Algorithms for separable convex optimization with linear ascending constraints

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Abstract. The paper considers the minimization of a separable convex function subject to linear ascending constraints. The problem arises as the core optimization in several resource allocation scenarios, and is a special case of an optimization of a separable convex function over the bases of a polymatroid with a certain structure. The paper generalizes a prior algorithm to a wider class of separable convex objective functions that need not be smooth or strictly convex. The paper also summarizes the state-of-the-art algorithms that solve this optimization problem. When the objective function is a so-called d -separable function, a simpler linear time algorithm solves the problem.

Keywords. Convex programming; OR in telecommunications; ascending constraints; linear constraints; polymatroid.

1. Introduction

In this paper we consider the following separable convex optimization problem with linear inequality constraints. The problem arises in a wide variety of resource allocation settings and we highlight several immediately after stating the abstract problem. Let $x = (x(1), x(2), \dots, x(n)) \in \mathbb{R}^n$. Let $w_e : [0, b_e] \rightarrow \overline{\mathbb{R}}$, $e = 1, 2, \dots, n$ be convex functions where $0 < b_e \leq \infty$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ be the extended real line. We wish to minimize a separable objective function $W : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ as in

Problem Π :

$$\text{minimize } W(x) := \sum_{e=1}^n w_e(x(e)) \quad (1)$$

$$\text{subject to } x(e) \in [0, \beta(e)], \quad e = 1, 2, \dots, n, \quad (2)$$

$$\sum_{e=1}^l x(e) \geq \sum_{e=1}^l \alpha(e), \quad l = 1, 2, \dots, n-1, \quad (3)$$

$$\sum_{e=1}^n x(e) = \sum_{e=1}^n \alpha(e). \quad (4)$$

We assume $\beta(e) \in (0, b_e]$ for $e = 1, 2, \dots, n$, and $\alpha(e) \geq 0$ for $e = 1, 2, \dots, n$. The inequalities in (2) impose positivity and upper bound constraints. The inequalities in (3) impose

a sequence of *ascending constraints* with increasing heights $\sum_{e=1}^l \alpha(e)$ indexed by l . We also assume

$$\sum_{e=1}^l \alpha(e) \leq \sum_{e=1}^l \beta(e), \quad l = 1, 2, \dots, n, \quad (5)$$

a necessary and sufficient condition for the feasible set to be nonempty (Lemma 2 of the appendix).

Problem Π arises in the following applications in operations research:

- An inventory problem with downward substitution (Wang [1]): A firm produces a product with n different grades. A higher grade of the product can be substituted for a lower grade. The firm has to make an inventory decision on the number of grade i products to stock before the demand is known. The objective is to minimize the loss incurred due to mismatch between the demand and the supply of products of each grade.
- The Clark–Scarf series multi-echelon inventory model (Clark and Scarf [2]): The inventory model consists of n facilities arranged in series. The demand is a random variable with known distribution of finite variance. The demand is first served at facility n with the excess demand at facility n passed on to facility $n-1$, and so on. There are costs involved in passing the demand to the next facility. There are also storage costs at the facilities. The problem is to find the amount to stock at each of the n facilities. The objective is to minimize

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the cost incurred due to the failures in meeting the demands at the facilities and the storage costs.

In addition to the afore-mentioned cases in operations research, problem Π arises in certain resource allocation problems in wireless communications where several mobiles simultaneously access a common medium. A high-level description is as follows. Each mobile transmits with a certain power (measured in joules per second) on a one-dimensional subspace of the available signalling vector space. The dimension of the signalling vector space is smaller than the number of mobiles, and orthogonalization of mobile transmissions is not possible. If two mobiles' signalling directions are not orthogonal, they will interfere with each other and affect each other's transmissions. Problem Π arises in the optimal allocation of directions (one-dimensional subspaces) to mobiles in each of the following settings.

- Mobiles have rate requirements (in bits per second) that must be met and the net transmitted power is to be minimized [3].
- Mobiles have quality of service requirements (in terms of minimum signal to interference and noise ratios) that must be met and again the net transmitted power is to be minimized [4, Sect. III].
- Mobiles have maximum transmission power constraints and the total rate achieved across all mobiles is to be maximized [4, Sect. II].

Problem Π also arises in the optimization of wireless multiple input multiple output (MIMO) systems as follows:

- Minimize power to meet mean-squared-error quality of service constraints on each of the datastreams in a point to point MIMO communication system [5].
- Minimize power in the context of linear transceiver design on MIMO networks with a single non-regenerative relay [6].

Problem Π also arises in an inequality-constrained maximum likelihood estimation problem where the parameters of a multi-nomial distribution are to be estimated subject to the constraint that the associated distribution is stochastically smaller than a given distribution [7].

The wide range of applications indicated earlier motivate us to study the abstracted problem Π in some detail.

The special case $\alpha(1) = \alpha(2) = \dots = \alpha(n-1) = 0$ makes the constraint in (3) irrelevant, and problem Π reduces to the well-studied minimization of a separable convex cost in (1) subject to boundedness and sum constraints, i.e., (2) and (4), respectively. See Patriksson [8] for a bibliographical survey of such problems with historical remarks and comments on solution approaches.

Morton *et al* [9] studied the special case of problem Π when $w_e(t) = \lambda(e)t^p$, $e = 1, 2, \dots, n$, where $p > 1$. They characterized the constraint set as the *bases* of a

polymatroid; we will define these terms soon. The algorithms to minimize a separable convex function over the bases of a polymatroid fall into two broad categories: greedy algorithms and decomposition algorithms. In the greedy category, the algorithm due to Federgruen and Groenevelt [10] has a complexity $O(B(\log n + F))$, where B is the total number of units to be allocated among the n variables and F is the number of steps required to check the feasibility at each step of the greedy update. Hochbaum [11] proposed a variant of the greedy algorithm that uses a scaling technique to reduce the complexity to $O(n(\log n + F) \log(B/(n\epsilon)))$. Hochbaum [11] points out that F takes $O(1)$ time for the case of linear ascending constraints.

The algorithms in the decomposition category use a divide and conquer approach. This class of algorithms divide the optimization problem into several subproblems that are easier to solve. Groenevelt [12] provided two algorithms of this variety. Fujishige in [13, Sect. 8] extended Groenevelt's algorithms to minimize over the bases of a more general "sub-modular system". Fujishige's decomposition algorithm requires an oracle for a particular step of the algorithm. The oracle gives a base satisfying certain conditions. This oracle depends on the particularity of the problem. For example, Groenevelt's decomposition algorithm [12] is for symmetric polymatroids. Morton *et al* [9] exploited the even more special structure in the polymatroid arising from the constraints in Eqs. (2)–(4) and identified an explicit algorithm that obviated the need for an oracle. Their procedure however was for the special case of $w_e(t) = \lambda(e)t^p$, $p > 1$, as indicated at the beginning of the previous paragraph. Padakandla and Sundaresan [14] provided an extension of this algorithm to strictly convex C^1 functions that satisfy certain slope conditions. Their proof of optimality is via a verification of Karush–Kuhn–Tucker conditions. Akhil *et al* [15] simplified the algorithm of [14] and relaxed some of the constraints imposed by Padakandla and Sundaresan [14] on the objective function. See D'Amico *et al* [16] for a similar extension of the algorithm of [14]. The complexity of the algorithms of Padakandla and Sundaresan [14] and Akhil *et al* [15] are at least $O(n^2)$ because a certain nonlinear equation is solved n^2 times. Zizhuo Wang's algorithm [1] reduces the number of times the non-linear equation is solved by a factor of n . Vidal *et al* [17] proposed a decomposition algorithm that solves problem Π in the same number of steps as that of Hochbaum's greedy algorithm [11].

In this paper, we provide a brief description of the greedy algorithm of Hochbaum [11] and the decomposition algorithm of Vidal *et al* [17]. We also extend the algorithm of [15] to a wider class of separable convex functions, such as negatives of piece-wise linear concave utility functions, which commonly arise in the economics literature. The extended algorithm proposed in this paper works for any convex w_e ; in particular, they need not be strictly convex or

differentiable. The proof of correctness of the algorithm employs the theory of polymatroids, which is summarized in the next section.

The complexity of our algorithm for the more general w_e is the same as that of Padakandla and Sundaresan [14], and indeed, our algorithm reduces to that of [14] when w_e are C^1 functions. We therefore refer the reader to that paper [14, Sect. 1] for a discussion on the relationship of the algorithm to those surveyed by Patriksson [8], and for performance comparisons with a standard optimization tool [14, Sect. 4].

The decomposition approach leads to an efficient algorithm for a special case of problem Π where the objective function has the following form:

$$W(x) = \sum_{e=1}^n d_e \phi\left(\frac{x(e)}{d_e}\right). \quad (6)$$

The minimizer of (6) over the bases of a polymatroid is known to be the *lexicographically optimal* base with respect to the weight vector $d = (d_1, d_2, \dots, d_n)$ (a notion introduced by Fujishige in [18]). This lexicographically optimal base optimizes W in (6) arising from ϕ that is strictly convex and continuously differentiable. Hence it suffices to consider (6) for the case of a quadratic ϕ . Veinott Jr. [19] proposed an elegant geometrical characterization of this optimal base when the polymatroid is defined by linear ascending constraints. The geometrical characterization is that of a *taut-string solution* for the minimizer of the optimization problem from the graph of the least concave majorant of a set of n points in x - y plane. Though Veinott Jr.'s computation of the least concave majorant of n points requires $O(n^2)$ steps, the string algorithm elucidated in Muckstadt and Sapro [20, Sect. 3.2.3], in the context of an inventory management problem, finds it in $O(n)$ steps. We briefly discuss the *taut-string solution* [19] and the string algorithm of Muckstadt and Sapro [20, Sect. 3.2.3].

The rest of the paper is organized as follows. In the next section, we discuss the preliminaries related to polymatroids. In section 3, we summarize the algorithm that solves problem Π and state the main results. In sections 4 and 5, we summarize Hochbaum's and Vidal's algorithms, respectively. In section 6, we discuss the *taut-string method*. Finally, in section 7, we summarize the string algorithm that finds the least concave majorant of a set of points.

While our paper is largely a survey, it also contains some novel contributions. The extension to general w_e (not necessarily strictly convex and/or not necessarily continuously differentiable everywhere in the domain), contained in section 3 and the Appendix, is new. The recognition that the string algorithm of Muckstadt and Sapro [20, Sect. 3.2.3] is applicable to the problem of Veinott Jr. [19] is also novel.

