Trade-credit modeling for deteriorating item inventory system with preservation technology under random planning horizon

DIPANA JYOTI MOHANTY¹, RAVI SHANKAR KUMAR² and A GOSWAMI^{1,*}

¹Department of Mathematics, Indian Institute of Technology, Kharagpur 721302, India

²National Institute of Technology Agartala, Agartala 799046, India

e-mail: dipanajyoti80@gmail.com; ravikhushi412@yahoo.co.in; goswami@maths.iitkgp.ernet.in

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Abstract. This paper deals with a stochastic deteriorating item inventory model with preservation technology and trade-credit finance. The planning horizon of many seasonal or fashionable items stochastically varies to some extent. On the other side, the demand of such items increases in initial phase followed by a constant. This type of demand pattern can be modeled as ramp-type demand. Deterioration of such products is a characteristic, which can be reduced by making the investment on latest equipment and technology. Consumption of such items within shelf life prevents to deterioration, which can be achieved by bulk sale. In order to stimulate the selling, trade-credit policy is also considered here. In these regards, this study examines the joint effect of preservation technology investment and trade-credit policy, wherein shortage is allowed and mixture of partial backlog and lost sales. Six cases may arise depending upon three parameters of time μ (demand increases up to μ), T (when on-hand inventory reaches to zero) and M (trade-credit period). The mathematical models are mainly categorized in two cases: (i) $\mu \leq M$ and (ii) $\mu > M$. The model is illustrated through numerical experiments, sensitivity analysis, and graphical representation.

Keywords. Inventory; stochastic review period; deterioration; preservation technology; trade-credit policy; partial backlogging.

1. Introduction

Planning horizon of many seasonal and fashionable items, in general, stochastically fluctuate in some extend. As for example, in India, winter starts with November and ends with February. But its duration is not always fixed. A little stochastic variability can be easily noticed over the years. Thus, planning horizon of seasonal products such as fruits, clothes, food grains, etc. is a random variable instead of a fixed deterministic constant. Demand for such products, in general, increases in initial phase followed by a constant. This type of demand pattern is referred as ramp-type in the literature [1]. This paper is intended to develop an deteriorating inventory model for ramp-type demand in stochastic framework, wherein deterioration rate is reduced through some technology called preservation technology (PT). Consumption of such item within shelf life is another way to avoid the deterioration. In order to stimulate the selling, we incorporate trade-credit finance in form of permissible delay in payment.

Demand is one of the most important key parameter of inventory/supply chain management. Multifariousness of deterministic demand such as uniform, linear, time

dependent and ramp-type has been modeled by many authors ([2, 3]). Hill [1] first considered ramp-type demand into an inventory problem and derived an EOQ formula. After that many authors have modeled it for deteriorating items' inventory problem such as [4–12] and [13].

Deterioration is the phenomena of vaporization, damage, spoilage, dryness, etc. that decrease the usefulness of the product from the original one [14]. Deterioration rate significantly influences inventory, sales, and order quantities, especially for fast deteriorating products, which finally results in higher total relevant cost and lower demand rate [15]. Ouyang *et al* [16] suggested that deterioration rate can be reduced by improving the storage facility, which finally reduces the overall relevant cost. In recent years, many enterprises invest on types of equipment such as refrigeration, drying and vacuum technology to reduce the deterioration rate and to extend the product expiration date [17]. Zauberman *et al* [18] developed a method for color retention of Litchi fruits with SO_2 fumigation. In order to reduce the deterioration rate and to extend the expiration date of the product, preservation technologies (PT) like procedural changes and specialized equipment acquisition have been mathematically modeled by many researchers. The sensitivity analysis on numerical studies by [19] said that a lower deterioration rate is beneficial from the economic

*For correspondence

viewpoint. Geetha and Uthayakumar [20] also justified [19] observation. Hsu *et al* [17] proposed a deteriorating inventory model considering constants deterioration and demand rates, wherein PT also included. They determine the retailer's replenishment and PT strategies in order to maximize the retailer's profit. Moreover, that model also considered fixed cost of PT which was independent of replenishment cycle length. Dye and Hsieh [14] extended [17] by considering time-varying deterioration and backlogging rates and replenishment cycle length dependent PT cost. Lee and Dye [21] extended [17] by considering stock-dependent demand rate and replenishment cycle length dependent PT cost. Hsieh and Dye [22] developed a production-inventory model with controllable deterioration rate by considering time-dependent demand rate and PT cost as a decision variable in conjunction with production policy. Dye [23] studied PT investment and inventory decisions for a retailer's non-instantaneous deteriorating items, and through a sensitivity analysis delineated its dependency on cost parameters. Mishra *et al* [24] developed a deteriorating inventory model for price and stock dependent demand by employing PT. In recent years, deteriorating inventory model employing PT under different business scenarios such as optimal pricing, service and demand patterns, has been modeled by [25, 26] and [27].

Trade-credit modeling in deteriorating inventory has received considerable attention in recent business scenarios, especially, for items having the finite shelf life. Trade-credit in term of permissible delay in payment is a time span between purchasing and payment, which is generally offered by an upper-stair partner to lower-stair partner in a supply chain (SC). The lower-stair SC partner can earn interest on the money during the trade-credit period and may pay interest for delayed settlement after the end of the credit period. Goyal [28] derived retailer's EOQ (Economic Order Quantity) when a supplier offers a permissible delay in payment. Aggarwal and Jaggi [29] extended [28] by incorporating deterioration. Chang *et al* [30] developed an EOQ model by considering order quantity linked trade-credit period and deterioration. Deteriorating inventory models with the permissible delay in payment under different business environment have been developed by [16] and [19]. Ouyang *et al* [31] extended [30] by considering an order quantity dependent partially permissible delay in payment. Skouri *et al* [8] modeled ramp-type demand pattern in a deteriorating inventory problem by incorporating PT. Dye and Hsieh [32] developed a deteriorating inventory model for finite time horizon, wherein PT and trade-credit are taken into account. Yang *et al* [33] extended [32] by considering time and trade-credit period dependent demand and risk. Chen and Teng [34] enhanced deteriorating inventory model under trade-credit finance by considering expiration date and discount cash flow analysis. Tiwari *et al* [35] addressed a two-warehouse inventory model by integrating the issues of trade-credit, non-instantaneous

deterioration, and inflation, wherein shortages are allowed. Recently, Tsao and Sheen [36] extended [23] by addressing the problem of joint location and PT investment decision making for non-instantaneous deteriorating items under trade-credit. All the discussed trade-credit modeling problems are developed by considering deterministic constant planning horizon. But as discussed at beginning of this section, in many cases planning horizon varies randomly.

