A novel numerical scheme for the analysis of effects of surface roughness on EHL line contact with couple stress fluid as lubricant

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Abstract. In this article, a Jacobian-free Newton Multigrid (JFNMG) method is used for obtaining the solution of isothermal, steady and compressible elastohydrodynamic lubrication (EHL) line contact problem with surface roughness. The lubricant is a couple stress fluid. A finite difference scheme is used for the solution of EHL equations. The proposed JFNMG method, for the solution of resulting nonlinear system of algebraic equations, comprises nonlinear Newton iterations on the outer loop and linear multigrid iterations on the inner loop. It overcomes the limitations of conventional schemes for the investigation of the problems covering wide range of physical parameters of interest. For increasing values of couple stress parameter, there is an increase in minimum film thickness and considerable decrease (in height as well as spread) of pressure spike. Also, the sensitivity of height and spread of pressure spike as a function of load, couple stress parameter and other parameters are investigated.

Keywords. Elastohydrodynamic lubrication; couple stress fluid; surface roughness; Newton multigrid; inexact Newton.

1. Introduction

In elastohydrodynamic lubrication (EHL), application of heavy load on the bearings develops high pressure in the lubricant and deforms contacting surfaces and as such film thickness becomes a function of pressure developed. Also, the viscosity and density of the lubricant depend nonlinearly on pressure developed. The pressure developed in the film region (0.3 Gpa to 3.0 Gpa) and the corresponding minimum film thickness attained is in a wide range of scale (micrometer). These aspects require special skills to conduct experiments for finding pressure and film profiles. Theoretical predictions made in EHL analysis are validated based on accurate experimental data. Early developments of EHL theory and experiments presented in [1] are of interest. Latest experiments include (improvements over optical interferometer), namely, image analysis procedure, introduction of spacer layer, high speed video camera, etc. Mathematical modeling of isothermal EHL problems comprises Reynolds equation, film thickness equation and force balance equation. In addition, viscosity-pressure and density-pressure (for compressible lubricant) relations have to be considered. The boundary conditions to be satisfied are inlet condition, outlet condition and cavitation condition on pressure. In the discrete form, the coupled nonlinear algebraic equations have to

be solved using suitable methods. Simple Gauss–Seidel iteration method [2] is used to solve EHL equations covering a variety of problems (elliptical parameter results, fully flooded results). Some of the salient features and earlier contributions are given in the book [3]. For the line contact EHL problem, governing equations are solved later using Gauss–Seidel scheme and expression for minimum and central film thickness are derived [4], which are widely used. More refined contributions in [5] and [6], on the solution scheme are widely used. The method (Gauss–Seidel) requires under-relaxation parameter for convergence and is unstable for large load condition. For solving difference equations simultaneously, Newton–Raphson method is used [7] and sound mathematical justifications of the pressure spike [8] as well as stability of the scheme treating it as nonlinear complementary problem are given in [9] and [10]. But the scheme has drawback in dealing with Jacobian matrix, requiring huge storage which is not stable for large load conditions. Finite element methods (and recent discontinuous Galerkin methods) are more versatile which can accommodate all aspects of the EHL problems (higher order cases, transient case, etc), but it requires bigger computing system and (tedious) programming skills. However, in one of the recent articles, [11] authors alleviate these fears and stress that finite element method is equally amenable to common users on a PC. An important development in these solution schemes of EHL problem is the emergence of

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multigrid techniques [12] combined with Gauss–Seidel relaxation scheme. It accelerates the convergence of solution scheme and enables implementation of the scheme on smaller computers. Later developments [13–15], give multi-level, multi-integration for the fast evaluation of elastic deflection equation. The striking strategy here is that multigrid eliminates both high and low frequency components of errors using sequence of grid points obtained via intergrid operators. Multigrid scheme for the solution of nonlinear system of equations is Full Approximation Scheme (FAS) with full multigrid [16]. The interest in the detailed investigation of pressure-spike, its location, height and spread is not only of academic nature, but also is of practical importance as it affects surface fatigue and early failures of rolling element bearings. Detail investigation of the nature of pressure spike (its height and spread) are made [17–20] using different methods. Review articles [21, 22] present current status of EHL studies. In recent studies [23, 24], authors demonstrate the robustness of Krylov subspace methods (restart GMRES) for the analysis of EHL line contact problems. The precise nature of Petrusevich pressure spike as a function of load and speed parameters are presented.