2. Preliminaries

In this section, we describe some preliminary results that reduce problem Π to an optimization over the bases of an appropriate polymatroid. The reduction is due to Morton *et al* [9] and is given here for completeness. We then state a result due to Groenevelt [12] for polymatroids, which was subsequently generalized to sub-modular functions by Fujishige [13, Sect. 8]. Groenevelt's result will provide a necessary and sufficient condition for optimality. The next section provides an algorithm to arrive at a base that satisfies the necessary and sufficient condition of Groenevelt [12]. We begin with some definitions.

Let $E = \{1, 2, \dots, n\}$. Let $f : 2^E \rightarrow \mathbb{R}_+$ be a *rank* function, i.e., a nonnegative real function on the set of subsets of E satisfying

$$f(\emptyset) = 0, \quad (7)$$

$$f(A) \leq f(B) \quad (A \subseteq B \subseteq E), \quad (8)$$

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad (A, B \subseteq E). \quad (9)$$

The pair (E, f) is called a *polymatroid* with ground set E . For an $x \in \mathbb{R}_+^E$ and $A \subseteq E$ define

$$x(A) := \sum_{e \in A} x(e).$$

A vector $x \in \mathbb{R}_+^E$ is called an *independent* vector if $x(A) \leq f(A)$ for every $A \subseteq E$. Let $P(f)$, the *polymatroidal polyhedron*, denote the set of all independent vectors of (E, f) . The *base* of the polymatroid (E, f) , denoted $B(f)$, is defined as

$$B(f) := \{x \in P(f) : x(E) = f(E)\}.$$

They are the maximal elements of $P(f)$ with respect to the partial order " $\leq$ " on the set $\mathbb{R}_+^E$ defined by component-wise domination, i.e., $x \leq y$ if and only if $x(e) \leq y(e)$ for every $e \in E$. We shall also refer to an element of the base of a polymatroid as a *base*.

For two real sequences $a = (a_1, a_2, \dots, a_k)$ and $b = (b_1, b_2, \dots, b_k)$ of same length k , a is lexicographically greater than or equal to b if for some $j \in \{1, 2, \dots, k\}$ we have

$$a_i = b_i \quad (i = 1, 2, \dots, j-1) \quad \text{and} \quad a_j > b_j$$

or

$$a_i = b_i \quad (i = 1, 2, \dots, k).$$

Let $x \in \mathbb{R}_+^E$ and let $T(x)$ be the $|E|$ -length sequence arranged in the increasing order of magnitude. Let $h_e : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a family of continuous and strictly increasing functions indexed by $e \in E$. Let $h : \mathbb{R}_+^E \rightarrow \mathbb{R}^E$ be the vector function

defined as $h(x) := (h_e(x(e)), e \in E)$. A base x of (E, f) is an h -lexicographically optimal base if the $|E|$ -tuple $T(h(x))$ is lexicographically maximum among all $|E|$ -tuples $T(h(y))$ for all bases y of (E, f) . Let $d \in \mathbb{R}_+^E$ with all components strictly positive. For the case $h_e = x(e)/d_e$, h -lexicographically optimal base is also known as the lexicographically optimal base with respect to the weight vector d .

For any $e \in E$, define

$$\text{dep}(x, e, f) = \cap \{A \mid e \in A \subset E, x(A) = f(A)\},$$

which in words is the smallest subset among those subsets A of E containing e for which $x(A)$ equals the upper bound $f(A)$. Fujishige shows that $\text{dep}(x, e, f) - \{e\}$ is $\emptyset$ if e and $x \in B(f)$ are such that $x(e)$ cannot be increased without making x dependent. Otherwise, $\text{dep}(x, e, f)$ is made up of all those $u \in E$ from which a small amount of mass can be moved from $x(u)$ to $x(e)$ yet keeping the new vector independent. Thus (u, e) is called an *exchangeable pair* if $u \in \text{dep}(x, e, f) - \{e\}$.

For a $\beta \in \mathbb{R}_+^E$, define the set function

$$f_\beta(A) = \min_{D \subseteq A} \{f(D) + \beta(A - D)\} \quad (A \subseteq E).$$

We now state without proof an interesting property of the subset of independent vectors of a polymatroid that are dominated by β . See Fujishige [13] for a proof.

Proposition 1 *The set function f_β is a rank function and (E, f_β) is a polymatroid. Furthermore, $P(f_\beta)$ is given by $P(f_\beta) = \{x \in P(f) : x \leq \beta\}$.*

We are now ready to relate the constraint set in problem Π to the base of a polymatroid, as done by Morton *et al* [9]. Define

$$\begin{aligned} c(0) &:= 0, \\ c(j) &:= \sum_{e=1}^j \alpha(e) \quad (1 \leq j \leq n), \end{aligned} \quad (10)$$

and further define, for $A \subseteq E$,

$$\zeta(A) := \max \{c(j) : \{1, 2, \dots, j\} \subseteq A, 0 \leq j \leq n\}, \quad (11)$$

$$f(A) := \zeta(E) - \zeta(E - A) = c(n) - \zeta(E - A). \quad (12)$$

Proposition 2 *The following statements hold.*

- The function f in (12) is a rank function and therefore (E, f) is a polymatroid.
- The set of $x \in \mathbb{R}_+^E$ that satisfy the ascending constraints (3)–(4) equals the base $B(f)$ of the polymatroid (E, f) .
- The set of $x \in \mathbb{R}_+^E$ that satisfy the ascending constraints (3)–(4) and the domination constraint (2) equals the base $B(f_\beta)$ of the polymatroid (E, f_β) .

See Morton *et al* [9] for a proof. Incidentally, this is shown by recognizing that (E, ζ) is a related object called

the contrapolymatroid, that the set of all vectors meeting the afore-mentioned constraints is the base of the contrapolymatroid, and that the base of the contrapolymatroid (E, ζ) and the base of the polymatroid (E, f) coincide. This proposition thus says that problem Π is simply a special case of Π_1 below with $g = f_\beta$.

Let (E, g) be a polymatroid. Our interest is in the following.

$$\begin{aligned} \text{Problem } \Pi_1 : \quad & \text{Minimize } \sum_{e \in E} w_e(x(e)) \\ & \text{subject to } x \in B(g). \end{aligned} \quad (13)$$

We next state a necessary and sufficient condition that an optimal base satisfies. For each $e \in E$, define w_e^+ and w_e^- to be the right and left derivatives of w_e .

Theorem 1 *A base $x \in B(g)$ is an optimal solution to problem Π_1 if and only if for each exchangeable pair (u, e) associated with base x (i.e., $u \in \text{dep}(x, e, g) - \{e\}$), we have $w_e^+(x(e)) \geq w_u^-(x(u))$.*

The result is due to Groenevelt [12]. See Fujishige [13, Th. 8.1] for a proof of the more general result on submodular systems. In the next section, we provide an algorithm to arrive at a base that satisfies Groenevelt's necessary and sufficient condition.

An alternate characterization of the optimal base when w_e is strictly convex and continuously differentiable is the following. Let h_e be the derivative of w_e . Note that h_e is continuous and strictly increasing.

Theorem 2 *For each $e \in E$, let w_e be strictly convex and continuously differentiable with h_e as its derivative. A base $x \in B(g)$ is an optimal solution to problem Π_1 if and only if x is an h -lexicographically optimal base.*

The result is due to Fujishige [13]. See Fujishige [13, Th. 9.1] for the proof.

3. Some new results

Fujishige [13] provides an algorithm called the *decomposition* algorithm to find an increasing chain of subsets of E as a key step to finding the optimal solution. However, in this section, we extend an algorithm of Padakandla and Sundaresan [14], which is itself an extension of the algorithm of Morton *et al* [9], to arrive directly at a chain and thence the optimal solution. This algorithm reduces to that of Padakandla and Sundaresan [14] for strictly convex $\mathcal{C}^1$ functions w_e , $e \in E$, and to that of Morton *et al* [9] for the case when $w_e(\zeta) = \lambda(e)\zeta^p$ for each $e \in E$, and $p > 1$.

The following algorithm seeks to identify the desired chain that will yield an x that meets the necessary and sufficient condition of Theorem 1. Define the generalized inverse functions of w_e as

$$i_e^-(\eta) := \inf \{ \zeta : w_e^+(\zeta) \geq \eta \},$$

$$i_e^+(\eta) := \sup \{ \zeta : w_e^-(\zeta) \leq \eta \}.$$

See figures 1 and 2. The function w_e^+ is a right-continuous nondecreasing function while its inverse i_e^- is a left-continuous nondecreasing function. On the other hand, w_e^- and i_e^+ are left-continuous and right-continuous, respectively.

It is straightforward to see that $i_e^-(\eta) \leq i_e^+(\eta)$ with strict inequality only for those η for which the interval $\{ \zeta : w_e^+(\zeta) = w_e^-(\zeta) = \eta \}$ is not a singleton. For readers unfamiliar with these notions, it may be beneficial to keep the simple case when $w_e \in C^1$ in mind, because in this case $w_e^+ = w_e^- = w_e'$ is the derivative of w_e , and $i_e^+ = i_e^- = (w_e')^{-1}$ is the inverse of w_e' .

Define their saturated and truncated counterparts as

Figure 1. The function $w_e^+(\zeta)$ as a function of ζ .

Figure 2. The functions $i_e^-(\eta)$ and $H_e^-(\eta)$ as a function of η . They coincide for $\eta \leq \eta_7$.

$$H_e^-(\eta) = \begin{cases} 0 & \eta \leq w_e^+(0) \\ i_e^-(\eta) & w_e^+(0) < \eta \leq w_e^-(\beta(e)) \\ \beta(e) & \eta > w_e^-(\beta(e)), \end{cases}$$

and

$$H_e^+(\eta) = \begin{cases} 0 & \eta < w_e^+(0) \\ i_e^+(\eta) & w_e^+(0) \leq \eta < w_e^-(\beta(e)) \\ \beta(e) & \eta \geq w_e^-(\beta(e)), \end{cases}$$

respectively. We now state the main algorithm with two associated subroutines.

Algorithm 1 • Let $s(0) = n+1$ and set $s(j) \in \{1, 2, \dots, n\}$ recursively for $j = 1, 2, \dots$ as follows.