We have considered random time horizon, instead of deterministic time horizon, which is the major contribution of the paper.

As for our knowledge and an evidence of the literature survey, no inventory model has been developed in stochastic framework considering time dependent ramp-type demand, preservation technology and trade-credit finance. Stochastic variability in planning horizon of seasonal and fashionable product is worthwhile, because it depends upon many environmental and other factors. Thus, purpose of this study is twofold: (1) Extend the deterministic inventory model for ramp-type demand in stochastic environment by considering planning horizon as a random variable. (2) To develop a methodology to account the opportunity cost and opportunity gain for stochastic period. The rest of the paper is organized as follows: Section 2 lists all the notations, and describes the assumption made in this study. Section 3 mathematically formulates the model under the assumptions made in section 2. Section 4 develops a methodology to account the opportunity gain and opportunity cost for stochastic period. Solution methodology is provided in sections 5 and 6. Section 7 provides some numerical illustrations in support of theoretical discussion. This section also finds sensitivity analysis by changing the value of key parameters, and through which delineates managerial insights. Section 8 is concluding remark section which also provides future direction of this study.

2. Notations and assumptions

The following notations are used throughout the paper.

2.1 Notation

Parameters:

- x random scheduling period with probability density function $f(x)$, where $(x_{\min} \leq x \leq x_{\max})$
- M permissible delay in payment
- θ deterioration rate
- s per unit selling price
- p per unit purchasing price
- A ordering cost
- h inventory holding cost per product per unit time

- d disposal cost per product
 b back-logging cost per product per unit time
 l the goodwill cost due to lost sale per unit product
 e interest earned per product per unit time
 c interest charged per product per unit time
 α parameter of demand function
 μ time within which ramp-type demand increases
 β percentage of partially back order
 δ parameter of exponential distribution

Decision variables:

- T time to finish the on-hand inventory after replenishment
 ξ the preservation technology cost per unit time

Dependant variables:

- $\bar{I}_c$ expected inventory for which interest paid
 $\bar{I}_e$ expected inventory for which interest earned
 $\bar{P}$ expected review period
 $\bar{D}$ expected deteriorating items
 $\bar{Q}$ expected order quantity
 $\bar{I}$ expected inventory
 $\bar{B}$ expected shortage
 $\bar{L}$ expected lost sale
 S maximum inventory level.

2.2 Assumptions

While developing the proposed model, the following assumptions are made.

1. Demand rate is time dependent ramp-type which is defined as

$$r(t) = \begin{cases} \alpha t, & t \leq \mu; \\ \alpha \mu, & t > \mu, \end{cases} \quad (2.1)$$

where $\mu > 0$ and $\alpha > 0$.

2. Shortage is allowed, and is mixture of partially back-logging and lost sales. The backlogging quantity is delivered at the end of the period.
3. A single period inventory system is considered, wherein planning period is a random variable.
4. The original rate of deterioration is constant, and is θ (a fraction of on-hand inventory), where $0 \leq \theta < 1$.
5. The deterioration rate is reduced through investment in preservation technology ξ . The proportion of reduced deterioration rate is $m(\xi) = \theta(1 - e^{-\gamma\xi})$, where $\gamma > 0$, $\xi > 0$, and γ is the simulation coefficient representing the percentage increase in $m(\xi)$ per dollar increase in ξ , is a continuous, concave, increasing function of retailer's capital investment, where $m(0) = 0$ and $\lim_{\xi \rightarrow \infty} m(\xi) = \theta$. Note that $m'(\xi) > 0$ and $m''(\xi) < 0$ imply the diminishing marginal productivity of capital.
6. Repairing or replacement for the deteriorated item is not permissible.
7. A trade-credit period M is offered to the retailer to settle the purchasing amount. The sale revenue of the period $[0, M]$ is deposited in an interest bearing account. If $M < T$, then the retailer has also to pay the interest on the investment of remaining inventory for the period $[M, T]$.

3. Mathematical formulation

In this section, we mathematically model the proposed inventory control problem. The inventory fluctuation over time is shown in figure 1. The figure also delineates the effect of PT. The smooth curve indicates inventory level when PT has applied while the dotted curve for without PT. At the beginning of the period, replenishment is made which bring the inventory level up to S . After that, it is depleted due to demand and deterioration up to T . At that time inventory level reaches at zero, then shortage starts and continues up to the end of the period. A fraction of the

Figure 1. Inventory positions with PT and without PT.

shortage is backlogged and remaining lost sale. As shown in the figure, inventory fluctuation depends upon values of μ and T . Hence, we consider two cases: (1) $\mu \leq T$ and (2) $\mu > T$.

3.1 Case I. When $\mu \leq T$

In this case, shortage starts before μ and continues up to the end of the period as shown in figure 1(a). The inventory level is governed by the following differential equation.

$$\begin{cases} \frac{di}{dt} + (\theta - m(\xi))i = -\alpha t, & 0 \leq t \leq \mu; \\ \frac{di}{dt} + (\theta - m(\xi))i = -\alpha\mu, & t \geq \mu \end{cases} \quad (3.1)$$

with initial condition

$$i = \begin{cases} i_1(t) = -\frac{\alpha}{\theta - m(\xi)} \left(t - \frac{1}{\theta - m(\xi)} \right) \\ \quad + \left(S_1 - \frac{\alpha}{(\theta - m(\xi))^2} \right) e^{-(\theta - m(\xi))t}, & 0 \leq t \leq \mu; \\ i_2(t) = \frac{\alpha\mu}{\theta - m(\xi)} \left(e^{-(\theta - m(\xi))(t-T)} - 1 \right), & t \geq \mu. \end{cases} \quad (3.4)$$

$i(T) = 0$ gives

$$S_1 = \frac{\alpha}{\theta - m(\xi)} \left(\mu e^{(\theta - m(\xi))T} + \frac{1 - e^{(\theta - m(\xi))\mu}}{\theta - m(\xi)} \right). \quad (3.5)$$