1.1 Couple stress fluid

For the improved performance of EHL contacts, the lubricant plays a vital role. Addition of long chain polymers to base oil enhances the quality of lubricants. Experimental studies [25–28], have demonstrated considerable load enhancement and friction reduction in bearings by this process. Later, film forming properties of lubricant-base oil mixtures in EHL contact are given in [29, 30]. The flow behavior of Newtonian lubricant with suitable additives can be described by the simplest theory of couple stress fluid, namely, Stokes' model [31]. The Stokes model accounts for polar effects, couple stresses and body couples on the fluid. Exhaustive review of literature [32] of recent articles on couple stress fluid as lubricant in hydrodynamic as well as in EHL studies is of much use. In [33] steady state analysis of incompressible line contact EHL problem with couple stress fluid model as lubricant is discussed. Accurate empirical formula for the film thickness of EHL elliptical contacts using couple stress lubricant [34] is a good substitute to earlier ones. The influence of couple stress on steady state thermal EHL line contact problem and, later investigation of compressible EHL line contact problem to demonstrate the combined influence of couple stress lubricant with wavy surface roughness on lubrication characteristics are presented in [35, 36] respectively. For point contact problem [37] the effects of surface roughness and couple stress lubricant on EHL characteristics using average flow model equation are obtained. In one of the recent studies [32] accurate asymptotic formula for central and minimum film thickness with couple stress lubricant for finite EHL problem are given.

1.2 Surface topology

The surface topology of heavily loaded (EHL) mechanical components (viz., gears, cams, followers, rolling bearings) are of rough structure even with most efficient surface finish. This surface asperity affects lubrication characteristics such as pressure, film thickness and friction. The influence of roughness is investigated either with stochastic models or deterministic models. In the former, roughness is modeled through stochastic components such as flow factors which predict average pressure, average film profiles, etc., but fail to provide details of localized peak pressure, exact film shape, etc., which are important in proposing suitable design parameters of bearings. Notable earlier stochastic studies on this topic are presented in references [38–40]. To overcome the drawbacks of stochastic models, useful studies on surface roughness effects in EHL contacts using deterministic model are made [41]. Recently, Carreau type shear thinning lubricant model is used [42] to present surface roughness effect in pure sliding EHL line contact. The influential work on fatigue life theory [43] of bearings establishes relation between surface features, the associated stress and fatigue life. To overcome the limitations of stochastic models, many deterministic models that account for real surface topology (for thermal, mixed type, etc.) have been proposed and developed. Some of the important contributions on this latter category are presented in [44–46]. These models are more useful for the design, performance analysis and life improvement of bearings.

Jacobian-free Newton multigrid (JFNMG) method

JFNMG method is a nested iterative scheme for the solution of nonlinear system of algebraic equations. We use Jacobian-free Newton multigrid method for solving nonlinear system of algebraic equations arising from discretization of equations modeling EHL problem. System of nonlinear equations is of the form

$$F(x) = 0 \quad (2.1)$$

where $F : R^n \rightarrow R^n$ is a nonlinear continuously differentiable function. Newton type methods are robust for the solution of equation (2.1) with suitable initial approximation. Newton method iteratively builds up a sequence of approximate solutions $\{x^k\}$ and the $(k + 1)$ th iterate is given by

$$x^{k+1} = x^k + \delta x^k \quad (2.2)$$

The update direction δx^k is the solution of system of linear equations

$$J(x^k) \delta x^k = -F(x^k), \quad k = 0, 1, \dots \quad (2.3)$$

$J(x^k)$ is the Jacobian of F at x^k , with the initial approximation x^0 and nonlinear iteration index k . In favorable cases, $\{x^k\}$ converges to the solution x of equation (2.1) quadratically. This local convergence depends