- For each l , $1 \leq l < s(j-1)$, pick η_l^j to be the smallest η satisfying

$$\sum_{e=l}^{s(j-1)-1} H_e^-(\eta) \leq \sum_{e=l}^{s(j-1)-1} \alpha(e) \leq \sum_{e=l}^{s(j-1)-1} H_e^+(\eta), \quad (14)$$

if such an η exists. If not, η_l^j is undefined. Thus, if η_l^j is defined, we have

$$\sum_{e=l}^{s(j-1)-1} H_e^-(\eta_l^j) \leq \sum_{e=l}^{s(j-1)-1} \alpha(e) \leq \sum_{e=l}^{s(j-1)-1} H_e^+(\eta_l^j). \quad (15)$$

- Let

$$\Gamma_j = \min \{ \eta_l^j \mid 1 \leq l < s(j-1) \text{ and } \eta_l^j \text{ exists} \}, \quad (16)$$

and let $l_1, l_2, \dots, l_r$ be the indices that attain the minimum in (16), ordered so that $s(j-1) > l_1 > l_2 > \dots > l_r \geq 1$. The quantity r denotes the number of tied indices.

- Set $s(j)$ as given by the output of Subroutine 1.
- Set values $x(e)$ for $e = s(j), s(j)+1, \dots, s(j-1)-1$ as given by the output of Subroutine 2.
- If $s(j) = 1$, exit. □

We next describe the two subroutines referred to in Algorithm 1.

The first one sets $s(j)$. One property that the obtained $s(j)$ from Subroutine 1 will have, and we will soon show this, is that constraints (3) are met with equality whenever l equals an $s(j)$ for some iterate j . We will also see that Subroutine 2 will set a variable $x(e)$ to $H_e^+(\Gamma_j)$ whenever possible. For feasibility then, the partial sums of $H_e^+(\Gamma_j)$ from $s(j)$ to each of the tied indices l_i should exceed the corresponding partial sums of $\alpha(e)$. Hence $s(j)$ should be chosen carefully, which is the purpose of the next-described Subroutine 1.

Subroutine 1 This subroutine takes as input $s(j-1)$, $r, l_1, l_2, \dots, l_r$ and Γ_j . It returns $s(j)$.

- If there is a unique index l_1 that achieves the minimum, set $s(j) = l_1$. Otherwise we have $s(j-1) > l_1 > l_2 > \dots > l_r$. Choose $s(j)$ as follows. Consider the inequality

$$\sum_{e=m}^{k-1} H_e^+(\Gamma_j) \geq \sum_{e=m}^{k-1} \alpha(e), \quad (17)$$

where k and m are iterates.

1. Initialize $i = 1, t = r, k = l_i, m = l_i$.
2. Repeat

if (14) holds, $i \leftarrow i + 1, k \leftarrow l_i$.
else $t \leftarrow t - 1, m \leftarrow l_t$

until $i = t$.

3. Set $s(j) = l_i$ and return $s(j)$. $\square$

The second subroutine sets the variables for a subset of E .

Subroutine 2 This subroutine takes as input the following quantities: $s(j), \Gamma_j, I = \{l_1, l_2, \dots, l_p\}$ with $s(j-1) > l_1 > l_2 > \dots > l_p > s(j)$. These l_i s are the set of tied indices that satisfy (15) at the j^{th} iteration, and are strictly larger than $s(j)$ set by Subroutine 1. If I is empty, p is taken to be zero. It sets $x(e)$ for $e = s(j), s(j) + 1, \dots, s(j-1) - 1$.

1. Initialize $m = 0, p_m = p, l_0^m = s(j-1), l_{p_m+1}^m = s(j), I_m = I, l_i^m = l_i (1 \leq i \leq p_m)$.
2. Calculate

$$\gamma = \min \left\{ \sum_{e=l_i^m}^{l_0^m-1} \alpha(e) - \sum_{e=l_i^m}^{l_0^m-1} H_e^-(\Gamma_j) \forall i = 1, \dots, p_m + 1 \right\}. \quad (18)$$

Let t be the largest index in (15) such that

$$\gamma = \sum_{e=l_t^m}^{l_0^m-1} \alpha(e) - \sum_{e=l_t^m}^{l_0^m-1} H_e^-(\Gamma_j).$$

3. Assign $x(e), e \in \{l_0^m - 1, l_0^m - 2, \dots, l_1^m\}$ such that $x(e) \in [H_e^-(\Gamma_j), H_e^+(\Gamma_j)]$ and

$$\sum_{e=l_1^m}^{l_0^m-1} x(e) = \gamma + \sum_{e=l_1^m}^{l_0^m-1} H_e^-(\Gamma_j). \quad (19)$$

For $e \in \{l_1^m - 1, l_1^m - 2, \dots, l_t^m\}$, assign

$$x(e) = H_e^-(\Gamma_j). \quad (20)$$

4. If $l_t^m = s(j)$, exit.
5. Let I_{m+1} be the indices l in $I \cap \{l : l < l_t^m\}$ that satisfy

$$\sum_{e=l}^{l_t^m-1} H_e^+(\Gamma_j) \geq \sum_{e=l}^{l_t^m-1} \alpha(e). \quad (21)$$

Set $p_{m+1} = |I_{m+1}|$. Further, set $l_0^{m+1} = l_t^m$ and $l_{p_{m+1}+1}^{m+1} = s(j)$. Denote the indices in I_{m+1} as $l_1^{m+1}, l_2^{m+1}, \dots, l_{p_{m+1}}^{m+1}$ such that $l_1^{m+1} > l_2^{m+1} > \dots > l_{p_{m+1}}^{m+1}$. Set $m \leftarrow m + 1$. Go to step 2. $\square$

We next formally state correctness and optimality properties of the algorithm and prove them in the following section. That the operations in the algorithm are all well defined can be gleaned from the proof given in the appendix.

Theorem 3 If the feasible set is nonempty, Algorithm 1 runs to completion and puts out a feasible vector. This output solves problem Π .

Observe that the hypothesis is the natural minimum requirement that the feasible set is nonempty. See Lemma 2 for a natural necessary and sufficient condition for a nonempty feasible set.

The algorithm for the special case when w_e are strictly convex and C^1 is given in [15]. The algorithm is much simpler in this case. The Subroutine 1 chooses l_r , the smallest index satisfying (15), as $s(j)$ since (17) holds true for $m = l_r$ and $k = l_i$ for all $i = 1, 2, \dots, r-1$. Since $H_e = H_e^- = H_e^+$, Subroutine 2 assigns $x(e) = H_e(\Gamma_j)$ for all $e \in \{s(j), s(j) + 1, \dots, s(j-1) - 1\}$.

See Appendix given in the supplementary material for the proof of Theorem 3.

4. A greedy algorithm

In section 3, we extended the algorithm of Akhil *et al* [15], which solves problem Π , to separable convex functions that are not strictly convex or differentiable. In this section, we describe a more efficient algorithm proposed by Hochbaum [11] (with a correction note by Moriguchi and Shioura [21]) that provides an ϵ -optimal solution to Π_1 . It is based on the greedy approach.

Consider the following discrete resource allocation problem that is problem Π_1 with variables restricted to integers.

$$\begin{aligned} \text{Problem } \Pi_2 : \quad & \text{minimize } \sum_{e \in E} w_e(x(e)) \\ & \text{subject to } x \in B(g), \\ & x \in \mathbb{Z}_+^E. \end{aligned} \quad (22)$$

The set of vectors satisfying the constraint set of problem Π_2 form the bases of the polymatroid (E, g) defined over integers. Problem Π_2 can be solved by the greedy algorithm [10]. Starting with an initial allocation $x = 0$, this algorithm increases the value of a variable by one unit if the corresponding decrease in the objective function is the largest among all possible feasible increments. The complexity of the greedy algorithm is $O(B(\log n + F))$, where F is the number

of operations required to check the feasibility of a given increment in a single variable. The log n complexity is to keep a sorted array of the marginal decrease in the objective function. Therefore, the complexity of the greedy algorithm is exponential in the number of input bits to the algorithm.

Hochbaum's algorithm, referred to as General Allocation Procedure (GAP) in [11], combines the greedy algorithm with a scaling technique. Hochbaum considers an additional constraint $x \geq l$, where $l = (l_1, l_2, \dots, l_n)$. The modified greedy algorithm consists of a subroutine $\text{greedy}(s, \tilde{l})$ that finds the variable that has the largest decrease in the objective function value among all variables that can be increased by one unit without violating feasibility, and increases it by s units. The subroutine starts with an initial allocation $x = \tilde{l}$. Define $\mathbf{e} \in \mathbb{R}^E$ as $\mathbf{e} = (1, 1, \dots, 1)$ and $\mathbf{e}^k \in \mathbb{R}^E$ as $\mathbf{e}^k(k) = 1$ and $\mathbf{e}^k(j) = 0, j \neq k$. Let the total number of units to be allocated among n variables be B , i.e., $g(E) = B$. The subroutine $\text{greedy}(s, \tilde{l})$ is described here (including the Moriguchi and Shioura [21] correction).

Algorithm 2 $\text{greedy}(s, \tilde{l})$

1. $x = \tilde{l}, \tilde{B} = B - \tilde{l} \cdot \mathbf{e}, \tilde{E} = \{1, 2, \dots, n\}$.
2. $k = \arg \min_{j \in \tilde{E}} \{w_j(x(j) + 1) - w_j(x(j))\}$.
3. **If** $x + \mathbf{e}^k$ *infeasible*
 $\tilde{E} \leftarrow \tilde{E} \setminus \{k\}, \delta_k = s$.
else, if $x + s \cdot \mathbf{e}^k$ *infeasible*
 $\tilde{E} \leftarrow \tilde{E} \setminus \{k\}, \alpha' = \hat{c}(x, k), x \leftarrow x + \alpha' \cdot \mathbf{e}^k$,
 $\tilde{B} \leftarrow \tilde{B} - \alpha', \delta_k = \alpha'$.
else
 $x \leftarrow x + s \cdot \mathbf{e}^k, \tilde{B} \leftarrow \tilde{B} - s, \delta_k = s$.
4. **If** $\tilde{B} = 0$ or $\tilde{E} = \emptyset$
Output x and *STOP*.
else
Go to step 2.

$\hat{c}(x, k)$ is the saturation capacity defined as the maximum amount by which $x(k)$ can be increased without violating feasibility and is given as follows:

$$\hat{c}(x, k) = \min \{g(A) - \sum_{j \in A} x(j) \mid k \in A \subseteq E\}.$$

The value is then recorded in the variable δ_k .

Hochbaum [11] and Moriguchi and Shioura [21] showed the proximity result that if $x^\star$ is the optimal solution to Π_2 and $x^{(s)}$ is the output of $\text{greedy}(s, \tilde{l})$, then, with $\delta = (\delta_1, \delta_2, \dots, \delta_n)$,

$$x^\star \stackrel{(a)}{>} x^{(s)} - \delta \geq x^{(s)} - s \cdot \mathbf{e}. \quad (23)$$

GAP executes $\text{greedy}(s, \tilde{l})$ in each iteration starting from $s = \lceil \frac{B}{2n} \rceil$ in the first iteration and halving it in each iteration till it reaches unity. The output of $\text{greedy}(s, \tilde{l})$, deducted by s units, provides an increasingly tighter lower bound to $x^\star$ in each iteration. The lower bound serves as initial allocation for the variable x in $\text{greedy}(s, \tilde{l})$ in the following iteration. When $s = 1$, $\text{greedy}(s, \tilde{l})$ puts out the optimum value of Π_2 . A formal description of GAP is given here.