Total accumulated inventory over the random interval $[0, x]$ is:

$$I_1(x) = \begin{cases} I_{11} = -\frac{\alpha}{\theta - m(\xi)} \left(\frac{x^2}{2} - \frac{x}{\theta - m(\xi)} \right) \\ \quad + \frac{1}{(\theta - m(\xi))^2} - \frac{S_1}{\alpha} \\ \quad - \left(S_1 - \frac{\alpha}{(\theta - m(\xi))^2} \right) \frac{e^{-(\theta - m(\xi))x}}{\theta - m(\xi)}, & 0 \leq x \leq \mu; \\ I_{12} = \frac{\alpha}{\theta - m(\xi)} \left(\frac{\mu^2}{2} + \frac{\mu}{\theta - m(\xi)} \right) \\ \quad + \frac{1}{(\theta - m(\xi))^2} + \frac{S_1}{\alpha} \\ \quad - \left(S_1 - \frac{\alpha}{(\theta - m(\xi))^2} \right) \frac{e^{-(\theta - m(\xi))\mu}}{\theta - m(\xi)} \\ \quad - \frac{\alpha\mu}{\theta - m(\xi)} \left(\frac{e^{-(\theta - m(\xi))(x-T)} - e^{-(\theta - m(\xi))(\mu-T)}}{\theta - m(\xi)} \right. \\ \quad \left. + x \right), & \mu \leq x \leq T; \\ I_{13} = \frac{\alpha}{\theta - m(\xi)} \left(\frac{\mu^2}{2} - \frac{1}{(\theta - m(\xi))^2} \right) \\ \quad + \frac{S_1}{\alpha} - \left(S_1 - \frac{\alpha}{(\theta - m(\xi))^2} \right) \frac{e^{-(\theta - m(\xi))\mu}}{\theta - m(\xi)} - \frac{\alpha\mu T}{\theta - m(\xi)} + \frac{\alpha\mu e^{-(\theta - m(\xi))(\mu-T)}}{(\theta - m(\xi))^2}, & x \geq T. \end{cases} \quad (3.6)$$

$$i(0) = S_1 \quad (3.2)$$

and boundary condition

$$i(T) = 0. \quad (3.3)$$

Eqs. (3.1) and (3.2) give

The planning period x is a random variable, consequently, inventory level, order quantity, back-order quantity, etc. are random variables. Therefore we obtain the expected values of these quantities as follows:

$$\begin{aligned}\bar{Q}_1 = E(Q_1) &= \int_{x_{\min}}^{\mu} (S_1 - i_1(x))f(x)dx \\ &+ \int_{\mu}^T (S_1 - i_2(x))f(x)dx \\ &+ \int_T^{x_{\max}} (S_1 + \alpha\mu\beta(x - T))f(x)dx,\end{aligned}\quad (3.7)$$

$$\begin{aligned}\bar{I}_1 = E(I_1) &= \int_{x_{\min}}^{\mu} I_{11}f(x)dx + \int_{\mu}^T I_{12}f(x)dx \\ &+ \int_T^{x_{\max}} I_{13}f(x)dx,\end{aligned}\quad (3.8)$$

$$\begin{aligned}\bar{D}_1 = E(D_1) &= \int_{x_{\min}}^{\mu} \left(S_1 - i_1(x) - \alpha \frac{x^2}{2} \right) f(x)dx \\ &+ \int_{\mu}^T \left(S_1 - i_2(x) + \alpha\mu \frac{\mu - 2x}{2} \right) f(x)dx \\ &+ \int_T^{x_{\max}} \left(S_1 + \alpha\mu \frac{\mu - 2T}{2} \right) f(x)dx,\end{aligned}\quad (3.9)$$

$$\begin{aligned}\bar{V}_1 = \bar{Q}_1 - \bar{D}_1 &= \frac{\alpha}{2} \int_{x_{\min}}^{\mu} x^2 f(x)dx \\ &+ \alpha\mu \int_{\mu}^T \left(x - \frac{\mu}{2} \right) f(x)dx \\ &+ \alpha\mu \int_T^{x_{\max}} \left(\left(T - \frac{\mu}{2} \right) + \beta(x - T) \right) f(x)dx,\end{aligned}\quad (3.10)$$

$$\bar{B}_1 = E(B_1) = \alpha\mu\beta \int_T^{x_{\max}} \frac{(x - T)^2}{2} f(x)dx, \quad (3.11)$$

$$\bar{L}_1 = E(L_1) = \alpha\mu(1 - \beta) \int_T^{x_{\max}} (x - T)f(x)dx, \quad (3.12)$$

$$\bar{P} = E(x) = \int_{x_{\min}}^{x_{\max}} xf(x)dx. \quad (3.13)$$

3.2 Case II. When $\mu > T$

As shown in figure 1(b), for this case, shortage starts before μ and continues up to the end of the random period x . The inventory level is governed by the following differential equation.

$$\frac{di}{dt} + (\theta - m(\xi))i = -\alpha t, 0 \leq t \leq T \quad (3.14)$$

with initial condition

$$i(0) = S_2 \quad (3.15)$$

and boundary condition

$$i(T) = 0. \quad (3.16)$$

Eqs. (3.14) and (3.15) give

$$\begin{aligned}i &= -\frac{\alpha}{\theta - m(\xi)} \left(t - \frac{1}{\theta - m(\xi)} \right) \\ &+ \left(S_2 - \frac{\alpha}{(\theta - m(\xi))^2} \right) e^{-(\theta - m(\xi))t}, \quad 0 \leq t \leq T.\end{aligned}\quad (3.17)$$

The boundary condition $i(T) = 0$ gives

$$S_2 = \frac{\alpha}{\theta - m(\xi)} \left(\left(T - \frac{1}{\theta - m(\xi)} \right) e^{(\theta - m(\xi))T} + \frac{1}{\theta - m(\xi)} \right). \quad (3.18)$$

