{rj} \ge \beta_r^1 y_{ro}; & \sum_{j=1}^{n}\mu_j^2 y_{rj} \ge \beta_r^2 y_{ro}, & r = 1,2,\ldots,S \\
& \sum_{j=1}^{n}\mu_j^1 = t_1; & \sum_{j=1}^{n}\mu_j^2 = t_2, & \\
& \sum_{j=1}^{n}\lambda_j^1 = t_1; & \sum_{j=1}^{n}\lambda_j^2 = t_2, & \\
& \frac{1}{D}\sum_{d=1}^{D}\varphi_d^1 = 1; & \frac{1}{s}\sum_{r=1}^{s}\beta_r^2 = 1 & \\
& \theta_i^1 \le t_1; & \theta_i^2 \le t_2, & i = 1,2,\ldots,m \\
& \varphi_d^1 \ge t_1; & \varphi_d^2 \ge t_2, & d = 1,2,\ldots,D \\
& \bar{\varphi}_d^1 \le t_1; & \bar{\varphi}_d^2 \le t_2, & d = 1,2,\ldots,D \\
& \beta_r^1 \ge t_1; & \beta_r^2 \ge t_2, & r = 1,2,\ldots,S \\
& \lambda_j^1 \ge 0; & \lambda_j^2 \ge 0, & j = 1,2,\ldots,n \\
& \mu_j^1 \ge 0; & \mu_j^2 \ge 0, & j = 1,2,\ldots,n
\end{array}$$

Model (5) demonstrates an LP for the two-stage VRS network model, and its optimal solution presents the overall efficiency of the two-stage system. This model evaluates the average efficiencies of the first and second stages $\left(E_o = \frac{1}{2}\left(E_o^1 + E_o^2\right)\right)$. The model together with the constraints related to intermediate products are considered in a convex set of products. It may be observed that Model (5) is always feasible and bounded; i.e., $0 < E_o^* \le 1$.

Theorem 3 *If* $\left(\theta_i^{1*}, \theta_i^{2*}, \varphi_d^{1*}, \varphi_d^{2*}, \bar{\varphi}_d^{1*}, \bar{\varphi}_d^{2*}, \beta_r^{1*}, \beta_r^{2*}, \lambda_j^{1*}, \lambda_j^{2*}, \mu_j^{1*}, \mu_j^{2*}\right)$ *is an optimal solution of Model (5), then*

$$\left(\theta_i^* = \frac{\theta_i^{1*}}{t_1} = \frac{\theta_i^{2*}}{t_2}, \varphi_d^* = \frac{\varphi_d^{1*}}{t_1} = \frac{\varphi_d^{2*}}{t_2}, \bar{\varphi}_d^* = \frac{\bar{\varphi}_d^{1*}}{t_1} = \frac{\bar{\varphi}_d^{2*}}{t_2}, \beta_r^* = \frac{\beta_r^{1*}}{t_1} = \frac{\beta_r^{2*}}{t_2}, \lambda_j^* = \frac{\lambda_j^{1*}}{t_1} = \frac{\lambda_j^{2*}}{t_2}, \mu_j^* = \frac{\mu_j^{1*}}{t_1} = \frac{\mu_j^{2*}}{t_2}\right)$$

is an optimal solution for Model (4) when $\hat{w}_1 = \hat{w}_2 = \frac{1}{2}$. Proof for this theorem is provided in the appendix.

Definition 1 DMU_o is overall efficient with convex hull in the intermediate products if the optimal value of Model (5) is equal to one; i.e., $E_o^* = 1$.

Example 2. To properly describe Model (5), it is rewritten to evaluate the overall efficiency score for DMU_B given in example 1 as follows:

$$E_o^B = \min\frac{1}{2}\left(\theta^1 + \bar{\varphi}^2\right) \tag{6}$$

$$
\begin{aligned}
\text{s.t.}\quad & \lambda_1^1 + 2\lambda_2^1 + 5\lambda_3^1 + 2\lambda_4^1 \le 2\theta^1; && \lambda_1^2 + 2\lambda_2^2 + 5\lambda_3^2 + 2\lambda_4^2 \le 2\theta^2 \\
& 2\lambda_1^1 + \lambda_2^1 + 4\lambda_3^1 + 2\lambda_4^1 = 1 && 2\lambda_1^2 + \lambda_2^2 + 4\lambda_3^2 + 2\lambda_4^2 = \varphi^2 \\
& 2\mu_1^1 + \mu_2^1 + 4\mu_3^1 + 2\mu_4^1 = \bar{\varphi}^1; && 2\mu_1^2 + \mu_2^2 + 4\mu_3^2 + 2\mu_4^2 = \bar{\varphi}^2 \\
& 4\mu_1^1 + \mu_2^1 + 2\mu_3^1 + 2\mu_4^1 \ge \beta^1; && 4\mu_1^2 + \mu_2^2 + 2\mu_3^2 + 2\mu_4^2 \ge 1 \\
& \lambda_1^1 + \lambda_2^1 + \lambda_3^1 + \lambda_4^1 = t_1; && \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = t_2 \\
& \mu_1^1 + \mu_2^1 + \mu_3^1 + \mu_4^1 = t_1 && \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 = t_2 \\
& \theta^1 \le t_1, \ \ \theta^2 \le t_2 && \bar{\varphi}^1 \le t_1, \bar{\varphi}^2 \le t_2 \\
& \beta^1 \ge t_1, \ \ \ 1 \ge t_2 && \varphi^2 \ge t_2, \ \ 1 \ge t_1 \\
& \lambda_j^1 \ge 0,\ \lambda_j^2 \ge 0,\ \mu_j^1 \ge 0,\ \mu_j^2 \ge 0, && j = 1, \ldots, 4.
\end{aligned}
$$

On solving Model (6), optimal overall efficiency score for DMU$_B$ is obtained as $E_o^{B*} = 0.625$. Similarly, the proposed model can evaluate the overall efficiency scores for other DMUs in example 1. Table 2 presents the results of this calculation.

Thus, DMU_A with convex hull in the intermediate products is overall efficient while other *DMUs* are inefficientall. Optimally, scaling coefficients show a value of 1 in evaluation of DMU_A under Model (5). Therefore, DMU_A is a member of the referenced set and may be introduced as a target for frontier projection of inefficient *DMUs*.

Regarding evaluation of DMU_B under Model (5), optimal values are obtained as $t_1^* = 0.5, t_2^* = 1, \theta^{1*} = 0.25, \bar{\varphi}^{2*} = 1, \varphi^{1*} = 1$ and $\beta^{2*} = 1$. Using the defined formulas, efficiencies of each stage and targets are calculated. Results indicate that efficiency of the first stage is equal to $E_B^{1*} = \theta^{1*} = 0.25$ and $\theta_B^* = \frac{\theta^{1*}}{t_1^*} = \frac{0.25}{0.5} = \frac{1}{2}$, $\varphi_B^* = \frac{\varphi^{1*}}{t_1^*} = \frac{1}{0.5} = 2$. This shows that the optimal solution for Model (5) is equal to the optimal solution of Model (4) $\left(E_B^{1*} = \frac{1}{2} = 0.25\right)$, and the obtained projection from results of the first stage from Model (4) could be determined as ($\theta_B^* x_B = x_B^*$, $\varphi_B^* z_B = z_B^*$), i.e., $x_B^* = 1$ and $z_B^* = 2$.

Thus, despite the fact that DMU_B is located on the boundary, it may be targeted on DMU_A due to the possibility of a change in the convex hull of the intermediate products. Therefore, DMU_B is not efficient in the first stage.

Additionally, the second stage scaling factors for Model (4) are $\bar{\varphi}_B^* = \frac{\bar{\varphi}^{2*}}{t_2^*} = 1$ and $\beta_B^* = \frac{\beta^{2*}}{t_2^*} = 1$ based on results. Thus, DMU_B is efficient in the second stage $\left(E_B^{2*} = \bar{\varphi}^{2*} = 1\right)$ and is located on the efficient frontier.