Algorithm 3 GAP

1. $i = 0, s_0 = \lceil \frac{B}{2n} \rceil, l^{(s_0)} = l$.
2. $x^{(s_i)} = \text{greedy}(s_i, l^{(s_i)})$.
3. **If** $s_i = 1$
Output $x^\star = x^{(s_i)}$ and *STOP*.
else
 $l^{(s_{i+1})} = \max \{l^{(s_i)}, x^{(s_i)} - s_i \cdot \mathbf{e}\}, s_{i+1} = \lceil \frac{s_i}{2} \rceil, i \leftarrow i + 1$.
Go to step 2.

As a consequence of the proximity theorem, $l^{(s_i)}, i = 1, 2, \dots$, form an increasing sequence of lower bounds of $x^\star$. Hence GAP tightens the lower bound on $x^\star$ in each iteration.

Let $x \in P(g), \tilde{E} = \{j \mid x + \mathbf{e}^j \text{ is feasible}\}$. If

$$k \in \tilde{E} \text{ and } k = \arg \min_{j \in \tilde{E}} \{w_j(x(j) + 1) - w_j(x(j))\}, \quad (24)$$

then [22, Cor. 1] tells that $x^\star(k) \geq x(k)$. Clearly, $\text{greedy}(s, \tilde{l})$ picks exactly such a k as in (24) to update. Hence deducting the last increment of each variable from the output of $\text{greedy}(s, \tilde{l})$ results in a lower bound to the optimal value $x^\star$, which explains the inequality (a) of (23).

The complexity of Hochbaum's algorithm is $O(n(\log n + F) \log \frac{B}{n})$. Hochbaum [11] showed that, for the case of linear ascending constraints, feasibility check is equivalent to the disjoint set union problem. A feasibility check step is the same as a "union-find" operation. A series of $n \log(\frac{B}{n})$ "union-find" operations can be done in $n \log(\frac{B}{n})$ time [23]. Hence F is $O(1)$.

Let $x_c^\star$ be the solution to Π_1 (with the continuous variables). Let $x^\star$ be the output of GAP. Moriguchi *et al* [24, Th. 1.3] showed that

$$\|x_c^\star - x^\star\|_\infty \leq n - 1.$$

By incrementing in steps of ϵ instead of 1, it is now clear that GAP can be used to find an ϵ -optimal solution in time $O(n(\log n + F) \log(\frac{B}{\epsilon}))$.

5. A decomposition algorithm

In this section, we describe a decomposition algorithm proposed by Vidal *et al* [17] that solves problem Π . The complexity of the algorithm in solving Π is the same as that of Hochbaum's algorithm. For the case when problem Π has n variables and m ($m < n$) ascending constraints, the performance of the algorithm beats that of Hochbaum's algorithm. The problem for this case is as follows. Let $s[0] = 0, s[m] = n$ and $s[i] \in \{1, 2, \dots, n\}$ with $s[1] < s[2] < \dots < s[m-1] < n$.

Problem Π_3 :

$$\text{minimize } W(x) := \sum_{e=1}^n w_e(x(e)) \quad (25)$$

$$\text{subject to } x(e) \in [0, \beta(e)] \cap \mathbb{Z}, e = 1, 2, \dots, n, \quad (26)$$

$$\sum_{e=1}^{s[l]} x(e) \leq a_l, l = 1, 2, \dots, m-1, \quad (27)$$

$$\sum_{e=1}^n x(e) = B. \quad (28)$$

The algorithm solves Π_3 by a recursive process that leads to a hierarchy of sub-problems spanning across $1 + \lceil \log m \rceil$ levels. At each level, multiple sub-problems are solved. Each sub-problem involves a subset of variables with optimization done over a single sum constraint and upper and lower bounds on variables. The solution to these sub-problems bounds the value of the respective variables in the sub-problems in the next higher level.

We now give the main procedure that involves the tightening of the ascending constraints in Π_3 using the upper bounds on the variables and a call to the main recursive procedure $\text{Nestedsolve}(1, m)$, which will be described soon.

Algorithm 4 General solution procedure

- *Tightening*:

1. $\bar{a}_0 = 0; \bar{a}_m = B$

2. **for** $i = 1$ **to** $m-1$ **do**

$$\bar{a}_i \leftarrow \min \left\{ \bar{a}_{i-1} + \sum_{e=s[i-1]+1}^{s[i]} \beta(e), a_i \right\}.$$

- *Hierarchical resolution*:

3. $(\bar{c}_1, \bar{c}_2, \dots, \bar{c}_n) = (0, 0, \dots, 0)$.

4. $(\bar{d}_1, \bar{d}_2, \dots, \bar{d}_n) = (\beta(1), \beta(2), \dots, \beta(n))$.

5. $(x(1), x(2), \dots, x(n)) = \text{Nestedsolve}(1, m)$.

6. **return** $(x(1), x(2), \dots, x(n))$.

The output of the recursive procedure $\text{Nestedsolve}(v, w)$ minimizes the following optimization problem.

$\text{Nested}(v, w)$:

$$\text{minimize } W(x) := \sum_{e=s[v-1]+1}^{s[w]} w_e(x(e)) \quad (29)$$

$$\text{subject to } 0 \leq x(e) \leq \beta(e), e = s[v-1] + 1, \dots, s[w], \quad (30)$$

$$\sum_{e=s[v-1]+1}^{s[l]} x(e) \leq \bar{a}_l - \bar{a}_{v-1}, l = v, v+1, \dots, w-1, \quad (31)$$

$$\sum_{e=s[v-1]+1}^{s[w]} x(e) = \bar{a}_w - \bar{a}_{v-1}. \quad (32)$$

The procedure $\text{Nestedsolve}(v, w)$ recursively solves the afore-mentioned problem by solving the sub-problems $\text{Nestedsolve}(v, t)$ and $\text{Nestedsolve}(t+1, w)$ where $t = \lfloor \frac{v+w}{2} \rfloor$. These sub-problems are further divided in the same manner and at the lowest level, the sub-problems involve optimization over a single sum constraint. Consider the following optimization problem with single sum constraint and bounds on variables.

$\text{RAP}(v, w)$:

$$\text{minimize } W(x) := \sum_{e=s[v-1]+1}^{s[w]} w_e(x(e)) \quad (33)$$

$$\text{subject to } \hat{c}_e \leq x(e) \leq \hat{d}_e, e = s[v-1] + 1, \dots, s[w], \quad (34)$$

$$\sum_{e=s[v-1]+1}^{s[w]} x(e) = \bar{a}_w - \bar{a}_{v-1}. \quad (35)$$

The subproblem at the lowest level of the recursion is of the form $\text{RAP}(v, v)$ with $\hat{c}_e = 0$ and $\hat{d}_e = \beta_e$ for $e = s[v-1] + 1, \dots, s[v]$. We now describe the procedure to obtain the optimal solution to $\text{Nested}(v, w)$ from the solutions to $\text{Nested}(v, t)$ and $\text{Nested}(t+1, w)$. Let the solution to $\text{Nested}(v, t)$ and $\text{Nested}(t+1, w)$ be $(x^\downarrow(s[v-1] + 1), \dots, x^\downarrow(s[t]))$ and $(x^\uparrow(s[t] + 1), \dots, x^\uparrow(s[w]))$. Theorems 1 and 2 of [17] show that the optimal solution to $\text{Nested}(v, w)$, $x^\star = (x^\star(s[v-1] + 1), \dots, x^\star(s[w]))$, satisfies the following inequalities with $x(e) = x^\star(e)$ for all $e \in \{s[v-1] + 1, \dots, s[w]\}$.

$$0 \leq x(e) \leq x^\downarrow(e) \quad e \in \{s[v-1] + 1, \dots, s[t]\}, \quad (36)$$

$$\beta(e) \geq x(e) \geq x^\uparrow(e) \quad e \in \{s[t] + 1, \dots, s[w]\}. \quad (37)$$

It is easy to see that any $x = (x(s[v-1] + 1), \dots, x(s[w]))$ that satisfies (36), (37) and (32) is a feasible solution to $\text{Nested}(v, w)$. Also, $x^\star$ satisfies these three constraints as

observed earlier. This shows that $x^\star$ can be obtained by solving $\text{RAP}(v, w)$ with the following bounds on variables:

$$\hat{c}_e = 0, \hat{d}_e = x^\dagger(e) \quad e \in \{s[v-1] + 1, \dots, s[t]\}, \quad (38)$$

$$\hat{c}_e = x^\dagger(e), \hat{d}_e = \beta(e) \quad e \in \{s[t] + 1, \dots, s[w]\}. \quad (39)$$

$\text{Nestedsolve}(v, w)$ is described here.

Algorithm 5 $\text{Nestedsolve}(v, w)$

1. if $v = w$ then

$$(x(s[v-1] + 1), \dots, x(s[v])) = \text{RAP}(v, v)$$

else

$$t \leftarrow \left\lfloor \frac{v+w}{2} \right\rfloor$$

$$(x(s[v-1] + 1), \dots, x(s[t])) = \text{Nestedsolve}(v, t)$$

$$(x(s[t] + 1), \dots, x(s[w])) = \text{Nestedsolve}(t + 1, w)$$

for $i = s[v-1] + 1$ to $s[t]$ do

$$(\hat{c}_i, \hat{d}_i) = (0, x(i))$$

for $i = s[t] + 1$ to $s[w]$ do

$$(\hat{c}_i, \hat{d}_i) = (x(i), \beta(i))$$

$$(x(s[v-1] + 1), \dots, x(s[w])) = \text{RAP}(v, w)$$

The initial tightening of the constraints and initialization steps takes $O(n)$ steps. The main recursive procedure, $\text{Nestedsolve}(1, m)$, is solved as a hierarchy of RAP sub-problems, with $h = 1 + \lceil \log m \rceil$ levels of recursion. Each level $i \in \{1, 2, \dots, h\}$ has 2^{h-i} RAP sub-problems. A $\text{RAP}(v, w)$ sub-problem is solved by the method of Frederickson and Johnson [25], which works in $O(n \log(B/n))$ steps. A straightforward calculation shows that $\text{Nestedsolve}(1, m)$ works in $O(n(\log m) \log(B/n))$ steps.