The random accumulated inventory over the random interval $[0, x]$ is:

$$I_2(x) = \begin{cases} I_{21} = -\frac{\alpha}{\theta - m(\xi)} \left(\frac{x^2}{2} - \frac{x}{\theta - m(\xi)} \right) \\ \quad + \frac{1}{(\theta - m(\xi))^2} - \frac{S_2}{\alpha} \\ - \left(S_2 - \frac{\alpha}{(\theta - m(\xi))^2} \right) \frac{e^{-(\theta - m(\xi))x}}{\theta - m(\xi)}, & 0 \leq x \leq T; \\ I_{22} = -\frac{\alpha}{\theta - m(\xi)} \left(\frac{T^2}{2} - \frac{T}{\theta - m(\xi)} \right) \\ \quad + \frac{1}{(\theta - m(\xi))^2} - \frac{S_2}{\alpha} \\ - \left(S_2 - \frac{\alpha}{(\theta - m(\xi))^2} \right) \frac{e^{-(\theta - m(\xi))T}}{\theta - m(\xi)}, & x \geq T. \end{cases} \quad (3.19)$$

The expected values of order quantity, inventory level, deterioration, sale quantity, backlogging quantity and lost sale are as follow:

$$\begin{aligned}\bar{Q}_2 = E(Q_2) &= \int_{x_{\min}}^T (S_2 - i)f(x)dx \\ &+ \int_T^{\mu} \left(S_2 + \frac{\alpha\beta}{2}(x - T)^2 \right) f(x)dx \\ &+ \int_{\mu}^{x_{\max}} \left(S_2 + \frac{\alpha\beta}{2}(\mu - T)^2 \right. \\ &\quad \left. + \alpha\mu\beta(x - \mu) \right) f(x)dx,\end{aligned}\quad (3.20)$$

$$\begin{aligned}\bar{I}_2 = E(I_2) &= \int_{x_{\min}}^T I_{21}f(x)dx \\ &+ \int_T^{x_{\max}} I_{22}f(x)dx,\end{aligned}\quad (3.21)$$

$$\bar{D}_2 = E(D_2) = \int_{x_{\min}}^T \left(S_2 - i - \frac{\alpha x^2}{2} \right) f(x) dx + \int_T^{x_{\max}} \left(S_2 - \frac{\alpha T^2}{2} \right) f(x) dx, \quad (3.22)$$

$$\begin{aligned} \bar{V}_2 = \bar{Q}_2 - \bar{D}_2 &= \frac{\alpha}{2} \int_{x_{\min}}^T x^2 f(x) dx \\ &+ \frac{\alpha}{2} \int_T^{\mu} (T^2 + \beta(x-T)^2) f(x) dx \\ &+ \frac{\alpha}{2} \int_{\mu}^{x_{\max}} (T^2 + \beta(\mu-T)^2 \\ &+ 2\beta\mu(x-\mu)) f(x) dx, \end{aligned} \quad (3.23)$$

$$\begin{aligned} \bar{B}_2 = E(B_2) &= \frac{\alpha\beta}{6} \int_T^{\mu} (x-T)^3 f(x) dx \\ &+ \frac{\alpha\beta}{2} \int_{\mu}^{x_{\max}} \left(\frac{(\mu-T)^3}{3} + \mu(x-\mu)^2 \right) f(x) dx, \end{aligned} \quad (3.24)$$

$$\begin{aligned} \bar{L}_2 = E(L_2) &= \frac{\alpha(1-\beta)}{2} \int_T^{\mu} (x-T)^2 f(x) dx \\ &+ \alpha(1-\beta) \int_{\mu}^{x_{\max}} \left(\frac{(\mu-T)^2}{2} \right. \\ &\left. + \mu(x-\mu) \right) f(x) dx. \end{aligned} \quad (3.25)$$

4. Opportunity cost and opportunity gain

As we discussed in introduction section, the retailer is offered a trade-credit period M to settle the purchasing cost. Accumulated sale revenue during this period is deposited in an interest bearing account. If shortage starting time T is such that $T > M$, then the retailer has to pay interest on investment of remaining inventory for the period $[T, M]$. Depending upon the values of M, μ and T , we consider six cases as: (i) $0 \leq M \leq \mu \leq T$, (ii) $0 \leq \mu < M \leq T$, (iii) $0 \leq \mu \leq T < M$, (iv) $0 \leq M \leq T \leq \mu$, (v) $0 \leq T < M \leq \mu$ and (vi) $0 \leq T < \mu < M$. Furthermore, since μ and M are known parameters, here only T is a decision variable. Depending upon the values of μ and T , we develop two models, namely, Model I: when $M \leq \mu$ and Model II: when $M > \mu$. First we consider six cases of permissible delay:

4.1 Case $0 \leq M \leq \mu \leq T$

For this case, the random interest earned and interest paid are obtained as follows:

$$I_{e1} = \begin{cases} \int_0^x \int_0^t \alpha \eta d\eta dt + (M-x) \int_0^x \alpha t dt, & 0 \leq x \leq M; \\ \int_0^x \int_0^t \alpha \eta d\eta dt, & M \leq x \leq \mu; \\ \int_0^{\mu} \int_0^t \alpha \eta d\eta dt + \int_{\mu}^x \int_0^{\mu} \alpha \eta d\eta dt \\ \quad + \int_{\mu}^x \int_{\mu}^t \alpha \mu d\eta dt, & \mu \leq x \leq T; \\ \int_0^{\mu} \int_0^t \alpha \eta d\eta dt + \int_{\mu}^T \int_0^{\mu} \alpha \eta d\eta dt \\ \quad + \int_{\mu}^T \int_{\mu}^t \alpha \mu d\eta dt, & x \geq T; \end{cases} \quad (4.1)$$

and

$$I_{c1} = \begin{cases} 0, & 0 \leq x \leq M; \\ \int_M^x i_1(t) dt, & M \leq x \leq \mu; \\ \int_M^{\mu} i_1(t) dt + \int_{\mu}^x i_2(t) dt, & \mu \leq x \leq T; \\ \int_M^{\mu} i_1(t) dt + \int_{\mu}^T i_2(t) dt, & x \geq T. \end{cases} \quad (4.2)$$