Table 2. The overall efficiency scores for four DMUs in example 1.

dmu	*A*	*B*	*C*	*D*
θ_o^*	1	0.625	0.625	0.5

On evaluating DMU_D under Model (5), optimal values are obtained as $t_1^* = 1, t_2^* = 0.5, \theta^{1*} = 0.5, \bar{\varphi}^{2*} = 0.5, \varphi^{1*} = 1$ and $\beta^{2*} = 1$. Efficiency of the first and second stages are equal to $E_D^{1*} = \theta^{1*} = 0.5$ and $E_D^{2*} = \bar{\varphi}^{2*} = 0.5$, respectively. Therefore, DMU_D is inefficient in both stages. Obtained results for scale factors in the first stage of Model (4) using these results are equal to $\theta_D^* = \frac{\theta^{1*}}{t_1^*} = \frac{0.5}{1} = 0.5$ and $\varphi_D^* = \frac{\varphi^{1*}}{t_1^*} = \frac{1}{1}$, and in the second stage, $\bar{\varphi}_D^* = \frac{\bar{\varphi}^{2*}}{t_2^*} = 1$ and $\beta_D^* = \frac{\beta^{2*}}{t_2^*} = \frac{1}{0.5} = 2$; thus $E_D^{1*} = E_D^{2*} = 0.5$. Furthermore, input and output of DMU_D can be projected on the input and output of DMU_A using Model (4).

It must be noted that Model (5) focuses on sets of convex hulls in intermediate products to evaluate performance. Thus, these products may be changed only in this bounded hull. Model (5) scales the intermediate products in the first stage with expansion coefficients, and in the second stage with coefficients of contraction.

Therefore, while Model (5) preserves intermediate products at the same levels in both stages, overall efficiency is evaluated so that the level of production of intermediate products in the first stage is higher than the rate of product consumption in the second stage. These performance evaluations are consistent with production conditions.

5. Determining divisional efficiency in T_{CHI}

A new network DEA model that determines divisional efficiency for DMUs is introduced in this section. For this purpose, production sets (1) and (2) are used in calculating divisional efficiency.

Efficiency scores for DMU_o in the first and second stages are estimated by the following models:

$$E_o^1 = \min \frac{\frac{1}{m}\sum_{i=1}^{m} \theta_i}{\frac{1}{D}\sum_{d=1}^{D} \varphi_d} \tag{7}$$

$$\text{s.t.}\quad \begin{array}{ll} \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta_i x_{io}, & i = 1,\ldots,m \\ \sum_{j=1}^{n} \lambda_j z_{dj} = \varphi_d z_{do}, & d = 1,\ldots,D \\ \sum_{j=1}^{n} \lambda_j = 1, & \\ \lambda_j \ge 0, & j = 1,\ldots,n \\ \theta_i \le 1, & i = 1,\ldots,m \\ \varphi_d \ge 1, & d = 1,\ldots,D \end{array}$$

$$E_o^2 = \min \frac{\frac{1}{D}\sum_{d=1}^{D} \bar{\varphi}_d}{\frac{1}{s}\sum_{r=1}^{s} \beta_r} \tag{8}$$

$$\text{s.t.}\quad \begin{array}{ll} \sum_{j=1}^{n} \mu_j z_{dj} = \bar{\varphi}_d z_{do}, & d = 1,\ldots,D \\ \sum_{j=1}^{n} \mu_j y_{rj} \ge \beta_r y_{ro}, & r = 1,\ldots,s \\ \sum_{j=1}^{n} \mu_j = 1, & \\ \mu_j \ge 0, & j = 1,\ldots,n \\ \beta_r \ge 1, & r = 1,\ldots,s \\ \bar{\varphi}_d \le 1, & d = 1,\ldots,D. \end{array}$$

Models (7) and (8) may be viewed as an enhanced Russell graph by assuming VRS. E_o^1 and E_o^2 represent efficiencies of the first and second stages, respectively, in these models. These fractional models are transformed to LPs by applying the Charnes–Cooper transformation to produce Models (9) and (10):

$$E_o^1 = \min \frac{1}{m}\sum_{i=1}^{m} \theta_i \tag{9}$$

$$\text{s.t.}\quad \begin{array}{ll} \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta_i x_{io}, & i = 1,\ldots,m \\ \sum_{j=1}^{n} \lambda_j z_{dj} = \varphi_d z_{do}, & d = 1,\ldots,D \\ \sum_{j=1}^{n} \lambda_j = t & \\ \frac{1}{D}\sum_{d=1}^{D} \varphi_d = 1, & \\ \lambda_j \ge 0, & j = 1,\ldots,m \\ \theta_i \le t, & i = 1,\ldots,m \\ \varphi_d \ge t, & d = 1,\ldots,D \end{array}$$