6. A taut-string solution in a special case

In this section, we consider the minimization of an interesting subclass of separable convex functions known as d -separable convex functions [19] subject to ascending constraints (3) and (4). We will begin with a slightly more general setting. Let (E, g) be a polymatroid. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differentiable and strictly convex function. Fix $d = (d_1, d_2, \dots, d_n) \in \mathbb{R}_+^E$. The objective function W of Π_1 now has $w_e(x(e)) = d_e \phi\left(\frac{x(e)}{d_e}\right)$. Such a W is called d -separable.

We now state and prove a result that, for a fixed d , the minimizer is the same for any ϕ that is continuously differentiable and strictly convex. Further, the minimizer has a special structure.

Lemma 1 Let W in (13) be separable with $w_e(x(e)) = d_e \cdot \phi\left(\frac{x(e)}{d_e}\right)$, where ϕ is continuously differentiable and strictly convex. The $x^\star$ that minimizes W over the bases of the polymatroid (E, g) is the lexicographically optimal base of the polymatroid with respect to the weight vector d .

Proof W in (13) is a separable convex function with

$$w_e(x(e)) = d_e \phi\left(\frac{x(e)}{d_e}\right).$$

Let h_e be the derivative of w_e . By Theorem 2, the minimizer of (13) is the h -lexicographically optimal base of the polymatroid (E, g) , where we have

$$h_e(x(e)) = \phi'\left(\frac{x(e)}{d_e}\right), \quad e \in E.$$

Since $h_e = \phi'$ for all e and ϕ' is increasing, h -lexicographically optimal base is the lexicographically optimal base with respect to the weight vector d . $\square$

Given the flexibility in ϕ , let us choose $\phi = (u^2 + 1)^{1/2}$ [19]. Then

$$W(x) = \sum_{i=1}^n d_i \left(\left(\frac{x(i)}{d_i} \right)^2 + 1 \right)^{1/2} = \sum_{i=1}^n (x(i)^2 + d_i^2)^{1/2}. \quad (40)$$

Let us further restrict attention to ascending constraints of (3) and (4). Then the objective function (40) and constraints (3) and (4) have the following geometric interpretation. Let $D_i = \sum_{e=1}^i d_e$, $E_i = \sum_{e=1}^i \alpha(e)$ and $X_i = \sum_{e=1}^i x(e)$. Also, let $D_0 = E_0 = 0$. The constraints (3) and (4) are then

$$\begin{aligned} X_i &\geq E_i, \quad i = 1, 2, \dots, n-1, \\ X_n &= E_n. \end{aligned}$$

A vector x is feasible iff (D_i, X_i) lies above (D_i, E_i) in x - y plane for $i = 1, 2, \dots, n-1$. Also, (D_0, X_0) and (D_n, X_n) must coincide with (D_0, E_0) and (D_n, E_n) , respectively. Define feasible path as the path formed by the line segments joining (D_{i-1}, X_{i-1}) and (D_i, X_i) for $i = 1, 2, \dots, n$ for a feasible $X_1, X_2, \dots, X_n$. The length of the path corresponding to a feasible point $X_1, X_2, \dots, X_n$ gives the value of the objective function in (40) at the point. It is now obvious that the following *taut-string method* finds the minimum length path among all feasible paths, and hence the optimal solution, as described here.

- Place pins at the points (D_i, E_i) , $i = 0, 1, \dots, n$.
- Tie a string to the pin at origin and run the string above the points (D_i, E_i) , $i = 1, 2, \dots, n$.
- Pull the string tight. The string traces the minimum length path from the origin to the point (D_n, E_n) .

Figure 3 shows the *taut-string solution* for the points (D_i, E_i) , $i = 1, 2, \dots, 5$ in $\mathbb{R}^2$. All the paths starting from the origin and ending at (D_5, E_5) that lie above the broken line correspond to feasible paths. The bold line in the figure traces the path of the tightened string. Since E_i is increasing in i , X_i corresponding to the shortest path is also increasing as can be observed from the figure and hence the optimal x belongs to $\mathbb{R}_+^E$. The *taut-string method* solves the special case of minimization of a d -separable objective subject to ascending constraints.

It is clear that the taut string traces the concave cover, denoted by $\mathcal{C}$, which has the following properties:

- $\mathcal{C}$ is piece-wise linear and concave.
- $\mathcal{C}(0) = 0, \mathcal{C}(D_n) = E_n$ and $\mathcal{C}(D_k) \geq E_k \forall k$.
- The slope-change points of the piece-wise linear function lie on a subset of the set of points $\{D_1, D_2, \dots, D_{n-1}\}$ and $\mathcal{C}(D_i) = E_i$ if D_i is a slope-change point.

Note that the concave cover of a set of points is completely specified by the slope-change points. The minimum length path is also the graph of the least concave majorant of the points (D_i, E_i) , $i = 1, 2, \dots, n$. Veinott Jr.'s [19] computation of the optimal x corresponding to the minimum length path requires the computation of the least concave majorant of the points (D_i, E_i) , $i = 0, 1, \dots, n$. The algorithm of [19] runs in $O(n^2)$ steps.¹

In the next section, we provide an $O(n)$ algorithm for finding the concave cover of the set of points $(D_0, E_0), (D_1, E_1), (D_2, E_2), \dots, (D_n, E_n)$.

7. String algorithm

In the previous section, we described the method proposed by Veinott Jr. that reduces problem Π for the case of a d -separable objective function to a geometrical problem of finding the concave cover of the set of points $(D_0, E_0), (D_1, E_1), (D_2, E_2), \dots, (D_n, E_n)$ in x - y plane. Let the points be denoted as $t_0, t_1, t_2, \dots, t_n$. In this section, we describe an algorithm that finds the concave cover of these points in $O(n)$ steps. The algorithm is an adaptation of the string algorithm of Muckstadt and Sapra [20] that finds a “convex envelope” of a set of points for an inventory management problem.

Consider a piece-wise linear function with the slope-change abscissa locations being a subset of $\{D_0, D_1, D_2, \dots, D_n\}$. The piece-wise linear function is concave if the slopes of the adjacent straight line segments

are decreasing; more precisely, if $t_i = (D_i, E_i)$ corresponds to a D_i where slope changes, then

$$\text{slope of } t_i - t_{LN(i)} > \text{slope of } t_{RN(i)} - t_i, \quad (41)$$

where $LN(i)$ is the index of the slope-change point closest to the left of D_i , and $RN(i)$ is the index of the slope-change point closest to the right of D_i . Figure 4a is plotted taking $i = 1$.

The algorithm takes as input $t_0, t_1, \dots, t_n$. The algorithm checks the concavity condition in (41) at every point. If the condition is not satisfied at a point t_i , then t_i is dropped. After performing a sequence of such eliminations, the algorithm puts out the slope-change points of the desired concave cover. The algorithm is formally described below.

Algorithm 6 String algorithm

1. **Input:** A sequence of points $t_0, t_1, \dots, t_n$ with increasing (or nondecreasing abscissas).
2. **Initialize:** $j = 0$, and $LN(i) = i - 1$, for $i = 1, 2, \dots, n$ and $RN(i) = i + 1$, for $i = 0, 1, \dots, n - 1$.

3. Forward Step:

```

j ← j + 1.
if j ≠ n
    i = j.
    Go to Backward Step.
else
    return RN.
```

4. Backward Step:

```

if condition (41) is satisfied at point  $t_i$ 
    go to Forward Step.
else
    LN(RN(i)) = LN(i),
    RN(LN(i)) = RN(i),
    RN(i) = null.
    if LN(i) ≠ 0
        i ← LN(i)
        go to Backward Step.
    else
        go to Forward Step.
```

Figure 4 shows the key steps of the string algorithm for the case $n = 6$. Figure 4b shows the backward step of the string algorithm at t_4 . As the concavity condition in (41) is violated at t_4 , the point is dropped. Now, the left adjacent slope-change point of D_5 is D_3 and the right adjacent point of D_3 is D_5 . These changes are made in the backward step of the algorithm by updating the LN and RN pointers.

At the completion of the algorithm, the indices corresponding to the valid entries (entries excluding *null*) in RN

¹Veinott Jr. [19] considered another problem as well, one with upper and lower bounds on the variables on top of the ascending constraints. Veinott Jr. [19] provided a taut-string solution and a common algorithm for both problems. It is this common algorithm that takes $O(n^2)$ steps.

Figure 3. The taut-string solution

are the indices of the slope-change points of the concave cover. This is straightforward to check and a formal proof is omitted. The piece-wise linear function formed by bold lines in figure 4d is the concave cover of the set of points $t_0, t_1, t_2, \dots, t_6$. All points except t_2 are dropped and D_2 is the only slope-change point for the concave cover.

The following lemma gives the complexity of string algorithm.

Proposition 3 The complexity of the string algorithm is $O(n)$.

Proof The number of executions of the forward step is n , each consisting of a constant number of operations. Hence the complexity of forward procedure is $O(n)$. The number of times the backward step is executed is equal to the number of points dropped, which is less than n . In a single execution of the backward step, the concavity condition is checked. If a point is dropped, then the values of $LN(RN(i))$ and $RN(LN(i))$ are modified. All these amount to a constant

Figure 4. String algorithm (a) Backward step at $j = 1$ (b) Backward step at $j = 4$ (c) Backward step at $j = 5$ at node t_5 (d) Backward step at $j = 5$ at node t_3 .

Table 1. Algorithm complexity.

Optimization problem	Algorithm	Complexity
Separable convex optimization over the bases of a polymatroid	Greedy algorithm [11]	$O(n(\log n + F) \log(B/(n\epsilon)))$
Separable convex optimization over linear ascending constraints, m constraints, $m \leq n$	Greedy algorithm [11]	$O(n(\log n) \log(B/(n\epsilon)))$
	Decomposition algorithm [17]	$O(n(\log m) \log(B/(n\epsilon)))$
	Algorithm 1	$O(n^2 H)$
d -separable convex function over linear ascending constraints	String algorithm [20]	$O(n)$

number of operations, which does not scale with n . Hence the total complexity of the backward procedure is also $O(n)$. This completes the proof of the proposition. $\square$

8. Comparison and discussion

In this section, we compare the complexity of Algorithm 1 to those of the best-in-class algorithms such as the scaling-based greedy algorithm proposed by Hochbaum [11] and the decomposition algorithm proposed by Vidal *et al* [17]. These algorithms have $O(n(\log n) \log(B/(n\epsilon)))$ complexity to get to an ϵ -optimal solution, with B being the total number of units to be allocated. When the constraint set of Problem Π is the base set of a polymatroid (of which the linear ascending constraints is a special case), Hochbaum's algorithm [11] solves the optimization problem incurring an additional complexity of $O(nF \log(B/(n\epsilon)))$ where F is the complexity to check the feasibility of the vector resulting from a given increment of a feasible vector. In the linear ascending constraints case, the algorithm of Vidal *et al* [17] outperforms Hochbaum's algorithm by a factor of $\frac{\log n}{\log m}$ when the number of constraints is m , $m < n$. However, no extension of the algorithm of Vidal *et al* [17] is known for the case of polymatroids.