The expected value of interest earned and interest paid are as follows:

$$\begin{aligned} \bar{I}_{e1} = E(I_{e1}) &= \alpha \int_{x_{\min}}^M \left(\frac{M}{2} - \frac{x}{3} \right) x^2 f(x) dx + \alpha \int_M^{\mu} \frac{x^3}{6} f(x) dx \\ &+ \frac{\alpha\mu}{2} \int_{\mu}^T \left(\frac{\mu^2}{3} - \mu x + x^2 \right) f(x) dx \\ &+ \frac{\alpha\mu}{2} \int_T^{x_{\max}} \left(\frac{\mu^2}{3} - \mu T + T^2 \right) f(x) dx \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} \bar{I}_{c1} = E(I_{c1}) &= -\frac{\alpha}{\theta - m(\xi)} \int_M^{\mu} \left(\left(\frac{x^2 - M^2}{2} - \frac{x - M}{\theta - m(\xi)} \right) \right. \\ &+ \left. \left(\frac{S_1}{\alpha} - \frac{1}{(\theta - m(\xi))^2} \right) (e^{-(\theta - m(\xi))x} - e^{-(\theta - m(\xi))M}) \right) f(x) dx \\ &+ \frac{\alpha}{\theta - m(\xi)} \int_{\mu}^T \left\{ \left(\frac{\mu^2 + M^2}{2} + \frac{\mu - M}{\theta - m(\xi)} \right) \right. \\ &- \left. \left(\frac{S_1}{\alpha} - \frac{1}{(\theta - m(\xi))^2} \right) (e^{-(\theta - m(\xi))\mu} - e^{-(\theta - m(\xi))M}) \right. \\ &- \left. \mu \left(\frac{e^{-(\theta - m(\xi))(x-T)} - e^{-(\theta - m(\xi))(\mu-T)}}{\theta - m(\xi)} + x \right) \right\} f(x) dx \\ &+ \frac{\alpha}{\theta - m(\xi)} \int_T^{x_{\max}} \left\{ \left(\frac{\mu^2 + M^2}{2} - \frac{M}{\theta - m(\xi)} \right) \right. \\ &+ \left. \mu \left(\frac{e^{-(\theta - m(\xi))(\mu-T)}}{\theta - m(\xi)} - T \right) - \left(\frac{S_1}{\alpha} - \frac{1}{(\theta - m(\xi))^2} \right) \right. \\ &\left. \left(e^{-(\theta - m(\xi))\mu} - e^{-(\theta - m(\xi))M} \right) \right\} f(x) dx. \end{aligned} \quad (4.4)$$

The cost function comprised with expected total revenue (ETR) consisting with sale revenue and opportunity gain, and expected total cost (ETC) consisting with opportunity loss, ordering cost, purchasing cost, holding cost, deterioration cost, backloging cost, lost sale and preservation

technology cost. For this case, we denote expected profit function as $ETP_{1,1}(T, \xi)$

$$\begin{aligned} ETP_{1,1}(T, \xi) &= ETR - ETC \\ &= s\bar{V}_1 + se\bar{I}_{e1} - A - p\bar{Q}_1 - h\bar{I}_1 - d\bar{D}_1 - b\bar{B}_1 \\ &\quad - l\bar{L}_1 - pc\bar{I}_{c1} - \xi\bar{P}. \end{aligned} \quad (4.5)$$

4.2 Case $0 \leq \mu < M \leq T$

The random opportunity gain and opportunity cost are as follows:

$$I_{e2} = \begin{cases} \int_0^x \int_0^t \alpha \eta d\eta dt + (M-x) \int_0^x \alpha t dt, & 0 \leq x \leq \mu; \\ \int_0^\mu \int_0^t \alpha \eta d\eta dt + \int_\mu^M \int_0^\mu \alpha \eta d\eta dt \\ + \int_\mu^x \int_\mu^t \alpha \mu d\eta dt + (M-x) \int_\mu^x \alpha \mu dt, & \mu \leq x \leq M; \\ \int_0^\mu \int_0^t \alpha \eta d\eta dt + \int_\mu^x \int_0^\mu \alpha \eta d\eta dt \\ + \int_\mu^x \int_\mu^t \alpha \mu d\eta dt, & M \leq x \leq T; \\ \int_0^\mu \int_0^t \alpha \eta d\eta dt + \int_\mu^T \int_0^\mu \alpha \eta d\eta dt \\ + \int_\mu^T \int_\mu^t \alpha \mu d\eta dt, & x \geq T; \end{cases} \quad (4.6)$$

and

$$I_{c2} = \begin{cases} 0, & 0 \leq x \leq \mu; \\ 0, & \mu \leq x \leq M; \\ \int_M^x i_2(t) dt, & M \leq x \leq T; \\ \int_M^T i_2(t) dt, & x \geq T. \end{cases} \quad (4.7)$$

Hence, the expected values of opportunity gain and opportunity cost are as follows:

$$\begin{aligned} \bar{I}_{e2} = E(I_{e2}) &= \alpha \int_{x_{\min}}^\mu \left(\frac{M}{2} - \frac{x}{3} \right) x^2 f(x) dx \\ &\quad + \alpha \mu \int_\mu^M \left\{ \left(\frac{M}{2} - \frac{\mu}{3} \right) \mu \right. \\ &\quad \left. + (x-\mu) \left(M - \frac{\mu+x}{2} \right) \right\} f(x) dx \\ &\quad + \frac{\alpha \mu}{2} \int_M^T \left(\frac{\mu^2}{3} - \mu x + x^2 \right) f(x) dx \\ &\quad + \frac{\alpha \mu}{2} \int_T^{x_{\max}} \left(\frac{\mu^2}{3} - \mu T + T^2 \right) f(x) dx \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} \bar{I}_{c2} = E(I_{c2}) &= -\frac{\alpha \mu}{\theta - m(\xi)} \\ &\quad \times \int_M^T \left(\frac{e^{-(\theta-m(\xi))(x-T)} - e^{-(\theta-m(\xi))(M-T)}}{\theta - m(\xi)} \right. \\ &\quad \left. + x - M \right) f(x) dx + \frac{\alpha \mu}{\theta - m(\xi)} \\ &\quad \int_T^{x_{\max}} \left(\frac{e^{-(\theta-m(\xi))(M-T)} - 1}{\theta - m(\xi)} \right. \\ &\quad \left. - T + M \right) f(x) dx. \end{aligned} \quad (4.9)$$