$$E_o^2 = \min \frac{1}{D}\sum_{d=1}^{D} \bar{\varphi}_d \tag{10}$$

$$\text{s.t.}\quad \begin{array}{ll} \sum_{j=1}^{n} \mu_j z_{dj} = \bar{\varphi}_d z_{do}, & d = 1,\ldots,D \\ \sum_{j=1}^{n} \mu_j y_{rj} \le \beta_r y_{ro}, & r = 1,\ldots,s \\ \sum_{j=1}^{n} \mu_j = t, & \\ \frac{1}{s}\sum_{r=1}^{s} \beta_r = 1, & \\ \mu_j \ge 0, & j = 1,\ldots,n \\ \beta_r \ge t, & r = 1,\ldots,s \\ \bar{\varphi}_d \le t, & d = 1,\ldots,D \end{array}$$

Interpretation of the Models (9) and (10) is similar to the interpretation of enhanced Russell graph by assuming VRS. The only difference would be the use of equality restriction in output constraints of the first stage of Model (9) and in the input constraints of Model (10). Note that the overall efficiency score is the arithmetic mean of efficiency scores of all stages.

Definition 2 First (or second) division of DMU_o is efficient with respect to the set of convex hull in the intermediate products, if and only if the optimal value of Model (9) or Model (10) is equal to one; i.e., $E_o^{1*} = 1$ (or $E_o^{2*} = 1$).

Table 3. Stage efficiency scores of four units in the first stage for example 1.

DMU	Stage 1	Stage 2
A	1	1
B	0.25	1
C	1	0.25
D	0.5	0.5

To examine divisional efficiency scores in example 1, Models (9) and (10) are applied and results are reported in table 3. It is observed that results with the interpretations presented for DMUs in the previous section are the same. Results indicate that DMU_A is efficient with convex hull in intermediate products at both stages and the remaining DMUs are inefficient at least at one stage. As observed, overall efficiency score is the average of the first and the second stage efficiency scores.

6. An illustrative example

This section concerns application of the produced models to assess overall and divisional efficiency scores for the data set used in Kao and Hwang [20]. Performance of 24 non-life insurance companies in Taiwan was assessed in that study using a two-stage system in series.

Kao and Hwang [20] processed this example in two stages: premium acquisition and profit generation. The first stage uses two inputs to produce two intermediate products, which are consumed by the second stage to produce two final outputs. Inputs are considered to be employee operational and insurance expenses; intermediate products are directly written and reinsurance premiums, and the final two outputs are the underwriting and investment profits. Table 4 presents the data set of this example in the Kao and Hwang study [20].

Table 5 lists the calculated overall and divisional efficiency scores of the DMUs using Models (5), (9) and (10). Note that in the newly proposed method for evaluation of non-life insurance companies, intermediate products (directly written and reinsurance premiums) are allowed to vary only in the convex hull of the observed data. Products are boundaries; therefore, these products must exist in an acceptable range for performance evaluation of the companies.

As observed, overall efficiency and divisional efficiency score values for both stages of DMU_5 and DMU_{22} are equal to one. Therefore, these DMUs are generally efficient in the new production set and lie on the VRS frontier when evaluated by the proposed network model. DMUs for 1, 2, 9, 12, 15, 18, 19 and 24 in the first division are efficient with convex hull in the intermediate products. DMUs for 3, 17 and 20 in the second division are efficient with respect to

Table 4. Data set of the example.