mostly on a good initial approximation for the solution of linear system of equations (2.3). Various methods are used for this purpose. We plan to use (linear) multigrid method. Once δx^k is obtained, the Newton direction is determined, which is controlled by backtracking in not going too far (i.e., sufficiently decrease δx^k to find Newton step). An inexact Newton method [47] for solving linear system of equations (2.3), up to an accuracy controlled by the forcing term $\eta_k \in (0, 1)$, is to find δx^k such that

$$\|F(x^k) + F'(x^k)\delta x^k\| \leq \eta_k \|F(x^k)\| \quad (2.4)$$

where $F(x^k) + F'(x^k)\delta x^k$ is the residual of equation (2.3) as well as local linear model of F evaluated at x^k . The forcing term η_k sets the accuracy of the update direction δx^k , with proper selection of $\{x^k\}$. Inexact Newton method converges locally and super linearly. Further, for the inexact Newton global convergence (with arbitrary initial condition), we use Armijo-line search method [47]. In EHL problems, Hertz's pressure serves as good initial approximation and as such it is easy to set the forcing term η_k for backtracking Newton direction.

1.3 Approximation of Jacobian (matrix)-vector product

For a scalar problem (2.1), with $n + 1$ equations in $n - 1$ unknowns, we let

$$F(x) = \{F_1, F_2, F_3, \dots, F_n, F_{n+1}\} \quad \text{and} \quad (2.5)$$

$$x = \{x_1, x_2, x_3, \dots, x_n, x_{n+1}\}$$

To find the exact expression for J , it is a tough job in many situations but the Jacobian (matrix)-vector product (which is required in each iteration of Gauss-Seidel) can be approximated by

$$Jv \cong \frac{[F(x + \epsilon v) - F(x)]}{\epsilon} \quad (2.6)$$

where ϵ is small and its optimum value is 10^{-9} in 64-bit double precision [48]. Main advantage is that, without forming or storing the true/exact Jacobian, it gives vector that approximates the matrix vector multiplication (required at each iteration).

The layout of remaining part of the article is as follows. In the next section, equations representing the proposed model and the relevant relations are given. This is followed by presenting discretized forms of the equations and solution procedure. Later, the solutions and the results obtained are presented in graphs with relevant discussion and conclusions.

Governing equations and other relations The isothermal EHL line contact problem models the lubricant flows between two cylinders rotating under an applied load. The physical problem is described by the coupling of Reynolds equation, for the flow of

lubricant, and elastic deformation equation of cylinders. For a steady isothermal couple stress lubricant flow, the dimensionless form of pressure P is described by Reynolds equation, [36],

$$\frac{d}{dX} \left(\epsilon(P) \frac{dP}{dX} \right) - K \frac{d}{dX} (\rho H) = 0 \quad (3.1)$$

Where $\epsilon = \frac{\rho H^2}{\eta \xi}$, $\xi = \left[1 - \frac{12}{\Omega^2} + \frac{24}{\Omega^3} \tanh\left(\frac{\Omega}{2}\right) \right]^{-1}$, $\Omega = \frac{8WH}{\pi L_m}$, $K = \frac{(3U\pi^2)}{4W^2}$, $P(X)$ and $H(X)$ are unknown pressure and film thickness, respectively, L_m is a dimensionless couple stress parameter. The film thickness function incorporating the sinusoidal roughness term in dimensionless form is given by

$$H(X) = H_0 + \frac{X^2}{2} + \bar{v} + \bar{a} \sin\left(\frac{2\pi X}{l}\right) \quad (3.2)$$