If we denote $ETP_{1,2}$ be the expected profit function for this case, then

$$\begin{aligned} ETP_{1,2}(T, \xi) &= ETR - ETC \\ &= s\bar{V}_1 + se\bar{I}_{e2} - A - p\bar{Q}_1 - h\bar{I}_1 - d\bar{D}_1 - b\bar{B}_1 \\ &\quad - l\bar{L}_1 - pc\bar{I}_{c2} - \xi\bar{P}. \end{aligned} \quad (4.10)$$

4.3 Case $0 \leq \mu \leq T < M$

In this case $T < M$. Hence, there is no interest charge on the retailer. The random opportunity gain and opportunity cost for this case are:

$$I_{e3} = \begin{cases} \int_0^x \int_0^t \alpha \eta d\eta dt + (M-x) \int_0^x \alpha t dt, & 0 \leq x \leq \mu; \\ \int_0^\mu \int_0^t \alpha \eta d\eta dt + \int_\mu^M \int_0^\mu \alpha \eta d\eta dt \\ + \int_\mu^x \int_\mu^t \alpha \mu d\eta dt + (M-x) \int_\mu^x \alpha \mu dt, & \mu \leq x \leq T; \\ \int_0^\mu \int_0^t \alpha \eta d\eta dt + \int_\mu^M \int_0^\mu \alpha \eta d\eta dt \\ + \int_\mu^T \int_\mu^t \alpha \mu d\eta dt + \int_T^M \int_\mu^T \alpha \mu d\eta dt \\ + (M-x) \int_T^x \beta \alpha \mu dt, & T \leq x \leq M; \\ \int_0^\mu \int_0^t \alpha \eta d\eta dt + \int_\mu^M \int_0^\mu \alpha \eta d\eta dt \\ + \int_\mu^T \int_\mu^t \alpha \mu d\eta dt + \int_T^M \int_\mu^T \alpha \mu d\eta dt, & x \geq M; \end{cases} \quad (4.11)$$

and

$$I_{c3} = 0. \quad (4.12)$$

The expected values of opportunity gain and opportunity cost are as follows:

$$\begin{aligned}
I_{e3}^- = E(I_{e3}) = & \alpha \int_{x_{\min}}^{\mu} \left(\frac{M}{2} - \frac{x}{3} \right) x^2 f(x) dx \\
& + \alpha \mu \int_{\mu}^T \left\{ \left(\frac{M}{2} - \frac{\mu}{3} \right) \mu \right. \\
& + (x - \mu) \left(M - \frac{\mu + x}{2} \right) \Big\} f(x) dx \\
& + \alpha \mu \int_T^M \left\{ \left(\frac{M}{2} - \frac{\mu}{3} \right) \mu \right. \\
& + (T - \mu) \left(M - \frac{\mu + T}{2} \right) \\
& + \beta(M - x)(x - T) \Big\} f(x) dx \\
& + \alpha \mu \int_M^{x_{\max}} \left\{ \left(\frac{M}{2} - \frac{\mu}{3} \right) \mu \right. \\
& + (T - \mu) \left(M - \frac{\mu + T}{2} \right) \Big\} f(x) dx,
\end{aligned} \quad (4.13)$$

$$I_{c3}^- = E(I_{c3}) = 0. \quad (4.14)$$

If we denote expected total profit for this case as $ETP_{1,3}$, then

$$\begin{aligned}
ETP_{1,3}(T) = & ETR - ETC \\
= & s\bar{V}_1 + seI_{e3}^- - A - p\bar{Q}_1 - h\bar{I}_1 - d\bar{D}_1 - b\bar{B}_1 \\
& - l\bar{L}_1 - pcI_{c3}^- - \xi\bar{P}.
\end{aligned} \quad (4.15)$$

4.4 Case $0 \leq M \leq T \leq \mu$

The random interest earned and interest paid for this case are obtained as follows:

$$I_{e4} = \begin{cases} \int_0^x \int_0^t \alpha \eta d\eta dt + (M - x) \int_0^x \alpha t dt, & 0 \leq x \leq M; \\ \int_0^x \int_0^t \alpha \eta d\eta dt, & M \leq x \leq T; \\ \int_0^T \int_0^t \alpha \eta d\eta dt, & x \geq T; \end{cases} \quad (4.16)$$

and

$$I_{c4} = \begin{cases} 0, & 0 \leq x \leq M; \\ \int_M^x i dt, & M \leq x \leq T; \\ \int_M^T i dt, & x \geq T. \end{cases} \quad (4.17)$$

The expected values of interest earned and interest paid are as follows:

$$\begin{aligned}
I_{e4}^- = E(I_{e4}) = & \alpha \int_{x_{\min}}^M \left(\frac{M}{2} - \frac{x}{3} \right) x^2 f(x) dx \\
& + \alpha \int_M^T \frac{x^3}{6} f(x) dx + \alpha \int_T^{x_{\max}} \frac{T^3}{6} f(x) dx,
\end{aligned} \quad (4.18)$$

$$\begin{aligned}
I_{c4}^- = E(I_{c4}) = & -\frac{\alpha}{\theta - m(\xi)} \int_M^T \left(\left(\frac{x^2 - M^2}{2} - \frac{x - M}{\theta - m(\xi)} \right) \right. \\
& + \left(\frac{S_2}{\alpha} - \frac{1}{(\theta - m(\xi))^2} \right) (e^{-(\theta - m(\xi))x} - e^{-(\theta - m(\xi))M}) \Big) f(x) dx \\
& - \frac{\alpha}{\theta - m(\xi)} \int_T^{x_{\max}} \left\{ \left(\frac{T^2 - M^2}{2} - \frac{T - M}{\theta - m(\xi)} \right) \right. \\
& + \left(\frac{S_2}{\alpha} - \frac{1}{(\theta - m(\xi))^2} \right) (e^{-(\theta - m(\xi))T} - e^{-(\theta - m(\xi))M}) \Big\} \\
& f(x) dx.
\end{aligned} \quad (4.19)$$

If we denote $ETP_{2,1}$ as the expected total profit for this case, then

$$\begin{aligned}
ETP_{2,1}(T, \xi) = & s\bar{V}_2 + seI_{e4}^- - A - p\bar{Q}_2 - h\bar{I}_2 \\
& - d\bar{D}_2 - b\bar{B}_2 - l\bar{L}_2 - pcI_{c4}^- - \xi\bar{P}.
\end{aligned} \quad (4.20)$$