We compute the complexity of Algorithm 1 under the assumption that an oracle provides $H_e^+(\eta)$ and $H_e^-(\eta)$ for any η in $O(1)$ complexity. The complexity of the algorithm is dominated by three steps, viz., the non-linear equation in (14), Subroutines 1 and 2. Observe that Algorithm 1 assigns at least one variable in each iteration and hence terminates in at most n iterations. In each iteration, the algorithm solves the non-linear equation in (14) at most $O(n)$ times. Denote the complexity of solving the non-linear equation as H , which is a function of the desired accuracy ϵ and the number of variables n . The total complexity of solving the non-linear equation over all iterations of the algorithm is then $O(n^2 H)$. It is straightforward to see that the complexity of Subroutine 1 and Subroutine 2 in the j th iteration is $O(s(j-1) - s(j))$ and $O((s(j-1) - s(j))^2)$, respectively. Hence their complexities over all the iterations are $O(n)$ and $O(n^2)$, respectively.

Table 1 lists the complexities of the algorithms described in this paper. Padakandla and Sundaresan [3] showed via

simulations that Algorithm 1, for the case of strictly convex functions, outperforms the standard CVX solver. They compared the performance of the algorithms for four different objective functions and for variable sizes ranging from 50 to 500. In each case, Algorithm 1 performed better than the CVX solver. See [3, Sect. 4] for the simulation results. Vidal *et al* [17, Sect. 6] performed simulations that compare their algorithm to Algorithm 1, Hochbaum's algorithm [11], Wang's algorithm [1] and a large scale optimization problem solver, MOSEK. The simulations were performed for several problem instances and variable sizes ranging from 10 to 10^6 . MOSEK showed an exponential increase in the running time with the increase in variable sizes. All other algorithms showed a linear increase in running time. The rate of increase was smaller for the algorithms of Hochbaum [11] and Vidal *et al* [17] compared with Algorithm 1, which is in accordance to their theoretically computed complexities.

9. Conclusion

We discussed several algorithms that solve problem Π , a separable convex optimization problem with linear ascending constraints, that arises as a core optimization in several resource allocation problems. The algorithms can be classified as greedy-type or decomposition-type algorithms. The best-in-class algorithms have $O(n(\log n) \log(B/(n\epsilon)))$ complexity. A d -separable objective was also considered. In this special case, the solution is the lexicographically optimal base of a polymatroid formed by the constraint set of Π . We then argued that finding the lexicographically optimal base is equivalent to finding the least concave majorant of a set of points on the $\mathbb{R}_+^2$ quadrant. We then described an $O(n)$ algorithm for this problem. This is significant because of its applicability to the minimization of popular d -separable functions such as α -fair utility functions, which are widely used as utility functions in network utility maximization problems.

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Appendix I. Proof of correctness and optimality

In this section, we give the proof of Theorem 3.

1 Feasibility

We begin by addressing a necessary and sufficient condition for feasibility.

Lemma 2 *The feasible set is nonempty if and only if (5) is satisfied.*

Proof Assume that (5) is not satisfied for some l and let l' be the smallest among such indices. This implies that even if we assign the largest possible value for $x(e)$, i.e., $x(e) = \beta(e)$ for $e = 1, 2, \dots, l'$, the constraint (3) for $l = l'$ cannot be satisfied, and hence the constraint set is empty.

To prove sufficiency, assume that (5) holds. Let $l = l'$ be the smallest index l for which

$$\sum_{e=1}^l \beta(e) \geq \sum_{e=1}^n \alpha(e) \quad (\text{A1})$$

holds. Now, assign $x(e) = \beta(e)$ for $e = 1, 2, \dots, l' - 1$; assign

$$x(l') = \sum_{e=1}^n \alpha(e) - \sum_{e=1}^{l'-1} \beta(e),$$

and $x(e) = 0$ for $e = l' + 1, l' + 2, \dots, n$. Clearly, x satisfies (3) and (4) and is therefore a feasible point. This proves the lemma. $\square$

2 Proof of Theorem 3

In order to prove Theorem 3, the following should be shown to ensure that the algorithm terminates and generates the desired allocation.

1. The set whose minimum is taken in (16) should be nonempty at each iteration step.
2. The assignments in (19) and (20) should yield a feasible allocation at each iteration step, and the reduced problem for the next iteration is a similar but smaller problem.
3. The output of the algorithm should be feasible.
4. The output should satisfy the sufficiency conditions for optimality in Theorem 1.

We begin by proving that the minimum in (16) is over a nonempty set. For this, we need the following lemma.

Lemma 3 *At any iteration step j , for any l with $1 \leq l < s(j-1)$, there exists an η_l^j satisfying (14) if and only if*

$$0 \leq \sum_{e=1}^{s(j-1)-1} \alpha(e) \leq \sum_{e=1}^{s(j-1)-1} \beta(e). \quad (\text{A2})$$

Proof Let $\bar{\eta} = \max \{w_e^-(\beta(e)) : l \leq e < s(j-1)\}$. From the definition of H_e^+ , for all $\eta \geq \bar{\eta}$, we have

$$H_e^+(\eta) = \beta(e) \quad (l \leq e < s(j-1)).$$

Hence

$$\sum_{e=l}^{s(j-1)-1} H_e^+(\eta) = \sum_{e=l}^{s(j-1)-1} \beta(e) \quad (\eta \geq \bar{\eta}). \quad (\text{A3})$$

Next, define $\underline{\eta} = \min \{w_e^+(0) : l \leq e < s(j-1)\}$. The definition of H_e^- implies that for all $\eta \leq \underline{\eta}$, we have

$$H_e^-(\eta) = 0 \quad (l \leq e < s(j-1)).$$

Hence

$$\sum_{e=l}^{s(j-1)-1} H_e^-(\eta) = 0 \quad (\eta \leq \underline{\eta}). \quad (\text{A4})$$

Necessity of (A2) is then obvious from (A3) and (A4), since H_e^+ and H_e^- are nondecreasing. For sufficiency, in addition to the nondecreasing nature of H_e^+ and H_e^- , we also have $H_e^+ = H_e^-$ at all points of continuity of H_e^+ , and H_e^- is the left continuous version of the right continuous H_e^+ . Given (A2) and these observations, it follows that we can find an η that satisfies (14). $\square$

Proposition 4 *If the feasible region is nonempty, then at any iteration step j , the index $l = 1$ satisfies (15). Hence the set over which the minimum is taken in (16) is nonempty.*

Proof If the feasible region is nonempty, we have from Lemma 2 that (5) holds for $l = s(j-1) - 1$, i.e.,

$$\sum_{e=1}^{s(j-1)-1} \alpha(e) \leq \sum_{e=1}^{s(j-1)-1} \beta(e).$$

By Lemma 3, η_l^j exists. Consequently, the set over which the minimum is taken now contains η_1^j and is therefore nonempty. $\square$

We shift attention to Subroutine 1. We show that the $s(j)$ put out by Subroutine 1 satisfies a property that is crucial to prove the feasibility of the output of Algorithm 1. This is the property that the partial sums of $H_e^+(\Gamma_j)$ from $s(j)$ to each of the tied indices exceeds the corresponding partial sums of $\alpha(e)$. Since we will show equality of the constraints at $s(j)$, this property is necessary for feasibility.

Proposition 5 *The index $s(j)$ chosen by Subroutine 1 satisfies the following property:*

Let $l_1, l_2, \dots, l_r$ be the indices that attain the minimum in (16) with $s(j-1) > l_1 > l_2 > \dots > l_r \geq 1$. Let l_{p+1} with $1 \leq p+1 \leq r$ be the index chosen by Subroutine 1 as $s(j)$. Denote $I = \{l_1, l_2, \dots, l_p\}$. When $p = 0$ and I is empty then

$$\sum_{e=s(j)}^{l-1} H_e^+(\Gamma_j) \geq \sum_{e=s(j)}^{l-1} \alpha(e) \quad (l \in I). \quad (\text{A5})$$

Proof If $s(j) = l_r$ at the end of Subroutine 1, i.e., the iterate t did not decrement at all, it follows that (17) holds for all $k \in \{l_1, l_2, \dots, l_{r-1}\} \cup \{s(j-1)\}$ and $m = l_r$. Hence (A5) is trivially true.

Now suppose $s(j) > l_r$. In Subroutine 1, the variables i and t have initial values 1 and r , respectively. If (17) is satisfied for $m = l_r$ and $k = l_i$, then the value of i is increased by unity, until (17) is not satisfied at some $i = p'$ and $t = r$, i.e.,

$$\sum_{e=l_r}^{l_{p'}-1} H_e^+(\Gamma_j) < \sum_{e=l_r}^{l_{p'}-1} \alpha(e), \quad (\text{A6})$$

and therefore (17) is satisfied for all s satisfying $1 \leq s < p'$, i.e.,

$$\sum_{e=l_r}^{l_s-1} H_e^+(\Gamma_j) \geq \sum_{e=l_r}^{l_s-1} \alpha(e) \quad (1 \leq s < p'). \quad (\text{A7})$$

Moreover, $l_{p'} \geq s(j)$. If (17) is not satisfied already for $m = l_r$ and $k = l_i$, then $p' = 1$ and (A7) is irrelevant. Now the algorithm reduces the value of t in steps of unity until (17) is satisfied for $m = l_{r'}$ and $k = l_{p'}$, i.e.,

$$\sum_{e=l_{p'}}^{l_{p'}-1} H_e^+(\Gamma_j) \geq \sum_{e=l_{p'}}^{l_{p'}-1} \alpha(e), \quad (\text{A8})$$

with $l_{p'} \leq s(j)$. From (A6) and (A7), we get (for $p' > 1$)

$$\sum_{e=l_{p'}}^{l_s-1} H_e^+(\Gamma_j) \geq \sum_{e=l_{p'}}^{l_s-1} \alpha(e) \quad (1 \leq s < p'). \quad (\text{A9})$$

Summing (A8) and (A9) when $p' > 1$, or simply considering (A8) when $p' = 1$, we get

$$\sum_{e=l_{p'}}^{l_s-1} H_e^+(\Gamma_j) \geq \sum_{e=l_{p'}}^{l_s-1} \alpha(e) \quad (1 \leq s \leq p'). \quad (\text{A10})$$

However, (A10) is just (17) for $m = l_{p'}$ and $k = l_s$ with $1 \leq s \leq p'$. Proceeding by induction, r' decrements and p' increments, and eventually they coincide at $s(j)$, and (A5) holds for all $1 \leq l < s(j-1)$. Validity of (A5) for $l = s(j-1)$ is clear from the definition of $\eta_{s(j)}^j$. $\square$

The following lemma is a corollary to Proposition 5 and is useful in proving optimality.