where $\bar{v} = \frac{vR}{b^2}$ is the dimensionless surface displacement given by $\bar{v} = -\frac{1}{2\pi} \int_{X_1}^{X_c} P \ln(X - S)^2 ds$, $\frac{X^2}{2}$ defines the undeformed contact shape, third term pertains to elastic deformation of the contacting surfaces and the last term represents the roughness term with $\bar{a}$ as amplitude and l as wavelength. H_0 is the integrating constant, i.e., the offset film thickness, determined indirectly by the load balance equation, given in dimensionless form as

$$\int_{X_1}^{X_c} P(X) dX = \frac{\pi}{2}. \quad (3.3)$$

The dimensionless forms of density $\rho = \rho(P)$ and viscosity $\eta = \eta(P)$, which are functions of pressure, are given by the relations [1],

$$\rho(P) = 1 + \frac{0.6e - 9Pp_h}{1 + 1.7e - 9Pp_h} \quad (3.4)$$

which is valid for both mineral and synthetic lubricants, and

$$\eta(P) = \exp[(\ln \eta_0 + 9.67) * \{-1 + (1 + 5.1e - 9Pp_h)^z\}] \quad (3.5)$$

(Roelands relation [49])

respectively, z is the viscosity index ($z = 0.68$), p_0 the ambient pressure ($p_0 = 1.96e + 8$), α the pressure viscosity relation ($\alpha = 2.165e - 8$), p_h is the maximum Hertzian pressure. The physical dimensionless parameters characterizing the EHL line contact problems are velocity (U), load (W) and elasticity (G) and couple stress (L_m) parameters. The corresponding boundary conditions are

$$P(X_1) = P(X_c) = 0 \quad \text{and} \quad \frac{dP}{dX} = 0 \quad \text{at} \quad X = X_c \quad (3.6)$$

where X_c corresponds to cavitation point.

Table 1. The sensitivity of convergence/non-convergence of numerical scheme to the physical parameters with couple stress parameter and waviness surface ($A = 0.2$ and $l = 0.12$) with grid points $n = 350$ (asymptotic value).

U	W	L_m	Iterations for convergence			
			Newton–Raphson Method		JFNMG method	
			Convergence	CPU time (s)	convergence	CPU time (s)
1.0E–11	2.0E–4	0.0	Yes (144)	2.6	Yes (103)	2.1
		1.0E–5	Yes (100)	1.90	Yes (77)	1.75
		1.0E–4	Yes (2025)	27	Yes (145)	3.92
		1.0E–3	No	...	Yes (190)	4.21
		1.0E–2	No	...	Yes (205)	5.3
1.0E–10	2.0E–5	1.0E–5	36100	312.04	Yes (376)	9.3
		1.0E–4	2075	18.23	Yes (147)	4.02
		1.0E–3	No	...	Yes (412)	10.92
		1.0E–2	No	...	Yes (472)	11.73

PRESSURE AND FILM THICKNESS DISTRIBUTION

Figure 1. Pressure and film thickness distribution for $W = 2.0E-4$, $U = 1.0E-11$, $G = 5.0E3$ and $L_m = 0.0$.

Finite difference discretization of the equations The equations (3.1)–(3.3) are discretized using second order finite differences with uniform grid of $n + 1$ points (i.e., $n - 1$ interior points) X_i , $1 \leq i \leq n + 1$, and the domain of interest, initially is $[X_1, X_2] = [-2, 1.5]$, X_c , the cavitation point to be determined in the solution process, is an internal point near exit. The discretized form of Reynolds equation is

$$\frac{\left(\frac{\varepsilon_i + \varepsilon_{i+1}}{2}\right) \left(\frac{P_{i+1} - P_i}{\Delta x}\right) - \left(\frac{\varepsilon_i + \varepsilon_{i-1}}{2}\right) \left(\frac{P_i - P_{i-1}}{\Delta x}\right)}{\Delta x} - K \left(\frac{\rho_i H_i - \rho_{i-1} H_{i-1}}{\Delta x} \right) = 0 \quad (4.1)$$

where $\varepsilon_i = \frac{\rho_i H_i^3}{\eta_i \zeta_i}$, $\zeta_i = \left[1 - \frac{12}{\Omega_i^2} + \frac{24}{\Omega_i^3} \tanh\left(\frac{\Omega_i}{2}\right) \right]^{-1}$, $\Omega_i = \frac{8WH_i}{\pi L_m}$, $K = \frac{(3U\pi^2)}{4W^2}$.