4.5 Case $0 \leq T < M \leq \mu$

In this case $T < M$, hence, no interest is charged on the retailer. Hence, random interest earned by the retailer is

$$I_{e5} = \begin{cases} \int_0^x \int_0^t \alpha \eta d\eta dt + (M - x) \int_0^x \alpha t dt, & 0 \leq x \leq T; \\ \int_0^T \int_0^t \alpha \eta d\eta dt + \int_T^M \int_0^t \alpha \eta d\eta dt \\ \quad + (M - x) \int_T^x \beta \alpha t dt, & T \leq x \leq M; \\ \int_0^T \int_0^t \alpha \eta d\eta dt + \int_T^M \int_0^T \alpha \eta d\eta dt, & x \geq M; \end{cases} \quad (4.21)$$

and interest paid is

$$I_{c5} = 0. \quad (4.22)$$

Hence, the expected values of interest terms are as follows:

$$\begin{aligned}
I_{e5}^- = E(I_{e5}) = & \alpha \int_{x_{\min}}^T \left(\frac{M}{2} - \frac{x}{3} \right) x^2 f(x) dx \\
& + \alpha \int_T^M \left\{ \left(\frac{M}{2} - \frac{T}{3} \right) T^2 \right. \\
& + \beta(M - x) \frac{x^2 - T^2}{2} \Big\} f(x) dx \\
& + \alpha \int_M^{x_{\max}} \left(\frac{M}{2} - \frac{T}{3} \right) T^2 f(x) dx
\end{aligned} \quad (4.23)$$

and

$$I_{c5}^- = E(I_{c5}) = 0. \quad (4.24)$$

If we denote $ETP_{2,2}$ as the expected total profit for this case, then

$$\begin{aligned}
ETP_{2,2}(T, \xi) = & s\bar{V}_2 + seI_{e5}^- - A - p\bar{Q}_2 - h\bar{I}_2 \\
& - d\bar{D}_2 - b\bar{B}_2 - l\bar{L}_2 - pcI_{c5}^- - \xi\bar{P}.
\end{aligned} \quad (4.25)$$

4.6 Case $0 \leq T < \mu \leq M$

In this case also there is no interest charged on the retailer, only earns. The random interest terms are as follows:

$$I_{e6} = \begin{cases} \int_0^x \int_0^t \alpha \eta d\eta dt + (M-x) \int_0^x \alpha t dt, & 0 \leq x \leq T; \\ \int_0^T \int_0^t \alpha \eta d\eta dt + \int_T^M \int_0^T \alpha \eta d\eta dt \\ \quad + (M-x) \int_T^x \beta \alpha t dt, & T \leq x \leq \mu; \\ \int_0^T \int_0^t \alpha \eta d\eta dt + \int_T^M \int_0^T \alpha \eta d\eta dt \\ \quad + \int_x^M \int_T^\mu \beta \alpha \eta d\eta dt \\ \quad + (M-x) \int_\mu^x \beta \alpha \mu dt, & \mu \leq x \leq M; \\ \int_0^T \int_0^t \alpha \eta d\eta dt + \int_T^M \int_0^T \alpha \eta d\eta dt, & x \geq M; \end{cases} \quad (4.26)$$

and the interest paid is

$$I_{c6} = 0. \quad (4.27)$$

Expected values of interest terms are as follows:

$$\begin{aligned} I_{e6}^- = E(I_{e6}) = & \alpha \int_{x_{\min}}^T \left(\frac{M}{2} - \frac{x}{3} \right) x^2 f(x) dx \\ & + \alpha \int_T^\mu \left\{ \left(\frac{M}{2} - \frac{T}{3} \right) T^2 \right. \\ & \quad \left. + \beta(M-x) \frac{x^2 - T^2}{2} \right\} f(x) dx \\ & + \alpha \int_\mu^M \left\{ \left(\frac{M}{2} - \frac{T}{3} \right) T^2 \right. \\ & \quad \left. - \frac{\beta}{2} (M-x)(T^2 - 2\mu x + \mu^2) \right\} f(x) dx \\ & + \alpha \int_M^{x_{\max}} \left(\frac{M}{2} - \frac{T}{3} \right) T^2 f(x) dx, \end{aligned} \quad (4.28)$$

$$I_{c6}^- = E(I_{c6}) = 0. \quad (4.29)$$

If we denote the expected total profit as $ETP_{2,3}$, then

$$\begin{aligned} ETP_{2,3}(T, \xi) = & s\bar{V}_2 + seI_{e6}^- - A - p\bar{Q}_2 \\ & - h\bar{I}_2 - d\bar{D}_2 - b\bar{B}_2 - l\bar{L}_2 - pcI_{c6}^- - \xi\bar{P}. \end{aligned} \quad (4.30)$$

5. Model I: When $M \leq \mu$

Cases 4.1, 4.4 and 4.5 satisfy the condition $M \leq \mu$. Hence, integrated cost function for this model is comprised with Eqs. (4.5), (4.20) and (4.25). If we denote the integrated cost by ETP_1 , then

$$ETP_1 = \begin{cases} ETP_{1,1}, & M \leq \mu \leq T; \\ ETP_{2,1}, & M \leq T \leq \mu; \\ ETP_{2,2}, & T \leq M \leq \mu. \end{cases} \quad (5.1)$$

5.1 Solution procedure

The cost function is a profit function. Hence, our aim is to maximize it under time and preservation technology cost constraints as follows:

$$\begin{cases} \text{Maximize : } ETP_1(T, \xi) \\ \text{subject to :} \\ 0 < T < x_{\max} \\ 0 < \xi < W. \end{cases}$$

Each component of ETP_1 has an optimal policy which maximizes the profit, but is local. In order to find the unique optimum policy, we here write a procedure.