Lemma 4 The sequence $\{\Gamma_i : i = 1, 2, 3, \dots, k\}$ put out by Algorithm 1 satisfies $\Gamma_1 < \Gamma_2 < \dots < \Gamma_k$.

Proof We will prove that at iteration $j+1$, the number η_l^{j+1} that is the (smallest) solution (if it exists) of

$$\sum_{e=l}^{s(j)-1} H_e^-(\eta) \leq \sum_{e=l}^{s(j)-1} \alpha_e \leq \sum_{e=l}^{s(j)-1} H_e^+(\eta) \quad (\text{A11})$$

for l satisfying $1 \leq l \leq s(j)$ is strictly greater than Γ_j . Hence their minimum Γ_{j+1} is also strictly greater than Γ_j , and the proof will be complete.

For indices l that satisfy (14) in the j th iteration with $l < s(j)$ and $\eta_l^j = \Gamma_j$ (i.e., l is a tied index), there is some l' satisfying $s(j-1) > l' \geq s(j)$ and

$$\sum_{e=l}^{l'-1} H_e^+(\Gamma_j) < \sum_{e=l}^{l'-1} \alpha(e); \quad (\text{A12})$$

otherwise index t in Subroutine 1 would have pointed to $s(j)$, and an $s(j) > l$ would not have been picked by the subroutine. However, from Proposition 5, we also have

$$\sum_{e=s(j)}^{l'-1} H_e^+(\Gamma_j) \geq \sum_{e=s(j)}^{l'-1} \alpha(e). \quad (\text{A13})$$

Subtract (A13) from (A12) to get

$$\sum_{e=l}^{s(j)-1} H_e^+(\Gamma_j) < \sum_{e=l}^{s(j)-1} \alpha(e). \quad (\text{A14})$$

Hence $\eta_l^{j+1} > \Gamma_j$ for all such tied l . For all non-tied indices $l < s(j)$, i.e., indices that satisfy (14) in j th iteration but with $\eta_l^j > \Gamma_j$ or η_l^j does not exist, we must have

$$\sum_{e=l}^{s(j)-1} H_e^+(\Gamma_j) < \sum_{e=l}^{s(j)-1} \alpha(e), \quad (\text{A15})$$

and therefore after noting (A14) for the tied indices, we conclude that (A15) holds for all $l < s(j)$. Furthermore, since $s(j)$ is a tied index and $\eta_{s(j)}^j = \Gamma_j$, we also have

$$\sum_{e=s(j)}^{s(j)-1} H_e^+(\Gamma_j) \geq \sum_{e=s(j)}^{s(j)-1} \alpha(e). \quad (\text{A16})$$

Subtract (A16) from (A15) to get

$$\sum_{e=l}^{s(j)-1} H_e^+(\Gamma_j) < \sum_{e=l}^{s(j)-1} \alpha(e) \quad (l < s(j)). \quad (\text{A17})$$

Since H_e^+ is nondecreasing, the solution η_l^{j+1} to (A11) must be strictly larger than Γ_j , i.e., $\eta_l^{j+1} > \Gamma_j$, if the solution exists. $\square$

We next move to Subroutine 2. This subroutine assigns values to variables (from higher indices to lower indices) in stages over possibly several iterations. After each iteration, the assignment problem reduces to a similar but smaller problem. The next lemma is a step to say that every index between $s(j)$ and $s(j-1)$, both included, can be assigned

successfully within Subroutine 2 without a need to execute Subroutine 1 after every substage. In particular, $s(j)$ and Γ_j would remain stable over the entire execution of Subroutine 2.

Lemma 5 In step 5 of Subroutine 2, the indices satisfying $s(j) < l < l_t^m$ and (21) is a subset of $I_{m+1} = I \cap \{l : l < l_t^m\}$. Hence the set I_{m+1} contains all the indices $l < l_t^m$ that satisfy (21).

Proof For any index l' satisfying $s(j) < l' < l_t^m$ with $l' \notin I \cap \{l : l < l_t^m\}$, either $\eta_{l'}^j$ does not exist or l' is not a tied index, i.e., $\eta_{l'}^j$ exists but satisfies $\eta_{l'}^j > \Gamma_j$. In the former case, by Lemma 3 and the fact that $\alpha(e)$ are positive, we have

$$\sum_{e=l'}^{s(j)-1} \beta(e) < \sum_{e=l'}^{s(j)-1} \alpha(e). \quad (\text{A18})$$

Hence

$$\sum_{e=l'}^{s(j)-1} H_e^+(\Gamma_j) \leq \sum_{e=l'}^{s(j)-1} \beta(e) < \sum_{e=l'}^{s(j)-1} \alpha(e). \quad (\text{A19})$$

In the latter case, we must have

$$\sum_{e=l'}^{s(j)-1} H_e^+(\Gamma_j) < \sum_{e=l'}^{s(j)-1} \alpha(e) \quad (\text{A20})$$

for otherwise Γ_j would be a strictly smaller choice for $\eta_{l'}^j$, contradicting the choice of $\eta_{l'}^j$. Hence, in either case, we have that (A20) holds. However, we also have

$$\sum_{e=l_t^m}^{s(j)-1} H_e^+(\Gamma_j) \geq \sum_{e=l_t^m}^{s(j)-1} \alpha(e). \quad (\text{A21})$$

Subtracting the two inequalities, we get

$$\sum_{e=l'}^{l_t^m-1} H_e^+(\Gamma_j) < \sum_{e=l'}^{l_t^m-1} \alpha(e), \quad (\text{A22})$$

i.e., l' will not satisfy (18), which is what we set out to prove. $\square$

The following lemma is a key inductive step to show that the assignment problem in step 3 of Subroutine 2 reduces the problem to a similar but smaller problem after each iteration.

Lemma 6 Suppose at the m th iteration in Subroutine 2, we have indices $l_0^m, l_1^m, \dots, l_{p_m}^m, l_{p_m+1}^m = s(j)$. Let $M_m = \{s(j), s(j) + 1, \dots, l_0^m - 1\}$ and with $I_m = \{l_1^m, l_2^m, \dots, l_{p_m}^m\}$, let $I_m \cup \{s(j)\}$ be the indices l that satisfy

$$\sum_{e=l}^{l_0^m-1} H_e^-(\Gamma_j) \leq \sum_{e=l}^{l_0^m-1} \alpha(e) \leq \sum_{e=l}^{l_0^m-1} H_e^+(\Gamma_j). \quad (\text{A23})$$

Note that $I_m \subseteq M_m$. For all indices $l \in M_m \setminus (I_m \cup \{s(j)\})$, we then have

$$\sum_{e=l}^{l_0^m-1} H_e^+(\Gamma_j) < \sum_{e=l}^{l_0^m-1} \alpha(e). \quad (\text{A24})$$

Let M_{m+1} be the set of indices in M_m corresponding to $x(e)$ that are not assigned in the m^{th} iteration and let I_{m+1} be the set obtained in step 5 of Subroutine 2. The set I_{m+1} and $M_{m+1} \setminus (I_{m+1} \cup \{s(j)\})$ satisfy properties (A23) and (A24), respectively, with m replaced by $m + 1$.

Proof Note that from step 2 of Subroutine 2, l_t^m is chosen so that

$$\gamma = \sum_{e=l_t^m}^{l_0^m-1} \alpha(e) - \sum_{e=l_t^m}^{l_0^m-1} H_e^-(\Gamma_j). \quad (\text{A25})$$

With $l_0^{m+1} = l_t^m$, we also have

$$M_{m+1} = \{s(j), s(j) + 1, \dots, l_0^{m+1} - 1\}, \\ I_{m+1} = \{l : l \in I, l < l_t^m, \text{ and satisfies (21)}\}.$$

We claim that

$$\sum_{e=l}^{l_t^m-1} H_e^+(\Gamma_j) \geq \sum_{e=l}^{l_t^m-1} \alpha(e) \quad (l \in I_{m+1} \cup \{s(j)\}). \quad (\text{A26})$$

Indeed, the inequalities hold true for $l \in I_{m+1}$ by construction and for $l = s(j)$ by Proposition 5. For $l \in I_{m+1} \cup \{s(j)\}$, we have

$$\gamma = \sum_{e=l_t^m}^{l_0^m-1} \alpha(e) - \sum_{e=l_t^m}^{l_0^m-1} H_e^-(\Gamma_j) \leq \sum_{e=l}^{l_0^m-1} \alpha(e) - \sum_{e=l}^{l_0^m-1} H_e^-(\Gamma_j) \quad (\text{A27})$$

which on rearrangement yields

$$\sum_{e=l}^{l_t^m-1} H_e^-(\Gamma_j) \leq \sum_{e=l}^{l_t^m-1} \alpha(e) \quad (l \in I_{m+1} \cup \{s(j)\}). \quad (\text{A28})$$

Since $l_t^m = l_0^{m+1}$, (A28) and (A26) yield (A23) with $m + 1$ in place of m . The analogue of property (A24) for $l \in M_{m+1} \setminus (I_{m+1} \cup \{s(j)\})$ follows from Lemma 5. $\square$

Lemma 7 Assume that I_m and M_m are as in Lemma 6 and such that (A23) holds for indices in $I_m \cup \{s(j)\}$ and (A24) for indices in $M_m \setminus (I_m \cup \{s(j)\})$. The assignments in (19) and (20) in Subroutine 2 in a particular iteration satisfy

$$\sum_{e=l}^{l_0^m-1} x(e) \leq \sum_{e=l}^{l_0^m-1} \alpha(e) \quad (l_t^m \leq l < l_0^m) \quad (\text{A29})$$

with equality for $l = l_t^m$. Furthermore, it is possible to assign values to $x(e)$ as in step 3 without violating the feasibility constraints.