The film thickness equation approximated at X_i on the regular grid is given by

$$H(X_i) = H_0 + \frac{X_i^2}{2} - \frac{1}{\pi} \sum_{j=1}^n K_{ij} P(X_j) + \bar{a} \sin\left(\frac{2\pi X_i}{l}\right) \quad (4.2)$$

where

$$K_{ij} = \left(X_i - X_j + \frac{\Delta X}{2} \right) \left(\log \left| X_i - X_j + \frac{\Delta X}{2} \right| - 1 \right) - \left(X_i - X_j - \frac{\Delta X}{2} \right) \left(\log \left| X_i - X_j - \frac{\Delta X}{2} \right| - 1 \right) \quad (4.3)$$

Figure 2. (a) Pressure and film thickness profiles for $W = 2.0\text{E}-4$, $U = 1.0\text{E}-11$, $G = 5.0\text{E}3$ and $L_m = 1.0\text{E}-5$. (b) Pressure and film thickness profiles for $W = 2.0\text{E}-5$, $U = 1.0\text{E}-10$, $G = 5.0\text{E}3$ and $L_m = 1.0\text{E}-5$.

For $i = 1, 2, \dots, n+1$ and $j = 1, 2, \dots, n+1$, and the force balance equation in discrete form is

$$\Delta X \sum_{j=1}^n \left(\frac{P_j + P_{j+1}}{2} \right) - \frac{\pi}{2} = 0 \quad (4.4)$$

In discretizing equations (3.2) and (3.3), a piece-wise constant form of $P(X)$ is assumed. The pressure boundary conditions are

$$P(X_1) = 0, \quad \frac{dP}{dX} = 0 \text{ at } X = X_c \quad (4.5)$$

The above described EHL line contact problem is investigated using Newton–Raphson technique [36] and convergence of the solution scheme for different combinations of high load–low speed and low load–high speed conditions are presented. The focus of the present paper is to analyze these findings in a more general computational set-up using Jacobian-free Newton multigrid method for obtaining fast converging and accurate solutions for various combinations of parameters of interest and for much larger values of couple stress parameter.

Figure 3. Pressure and film thickness distribution for $W = 2.0\text{E-}4$, $U = 1.0\text{E-}11$, $G = 5.0\text{E}3$ and $L_m = 1.0\text{E-}4$.

Figure 4. Pressure and film thickness distribution for $W = 2.0\text{E-}4$, $U = 1.0\text{E-}11$, $G = 5.0\text{E}3$ and $L_m = 1.0\text{E-}3$.

2. Numerical implementation

The discretized systems of equations (4.1)–(4.2) and (4.4) are solved using Jacobian-free Newton Multigrid method. For each Newton iteration (outer loop), there is a linear system to be solved using multigrid method (inner loop). The required Jacobian matrix–vector product is computed using relation (2.6). With this, the pressure–viscosity and pressure–density relations are used in the Jacobian-free computation with much ease. The solution scheme consists of the following steps [50].

Step i Take initial values of P , X_c and H_0

Step ii Evaluate H from film thickness equation (4.2)

Step iii Solve the linear system using multigrid (V-cycles) and use Gauss–Seidel relaxation

Step iv Solve Reynolds equation (4.1) for P and use numerical under-relaxation for stability and convergence of iteration
 $P_{n+1}^{new} = P_n^{old} + C_1(P_n^{new\ predictor} - P_n^{old})$, C_1 being the under-relaxation parameter

Step v Update H_0 using force balance equation (4.4) and $H_0 = H_0 + RC_2$, where R is residual of the force balance equation, C_2 is the under-relaxation parameter

Figure 5. Pressure and film thickness distribution for $W = 2.0\text{E}-4$, $U = 1.0\text{E}-11$, $G = 5.0\text{E}3$ and $L_m = 1.0\text{E}-2$.