DMU	Operational expenses x_1	Insurance expenses x_2	Direct written premiums z_1	Reinsurance premiums z_2	Underwriting profit y_1	Investment profit y_2
1	1,178,744	673,512	7,451,757	856,735	984,143	681,687
2	1,381,822	1,352,755	10,020,274	1,812,894	1,228,502	834,754
3	1,177,494	592,790	4,776,548	560,244	293,613	658,428
4	601,320	594,259	3,174,851	371,863	248,709	177,331
5	6,699,063	3,531,614	37,392,862	1,753,794	7,851,229	3,925,272
6	2,627,707	668,363	9,747,908	952,326	1,713,598	415,058
7	1,942,833	1,443,100	10,685,457	643,412	2,239,593	439,039
8	3,789,001	1,873,530	17,267,266	1,134,600	3,899,530	622,868
9	1,567,746	950,432	11,473,162	546,337	1,043,778	264,098
10	1,303,249	1,298,470	8,210,389	504,528	1,697,941	554,806
11	1,962,448	672,414	7,222,378	643,178	1,486,014	18,259
12	2,592,790	650,952	9,434,406	1,118,489	1,574,191	909,295
13	2,609,941	1,368,802	13,921,464	811,343	3,609,236	223,047
14	1,396,002	988,888	7,396,396	465,509	1,401,200	332,283
15	2,184,944	651,063	10,422,297	749,893	3,355,197	555,482
16	1,211,716	415,071	5,606,013	402,881	854,054	197,947
17	1,453,797	1,085,019	7,695,461	342,489	3,144,484	371,984
18	757,515	547,997	3,631,484	995,620	692,731	163,927
19	159,422	182,338	1,141,951	483,291	519,121	46,857
20	145,442	53,518	316,829	131,920	355,624	26,537
21	84,171	26,224	225,888	40,542	51,950	6491
22	15,993	10502	52,063	14,574	82,141	4181
23	54,693	28,408	245,910	49,864	0.1	18,980
24	163,297	235,094	476,419	644,816	142,370	16,976

Table 5. The overall and divisional efficiency scores.

DMU	Overall E_o^*	Stage 1 E_o^{1*}	Stage 2 E_o^{2*}
1	0.73607	1	0.47214
2	0.68733	1	0.37467
3	0.80172	0.60345	1
4	0.35465	0.47730	0.23199
5	1	1	1
6	0.66640	0.92913	0.40367
7	0.48009	0.45914	0.50104
8	0.57433	0.69686	0.45179
9	0.64469	1	0.28939
10	0.53983	0.42404	0.65562
11	0.35321	0.67016	0.036259
12	0.78385	1	0.56770
13	0.49211	0.71566	0.26856
14	0.47144	0.43575	0.50712
15	0.84427	1	0.68855
16	0.56785	0.75571	0.37998
17	0.65946	0.31893	1
18	0.66768	1	0.33537
19	0.69655	1	0.39310
20	0.83523	0.67046	1
21	0.50402	0.74107	0.26697
22	1	1	1
23	0.48323	0.96647	0.10377
24	0.60351	1	0.20703

the convex hull in the intermediate products. Other units are inefficient in both divisions.

Overall efficiency scores are calculated by averaging the efficiency of all stages, i.e., $\left(\left(E_o^* = \frac{1}{2}\left(E_o^{1*} + E_o^{2*}\right)\right)\right.$.

Analysing the results in this example, it is observed that the proposed models in this work are effective in assessing performance of DMUs.

Additionally, the new models are capable of identifying the sources of inefficiencies in a two-stage network and help managers in addressing them. For example, results obtained from Models (5), (9) and (10) for DMU_1 are, respectively, $E_1^* = 0.73607, E_1^{1*} = 1$ and $E_1^{2*} = 0.47214$. Therefore, inefficiency of DMU_1 is due to poor performance in the second stage. Obtaining scaling coefficients from Model (10) enables a manager to take necessary actions to correct inefficiencies. Therefore, the newly introduced production set and the associated performance evaluation method presented in this paper are valuable and have practical utility in terms of production evaluation and performance management.

7. Conclusion

Two-stage network models are important in DEA due to their utility in complex models. Conventional DEA models with a two-stage network structure treat intermediate products in a dual fashion as input or output. This duality in roles can lead to inconsistencies in efficiency evaluation and Pareto–Koopmans dominance concepts.

This work investigates imposition of a uniform behaviour on intermediate products with an axiomatic approach. Further, a new PPS in two-stage production systems is proposed by focusing on dominance notion and convexity axiom using a set of convex hulls for intermediate products under the assumption of VRS.