Proof Since $l_t^m \in I_m$, the left inequality in (A23) of Lemma 6 implies

$$\sum_{e=l_t^m}^{l_0^m-1} H_e^-(\Gamma_j) \leq \sum_{e=l_t^m}^{l_0^m-1} \alpha(e). \quad (\text{A30})$$

After rearrangement, we get

$$0 \leq \gamma = \sum_{e=l_t^m}^{l_0^m-1} \alpha(e) - \sum_{e=l_t^m}^{l_0^m-1} H_e^-(\Gamma_j) \leq \sum_{e=l_1^m}^{l_0^m-1} \alpha(e) - \sum_{e=l_1^m}^{l_0^m-1} H_e^-(\Gamma_j) \quad (\text{A31})$$

because of the choice of t attaining the minimum in (15).

Adding $\sum_{e=l_1^m}^{l_0^m-1} H_e^-(\Gamma_j)$, we get

$$\sum_{e=l_1^m}^{l_0^m-1} H_e^-(\Gamma_j) \leq \gamma + \sum_{e=l_1^m}^{l_0^m-1} H_e^-(\Gamma_j) \leq \sum_{e=l_1^m}^{l_0^m-1} \alpha(e) \stackrel{(a)}{\leq} \sum_{e=l_1^m}^{l_0^m-1} H_e^+(\Gamma_j) \quad (\text{A32})$$

where (a) follows because $l_1^m \in I_m$. Also, any l satisfying $l_1^m < l < l_0^m$ is not in I_m and hence by Lemma 6

$$\sum_{e=l}^{l_0^m-1} H_e^+(\Gamma_j) \leq \sum_{e=l}^{l_0^m-1} \alpha(e) \quad (l_1^m < l < l_0^m). \quad (\text{A33})$$

From (A32) and (A33), it is evident that there exists an assignment for $x(e) \in [H_e^-(\Gamma_j), H_e^+(\Gamma_j)]$, when $l_1^m \leq e < l_0^m$, that gives equality in (19) without violating the feasibility constraints for $l_1^m \leq e < l_0^m$, i.e., without violating (A29).

Now, for indices l with $l_t^m < l < l_1^m$ and $l \notin I_m$ assigning $x(e)$ according to (20) does not violate feasibility constraints (A29) because we have

$$\sum_{e=l}^{l_0^m-1} H_e^+(\Gamma_j) < \sum_{e=l}^{l_0^m-1} \alpha(e). \quad (\text{A34})$$

Indeed, $x(e) = H_e^-(\Gamma_j)$ for $l_t^m \leq e < l_1^m$ and $x(e) \leq H_e^+(\Gamma_j)$ for $l_1^m \leq e < l_0^m$ and therefore (A34) implies

$$\sum_{e=l}^{l_0^m-1} x(e) \leq \sum_{e=l}^{l_0^m-1} H_e^+(\Gamma_j) < \sum_{e=l}^{l_0^m-1} \alpha(e), \quad (\text{A35})$$

which shows (A29).

For indices l with $l_t^m < l < l_1^m$ and $l \in I_m \cup \{s(j)\}$, assignment of $x(e)$ according to (20) is also feasible because

$$\begin{aligned} \sum_{e=l}^{l_0^m-1} x(e) &= \left(\gamma + \sum_{e=l_1^m}^{l_0^m-1} H_e^-(\Gamma_j) \right) + \sum_{e=l}^{l_1^m-1} H_e^-(\Gamma_j) \\ &= \gamma + \sum_{e=l}^{l_0^m-1} H_e^-(\Gamma_j) \\ &\stackrel{(b)}{\leq} \sum_{e=l}^{l_0^m-1} \alpha(e) - \sum_{e=l}^{l_0^m-1} H_e^-(\Gamma_j) + \sum_{e=l}^{l_0^m-1} H_e^-(\Gamma_j) \\ &= \sum_{e=l}^{l_0^m-1} \alpha(e) \end{aligned}$$

where (b) follows because γ is the minimum in (18) among all $l \in I_m \cup \{s(j)\}$. Moreover, the inequality is an equality when $l = l_t^m$ because the minimum is then attained. This proves that the assignment in step 3 is feasible. $\square$

Lemma 8 (Correctness of the algorithm) *If the feasible set is nonempty, Algorithm 1 runs to completion and puts out a feasible vector.*

Proof Since H_e^+ and H_e^- assume values between 0 and $\beta(e)$, constraint (2) is trivially satisfied. It is also straightforward to see that constraint (3) is satisfied if and only if

$$\sum_{e=l}^n x(e) \leq \sum_{e=l}^n \alpha(e) \quad (1 < l \leq n) \quad (\text{A36})$$

hold. Inequalities (A36) are obtained by subtracting (3) from (4). Now we show that the vector put out by Algorithm 1 satisfies (A36) and (4).

Consider the first iteration of Algorithm 1. Since the feasible set is nonempty, from Lemma 3, it follows that there exists an η that solves inequality (14). Hence the set over which minimum is taken in (16) is nonempty. Subroutine 1 sets the value of $s(1)$. In the first iteration of Subroutine 2, observe that $M_0 = \{s(1), s(1) + 1, \dots, n\}$ and $I_0 = I = \{l_1, l_2, \dots, l_p\}$, a subset of the indices that attain the minimum in (16), with p the smallest index such that l_p is strictly greater than $s(1)$. Clearly $I_0 \cup \{s(1)\}$ and $M_0 \setminus (I_0 \cup \{s(1)\})$ satisfy (A23) and (A24), respectively. By Lemma 6 and by induction, in every iteration of Subroutine 2, $I_m \cup \{s(1)\}$ and $M_m \setminus (I_m \cup \{s(1)\})$ satisfy (A23) and (A24), respectively. Moreover by Lemma 7, in every iteration of Subroutine 2, we allocate $x(e)$ for $l_t^m \leq l < l_0^m$ such that

$$\sum_{e=l}^{l_0^m-1} x(e) \leq \sum_{e=l}^{l_0^m-1} \alpha(e) \quad (\text{A37})$$

with equality for $l = l_t^m$. By induction on m , and observing that $l_0^0 = n + 1$, the output of Subroutine 2 satisfies

$$\sum_{e=l}^n x(e) \leq \sum_{e=l}^n \alpha(e) \quad (s(1) \leq l \leq n), \quad (\text{A38})$$

with equality for $l = s(1)$.

Now assume that (A38) is true for l satisfying $s(j-1) \leq l \leq n$, i.e.

$$\sum_{e=l}^n x(e) \leq \sum_{e=l}^n \alpha(e) \quad (s(j-1) \leq l \leq n), \quad (\text{A39})$$

with equality for $l = s(j-1)$. Consider the j th iteration of Algorithm 1. Observe that $M_0 = \{s(j), s(j) + 1, \dots, s(j-1) - 1\}$ and $I_0 = I = \{l_1, l_2, \dots, l_p\}$, a subset of indices that attain the minimum in (16) with p the smallest index such that l_p is strictly larger than $s(j)$. Clearly indices in $I_0 \cup \{s(j)\}$ and $M_0 \setminus (I_0 \cup \{s(j)\})$ satisfy (A23) and (A24),

respectively. By Lemma 6 and by induction, in every iteration of Subroutine 2, $I_m \cup \{s(j)\}$ and $M_m \setminus (I_m \cup \{s(j)\})$ satisfy (A23) and (A24), respectively.

Moreover, by Lemma 7, in every iteration of Subroutine 2, we allocate $x(e)$ for $l_t^m \leq e < l_0^m$ such that

$$\sum_{e=l}^{l_0^m-1} x(e) \leq \sum_{e=l}^{l_0^m-1} \alpha(e) \quad (l_t^m \leq l < l_0^m) \quad (\text{A40})$$

with equality for $l = l_t^m$. By induction, the output of Subroutine 2 satisfies

$$\sum_{e=l}^{s(j-1)-1} x(e) \leq \sum_{e=l}^{s(j-1)-1} \alpha(e) \quad (s(j) \leq l < s(j-1)). \quad (\text{A41})$$

Combining (A39) and (A41), we see that (A41) holds for l satisfying $s(j) \leq l \leq n$ with equality for $l = s(j)$. By induction once again on the j iterations we have

$$\sum_{e=l}^n x(e) \leq \sum_{e=l}^n \alpha(e) \quad (1 \leq l \leq n) \quad (\text{A42})$$

with equality for $l = 1$. We have thus verified feasibility. $\square$

What remains is the proof of optimality.

Lemma 9 The vector $x(e)$, $e \in E$, put out by Algorithm 1 is optimal.

Proof We use Theorem 1 to prove the optimality of x . Let $g = f_\beta$. In each iteration, at least one variable gets set. Hence, Algorithm 1 stops after $\tau \leq n$ steps. Define $A_0 = \emptyset$ and set $A_j = \{s(j), \dots, n\}$ for $j = 1, 2, \dots, \tau$. Observe that $A_\tau = E$. It is then an immediate consequence that $x(A_j) = f(A_j)$. By the definition of dep , we have $\text{dep}(x, e, f) \subseteq A_j$ for every $e \in A_j - A_{j-1}$, $j = 1, \dots, \tau$, and $\text{dep}(x, e, g) \subseteq \text{dep}(x, e, f)$ for an $x \in P(f) \cap P(g)$. Also, observe that $\text{dep}(x, e, g) = \{e\}$ for every e satisfying $x(e) = \beta(e)$, and a $u \neq e$ with $x(u) = 0$ cannot belong to $\text{dep}(x, e, g)$. This observation implies that for any $u \in \text{dep}(x, e, g)$, $u \neq e$, we must have $x(e) < \beta(e)$ and $0 < x(u)$. We then claim that

$$\begin{aligned} w_e^+(x(e)) &\stackrel{(a)}{\geq} w_e^+(H_e^-(\Gamma_j)) \stackrel{(b)}{\geq} \Gamma_j \stackrel{(c)}{\geq} \Gamma_i \stackrel{(d)}{\geq} w_u^-(H_u^+(\Gamma_i)) \\ &\stackrel{(e)}{\geq} w_u^-(x(u)). \end{aligned}$$

Inequality (a) holds since w_e^+ is nondecreasing and $x(e) \geq H_e^-(\Gamma_j)$. Inequality (b) follows from the definition of H_e^- , after noting that $x(e) < \beta(e)$. Since (e, u) is an exchangeable pair and $e \in A_j - A_{j-1}$, we must have $u \in A_j - A_{i-1}$ for some $i \leq j$, and (c) follows from Lemma 4. Inequality (d) follows from the definition of w_u^- after noting that $x(u) > 0$. Finally (e) holds since w_u^- is nondecreasing and $x(u) \leq H_u^+(\Gamma_i)$. The sufficient condition of Theorem 1 for optimality holds, and the proof is complete.

Lemmas 8 and 9 imply Theorem 3, and its proof is now complete. $\square$

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