- Step vi Fix the cavitation point X_c using
 $\frac{dP}{dX} = 0$ at $X = X_c$
 Step vii While not converged, go to Step ii

Hertz's pressure distribution is used as initial approximation for pressure P . An elegant procedure [51] enables in finding optimal values of the under-relaxation parameters C_1 and C_2 . The numerical values of these constants are in the range $0.4 < C_1 < 0.8$ and $0 < C_2 < 0.1$ for different sets of physical parameters. For the termination of iteration (inner loop), the error tolerance value is $1.0\text{E}-6$. At the Step iii, for the solution of nonlinear system of equations (4.1) for P , we use Gauss–Seidel method. For locating cavitation point, the following criteria is adopted (valid for one-dimensional case). To preserve the regularity of the pressure at the cavitation point X_c , $\frac{dP}{dX} = 0$. If the pressure gradient is zero at the initial guess X_c , it is fine. Otherwise, move X_c one interval to the left or right depending on whether $\frac{dP}{dX}|_{X_c} > 0$ or $\frac{dP}{dX}|_{X_c} < 0$, respectively. Approximate new cavitation point $X_{c'}$ and solve EHL problem. If pressure gradient is zero at $X_{c'}$, then we get converged solution. Otherwise, check for the condition whether

$$\left. \frac{dP}{dX} \right|_{X_c} \times \left. \frac{dP}{dX} \right|_{X_{c'}} < 0 \quad (5.1)$$

If condition (5.1) is not met, repeat the above process and locate two approximate values $X_{c'}$ and $X_{c''}$ for which (5.1) is satisfied. Interpolate the cavitation point between the two points $X_{c'}$ and $X_{c''}$ and call this new point by X_{IC} . Relocate the grid points so that X_{IC} is a node point. Solve the system of EHL equations. This ensures cavitation point pressure gradient to be nearly zero at X_{IC} and results into the final

required smooth solution of EHL problem. Alternatively, a more general procedure involves representing $\frac{dP}{dX}$ in difference form and clubbing it as an additional equation with difference schemes (4.1), where X_c (unknown cavitation point) is to be located suitably. Once accurate solution of linear system is obtained, Newton-inexact scheme [47] (requiring just 4–5 iterations) enables in finding solution for pressure P and height H accurately. These computations are repeated for all sets of physical parameters using Roelands viscosity-pressure relation.

The solution procedure given in the above section is implemented. At the Step iii, Gauss–Seidel relaxation is used to solve the linear system (inner loop) and the convergence criterion adapted at this step is that the multigrid V-cycles are carried until the norm of the residual is less than prescribed tolerance ($1.0\text{E}-06$). Using this solution (of linear system) of the inner loop and suitable under-relaxation parameters, the inexact Newton method (Newton line search in outer loop) gives solution of the nonlinear system, which requires 4–5 iterations to yield solution with error norm less than $1.0\text{E}-06$. The offset film thickness H_0 is calculated from the force balance equation using latest pressure values. The cavitation point is then updated using the procedure stated earlier. These computations are repeated, with mesh size $\Delta X = 0.01$ for all sets of values of physical parameters of interest. In fact, for smaller values of L_m , (couple stress parameter) and lower load, much coarser grid is enough to obtain accurate solution. As load and couple stress parameters increase, it requires finer grids for the converging solution. The entire numerical computation is done using MATLAB in double precision. It is of interest to observe that JFNMG performs well compared to Multigrid methods (FMG-FAS) in attaining prescribed

Figure 6. (a) Pressure profiles for $W = 2.0\text{E-}4$, $U = 1.0\text{E-}10$, $G = 5.0\text{E}3$ and different values of L_m . (b) Film profiles for $W = 2.0\text{E-}4$, $U = 1.0\text{E-}10$, $G = 5.0\text{E}3$ and different values of L_m .

accuracy if the inner and outer iterations are suitably selected. Similar predictions are made [16] (page: 105–106) while presenting basic structure of NMG and Multigrid (FMG-FAS) schemes in solving representative nonlinear equations using these schemes. These are the representative nonlinear examples and, in the absence of concrete theory, general claim cannot be made. The mesh size, required for the converging solution, depends on the parameters U , W and G . In fact, for heavy load (and also for high speed), it requires finer grids to solve the system up to desired accuracy.