An enhanced Russell graph measure is proposed to assess overall efficiency scores based on a new PPS. A method for converting the new non-linear model to a linear programming equivalent is next introduced using Charnes–Cooper transformation. The proposed enhanced Russell graph models also provide assessment of divisional efficiency under the assumption of VRS. In addition, it is proposed to use an average of stage efficiencies to represent overall efficiency of units. Finally, illustrative examples are presented to demonstrate and examine the new approach.

A significant advantage of the proposed models is their capability in utilization of input, output and intermediate products as complete vectors in evaluation of respective units. Therefore, as results indicate, the proposed models are consistent with the concept of Pareto–Koopmans efficiency because they consider sets of convex hulls for intermediate products in performance assessment of two-stage network units. Additionally, using the same proposed models, it may be possible to determine sources of inefficiency for specific production units. This may prove valuable and beneficial from an organizational management perspective.

Appendix

(a) **Theorem 1.** The first-stage technology or T_1, which is defined in set (4), is the minimal set that contains all observations and satisfies the axioms of strong disposability of inputs and convexity.

Proof Assume that technology T satisfies the axioms (A1)–(A3). It is shown that $T_1 \subseteq T$.

Let $((\boldsymbol{x}_1, z_1) \in T_1)$ so there exist non-negative intensity weights $\lambda_1, \ldots, \lambda_n$ for which $\sum_j \lambda_j = 1$ and $\begin{cases} x_1 \geq \sum_j \lambda_j x_j \\ z_1 = \sum_j \lambda_j z_j \end{cases}$. Since T satisfies (A1), it includes any observed $(\boldsymbol{x}_j, \boldsymbol{z}_j)$, $j = 1, \ldots, N$ and from (A2), therefore, $\sum_j \lambda_j (x_{j,} z_j) = \left(\sum_j \lambda_j x_j, \sum_j \lambda_j z_j\right) \in T$. T clearly satisfies (A3) in input side and that in turn indicates that $(x_1, z_1) \in T$.

Proof is complete. Note that proof for Theorem 2 is identical to proof for Theorem 1.

(b) **Theorem 3.** If $\left(\theta_i^{1*}, \theta_i^{2*}, \varphi_d^{1*}, \varphi_d^{2*}, \bar{\varphi}_d^{1*}, \bar{\varphi}_d^{2*}, \beta_r^{1*}, \beta_r^{2*}, \lambda_j^{1*}, \lambda_j^{2*}, \mu_j^{1*}, \mu_j^{2*}\right)$ is an optimal solution for Model (5), then $\left(\theta_i^* = \frac{\theta_i^{1*}}{t_1} = \frac{\theta_i^{2*}}{t_2}, \varphi_d^* = \frac{\varphi_d^{1*}}{t_1} = \frac{\varphi_d^{2*}}{t_2}, \bar{\varphi}_d^* = \frac{\bar{\varphi}_d^{1*}}{t_1} = \frac{\bar{\varphi}_d^{2*}}{t_2}, \quad \beta_r^* = \frac{\beta_r^{1*}}{t_1} = \frac{\beta_r^{2*}}{t_2}, \lambda_j^* = \frac{\lambda_j^{1*}}{t_1} = \frac{\lambda_j^{2*}}{t_2}, \mu_j^* = \frac{\mu_j^{1*}}{t_1} = \frac{\mu_j^{2*}}{t_2}\right)$ is an optimal solution for Model (4) when $\hat{w}_1 = \hat{w}_2 = \frac{1}{2}$.

Proof Suppose that $\left(\theta_i^{1*}, \theta_i^{2*}, \varphi_d^{1*}, \varphi_d^{2*}, \bar{\varphi}_d^{1*}, \bar{\varphi}_d^{2*}, \beta_r^{1*}, \beta_r^{2*}, \lambda_j^{1*}, \lambda_j^{2*}, \mu_j^{1*}, \mu_j^{2*}\right)$ is an optimal solution for Model (5). Obviously, $\left(\theta_i^*, \varphi_d^*, \bar{\varphi}_d^*, \beta_r^*, \lambda_j^*, \mu_j^*\right)$ is a feasible solution for Model (4) when $\hat{w}_1 = \hat{w}_2 = \frac{1}{2}$. For the selected optimal solution of Model (5):

$$\exists \lambda_j^{1*}, \lambda_j^{2*} \geq 0, 1\lambda_j^{1*} = t_1, 1\lambda_j^{2*} = t_2 \text{ and } \exists \mu_j^{1*}, \mu_j^{2*} \geq 0, 1\mu_j^{1*} = t_1, 1\mu_j^{2*} = t_2,$$

$$\text{s.t.} \quad \theta_i^{1*} \leq t_1, \theta_i^{2*} \leq t_2, \varphi_d^{1*} \geq t_1, \varphi_d^{2*} \geq t_2, \bar{\varphi}_d^{1*} \leq t_1, \bar{\varphi}_d^{2*} \leq t_2, \beta_r^{1*} \geq t_1, \beta_r^{2*} \geq t_2$$

$$\frac{1}{2}\left(\frac{1}{m}\sum\nolimits_{i=1}^{m} \theta_i^{1*} + \frac{1}{D}\sum\nolimits_{d=1}^{D} \bar{\varphi}_d^{2*}\right) \leq \frac{1}{2}\left(\frac{1}{m}\sum\nolimits_{i=1}^{m} \theta_i^{1} + \frac{1}{D}\sum\nolimits_{d=1}^{D} \bar{\varphi}_d^{2}\right).$$

Additionally,

$$t_1 = \frac{1}{\frac{1}{D}\sum_{d=1}^{D} \varphi_d} \geq 0, t_2 = \frac{1}{\frac{1}{s}\sum_{r=1}^{s} \beta_r} \geq 0, \theta_i^* = \frac{\theta_i^{1*}}{t_1} \text{ and } \bar{\varphi}_d^* = \frac{\bar{\varphi}_d^{2*}}{t_2}.$$

It is noted that

$$\frac{1}{2}\left(\frac{1}{m}\sum\nolimits_{i=1}^{m} t_1\theta_i^* + \frac{1}{D}\sum\nolimits_{d=1}^{D} t_2\bar{\varphi}_d^*\right) \leq \frac{1}{2}\left(\frac{1}{m}\sum\nolimits_{i=1}^{m} t_1\theta_i + \frac{1}{D}\sum\nolimits_{d=1}^{D} t_2\bar{\varphi}_d\right)$$
$$\Rightarrow \frac{1}{2}\left(\frac{\frac{1}{m}\sum_{i=1}^{m}\theta_i^*}{\frac{1}{D}\sum_{d=1}^{D}\varphi_d^*} + \frac{\frac{1}{D}\sum_{d=1}^{D}\bar{\varphi}_d^*}{\frac{1}{s}\sum_{r=1}^{s}\beta_r^*}\right) \leq \frac{1}{2}\left(\frac{\frac{1}{m}\sum_{i=1}^{m}\theta_i}{\frac{1}{D}\sum_{d=1}^{D}\varphi_d} + \frac{\frac{1}{D}\sum_{d=1}^{D}\bar{\varphi}_d}{\frac{1}{s}\sum_{r=1}^{s}\beta_r}\right).$$

Let $\hat{w}_1 = \hat{w}_2 = \frac{1}{2}$; thus

$$\left(\hat{w}_1\frac{\frac{1}{m}\sum_{i=1}^{m}\theta_i^*}{\frac{1}{D}\sum_{d=1}^{D}\varphi_d^*} + \hat{w}_2\frac{\frac{1}{D}\sum_{d=1}^{D}\bar{\varphi}_d^*}{\frac{1}{s}\sum_{r=1}^{s}\beta_r^*}\right) \leq \left(\hat{w}_1\frac{\frac{1}{m}\sum_{i=1}^{m}\theta_i}{\frac{1}{D}\sum_{d=1}^{D}\varphi_d} + \hat{w}_2\frac{\frac{1}{D}\sum_{d=1}^{D}\bar{\varphi}_d}{\frac{1}{s}\sum_{r=1}^{s}\beta_r}\right).$$

Therefore $\left(\theta_i^*, \varphi_d^*, \bar{\varphi}_d^*, \beta_r^*, \lambda_j^*, \mu_j^*\right)$ is an optimal solution for Model (4). Proof is complete.

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