3. Results and discussions

In section 1.1, the importance of couple stress fluid as lubricant is presented. The governing isothermal steady EHL line contact equations are solved using Finite difference scheme with uniform grid. The detail analysis is both for smooth surface as well as rough surface (waviness). For the analysis of rough surface, influence of thermal and transient aspects are important, but here the focus is on demonstrating the robustness of the proposed JFNMG method in obtaining accurate solution of the nonlinear

Figure 7. (a) Spread of pressure profiles for $W = 2.0\text{E-}4$, $U = 1.0\text{E-}10$, $G = 5.0\text{E}3$ and different values of L_m . (b) Spread of film profiles for $W = 2.0\text{E-}4$, $U = 1.0\text{E-}10$, $G = 5.0\text{E}3$ and different values of L_m .

system of algebraic equations resulting from the use of finite difference scheme for wide range of parameters of interest.

Table 1 contains the comparison of the sensitivity of conventional NR method with that of JFNMG scheme for various values of U , W and L_m . The NR method does not converge even for moderately low values of U , W or L_m whereas JFNMG is fast converging and works for wide range of EHL parameters of interest. Figure 1 shows the pressure and film profiles for Newtonian case ($L_m = 0$) with $A = 0.2$, $l = 0.12$, $U = 1.0\text{E-}11$ and $W = 2.0\text{E-}4$. The non-Newtonian results depicted in figures 2(a), 2(b), 3, 4 and 5 (for different values of L_m) show increase in film thickness

and suppression of the wriggles in the profiles due to surface roughness in comparison with the Newtonian results (figure 1).

Smooth surface with couple stress fluid as lubricant: The solution obtained using JFNMG, for the pressure and film shape are for various combinations of parameters, namely, U , W , G , L_m (couple stress), $\bar{a}$ (amplitude) and l (wavelength) for smooth as well as wavy rough surface. Earlier investigators [36] on this problem used Newton–Raphson method and presented useful findings. The method converges only for moderate values of couple stress parameter L_m , whereas in the present analysis it is possible to obtain stable and converging solution of the problem for larger

values of L_m as well as for various other combinations of parameters involved. In figure 6(a), pressure profiles are shown for smooth surfaces, for one set of U , W , G and for various values of L_m while in figure 6(b), profiles of associated film thickness are given. The pressure spike (spread as well as its height) almost shrinks (diminishes) for increasing values of L_m . These are associated with increase in minimum film thickness and corresponding dip at the spike (location) of the film profile. In figures 7(a) and (b), the spread forms of pressure spike and film shapes, respectively are shown for various values of L_m . These findings validate experimental observations, on lubricants with additives, by earlier studies.

3.1 Rough surface with couple stress fluid as lubricant

For the wavy rough surface, the pressure and film shapes are functions of amplitude $\bar{a}$ and wavelength l , in addition to their sensitivity to L_m , U , W and G . Figures 2(a) and (b) pertain to pressure and film profiles for high load–low speed and low load–high speed, respectively, for $\bar{a} = 0.2$, $l = 0.12$ and $L_m = 1.0\text{E}-5$. For the given values of $\bar{a} = 0.2$, $l = 0.12$, and one set of U, W, G , the pressure and film profiles are shown in figures 3, 4 and 5 for increasing values of L_m ($1.0\text{E}-4$ to $1.0\text{E}-2$). We find considerable decrease in wriggles of the profiles compared to those corresponding to figures for $L_m = 1.0\text{E}-5$. These findings confirm that the lubricant with additives suppress the wriggles in film profiles corresponding to larger values of the couple stress parameter L_m . These predictions enable designers in arresting bearing fatigue life.

4. Conclusion

An important issue of computing the converging solution of EHL line contact problem using JFNMG method is addressed here. It is observed that, for the investigation of the influence of pressure-viscosity and pressure-density nonlinear relations on EHL characteristics, Jacobian-free numerical scheme is the most effective one. It is possible to cover wide range of load conditions (of physical parameters) using Roelands pressure-viscosity relations. The proposed JFNMG scheme enables one to compute Jacobian matrix–vector product, at each iteration, of the inner loop (linear multigrid) accurately with much ease without requiring any additional storage for the Jacobian. It also overcomes other drawbacks of Newton–Raphson method, namely, the difficulty of implementing cavitation condition, and avoids huge requirement of computer memory and CPU time (whenever grid size is halved in order to minimize discretization error). Also, implementation strategy such as finding optimal relaxation parameters leading to the convergence of the scheme is given. Besides obtaining

accurate pressure distribution and pressure–height relation, the nature of pressure spike as a function of load and couple stress parameter is demonstrated. The proposed JFNMG method inherits convergence features of Newton method as well as convergence strategy (up to discretization error) of Multigrid. This work motivates for the future study in applying the proposed scheme (or its variants) to more involved EHL problems, namely, for the analysis of thermal and non-Newtonian rheology of lubricant on EHL characteristics. Further, it can be extended to the investigation of point contact problem, starvation and other current topics of interest in EHL.

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List of symbols

a	Amplitude of roughness (μm)
$\bar{a}$	Dimensionless amplitude of roughness, $\bar{a} = aR/b^2$
A	Normalized surface roughness amplitude, $A = \bar{a}/H_{\min}^{DH}$
b	Half width of the Hertzian contact, $b = 4R\sqrt{W/2\pi}$
E'	Effective elastic modulus of rollers 1 and 2 (Pa)
G	Dimensionless materials parameter, $\alpha E'$
h	Film thickness (m)
H	Dimensionless film thickness, $H = hR/b^2$
h_0	Offset film thickness (m)
H_0	Dimensionless constant/offset film thickness $H_0 = h_0R/b^2$
K_{ij}	Discrete approximation of K logarithmic kernel Eq. (4.3)
l	Dimensionless wavelength of roughness, $l = \lambda/b$
L_m	Dimensionless couple stress parameter, $L_m = \lambda_a/R$
N	Number of nodes on grid
p	Pressure (pa)
P_h	Maximum Hertzian press $P_h = (2w)/(\pi b)$
P	Dimensionless pressure, $P = p/P_h$
P_0	Ambient pressure
R	Equivalent radius of contact
u	Velocity component
u_1, u_2	Velocities of lower and upper surfaces respectively (m/s)
u_s	Sum velocity, $u_s = (u_1 + u_2)/2$
U	Dimensionless speed parameter, $U = (\eta_0 u_s)/(E'R)$

v	Surface displacement (m)
$\bar{v}$	Dimensionless displacement, $\bar{v} = vR/b^2$
w	External load per unit width
W	Dimensionless load parameter, $W = w/(E'R)$
ΔX	Mesh size
x	Abscissa coordinate
X	Dimensionless coordinate x/b
$[X_1, X_2]$	Domain of interest $[-2, 1.5]$
X_c	Dimensionless location of pressure spike
z	Pressure viscosity parameter
α	Pressure viscosity index
η	Fluid viscosity
η_0	Viscosity at ambient pressure
$\bar{\eta}$	Dimensionless viscosity, η/η_0
λ	Surface roughness wavelength (μm)
λ_a	Molecular length of additives (m)
ξ	Viscosity modification factor
ρ	Lubricant density at local pressure
ρ_0	Inlet density of the lubricant
$\bar{\rho}$	Dimensionless fluid density, ρ/ρ_0

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