Algorithms 1

- Step 1.** Maximize $ETP_{1,1}$ subject to the constraints $0 < T < x_{\max}$, $0 < \xi < W$ and $\mu \leq T$ by using the command FindMaximum in Mathematica.
- Step 2.** Maximize $ETP_{2,1}$ subject to the constraints $0 < T < x_{\max}$, $0 < \xi < W$ and $M \leq T \leq \mu$ by using the command FindMaximum in Mathematica.
- Step 3.** Maximize $ETP_{2,2}$ subject to the constraints $0 < T < x_{\max}$, $0 < \xi < W$ and $T \leq M$ by using the command FindMaximum in Mathematica.
- Step 4.** Set $(i^*, j^*) = \arg(\max_{i,j \in \{1,2\}} ETP_{i,j}^*)$, where $ETP_{i,j}^*$ is the optimal value of $ETP_{i,j}$.
- Step 5.** Optimal Solution is, $ETP_1^* = ETP_{i^*,j^*}$, $T = T_{i^*,j^*}$ and $\xi^* = \xi_{i^*,j^*}$.

6. Model II: When $\mu < M$

Since the cases 4.2, 4.3 and 4.6 satisfy the condition $\mu < M$. Hence the integrated cost function for this model is consisted with Eqs. (4.10), (4.15) and (4.30) as follows. If we denote the integrated profit by $ETP_2(T, \xi)$, then

$$ETP_2(T, \xi) = \begin{cases} ETP_{1,2}, & \mu \leq M \leq T; \\ ETP_{1,3}, & \mu \leq T \leq M; \\ ETP_{2,3}, & T \leq \mu \leq M. \end{cases} \quad (6.1)$$

6.1 Solution procedure

Similar to Model I, our aim is to maximize the profit function $ETP_2(T, \xi)$ subject to the constraints $0 < T < x_{\max}$, $0 < \xi < W$. The following procedures give the unique optimal policy.

Algorithms 2

- Step 1.** Maximize $ETP_{1,2}$ subject to the constraints $0 < T < x_{\max}$, $0 < \xi < W$ and $M \leq T$ by using the command FindMaximum in Mathematica.

Figure 2. Concavity of profit function for exponentially distributed planning horizon.

- Step 2.** Maximize $ETP_{1,3}$ subject to the constraints $0 < T < x_{\max}$, $0 < \xi < W$ and $\mu \leq T \leq M$ by using the command FindMaximum in Mathematica.
- Step 3.** Maximize $ETP_{2,3}$ subject to the constraints $0 < T < x_{\max}$, $0 < \xi < W$ and $T \leq \mu$ by using the command FindMaximum in Mathematica.
- Step 4.** Set $(i^*, j^*) = \arg(\max_{i,j \in \{1,2,3\}} ETP_{i,j}^*)$, where $ETP_{i,j}^*$ is the maximum value of $ETP_{i,j}$.
- Step 5.** Optimal Solution is, $ETP_2^* = ETP_{i^*,j^*}$, $T = T_{i^*,j^*}$ and $\xi^* = \xi_{i^*,j^*}$.

7. Illustrative example

In this section, we present some numerical examples to illustrate the mathematical formulation. We also carry out sensitivity analysis by changing the value of key parameters to test the robustness of the model, and to delineate the managerial insights also.

Example 1 Let us consider an inventory problem with input parameters in appropriate units as $\alpha = 25$, $\beta = 0.25$, $\gamma = 0.01$, $\theta = 0.2$, $A = 120$, $W = 500$, $s = 40$, $p = 10$, $h = 0.05$, $d = 0.1$, $b = 2$, $l = 3$, $e = 0.09$, $c = 0.1$, $M = 1$, $\mu = 2$. Again let us consider that the planning horizon is exponentially distributed with mean $\delta = 1$, i.e., the probability density function of planning horizon is $f(x) = e^{-\delta x}$, $0 \leq x \leq \infty$.

Since $M < \mu$, Model I will give the optimal policy. Using the Algorithm 1, we find the optimal policy as follows: $(i^*, j^*) = (1, 1)$, $ETP_1^* = ETP_{1,1} = 319.142$, $\xi^* = 61.6252$, $T^* = 2.74043$, $Q^* = 28.458$.

If $M = 2.5$, $\mu = 2$, then optimal policy can be obtained from Model II as follows: $(i^*, j^*) = (1, 2)$,

$ETP_2^* = ETP_{1,2} = 372.666$, $\xi^* = 74.0785$, $T^* = 2.96985$, $Q^* = 29.554$. Concavity of the profit functions for given data set are shown in figure 2, that ensure the uniqueness of unique optimal policy.

Sensitivity analysis. The effect of key parameters θ , β , γ , p , e , c , M , μ and δ on the optimal policy are examined by changing its values as -50% , -25% , $+25\%$ and $+50\%$, respectively. The effects on the both models for this changes are shown in table 1, figure 3 and figure 4. When θ is increased, ξ and ETP increase, while Q is almost constant for both the models. ξ increases because for larger deterioration rate. Thus one has to invest more amount in preservation technology. When γ is increases, $m(\xi)$ i.e., reduction in deterioration rate increases, consequently, ETP increases and Q decreases. This improvement in optimal policy is achieved by increasing ξ . Sensitivity analysis indicates that ξ is significantly affected by others parameters, also. Increasing value of β increases ETP in both the models because of larger β increases the sale revenue. Both the figures indicate that mean value δ of exponentially distributed planning horizon is highly sensitive. It is because, for a longer planning horizon (i.e., longer business span), one can earn more profit. Case $\mu < M$ is shown in figure 4. It indicates that μ is highly sensitive. Table 1 indicates that model $M \leq \mu$ provides more profit compare to model $\mu < M$. It is because, for longer μ , demand rate linearly increases rapidly, consequently, the total demand for the planning horizon increases, hence, provides more profit. The dots ‘-’ in both the tables indicate infeasible solutions.

The other observations for cost parameters are straightforward which ensure the robustness of the mathematical formulation.

Table 1. Sensitivity analysis under variation of various parameter for exponential distribution.

Parameter	%change	(i^*, j^*)	ζ^*	$\bar{T}^*$	ETP^*	$\bar{Q}^*$
$M \leq \mu$						
θ	-50	(1,1)	0.000056	2.78897	388.237	28.229
	-25	(1,1)	32.857	2.74043	347.91	28.458
	25	(1,1)	83.9395	2.74043	296.828	28.458
	50	(1,1)	102.172	2.74043	278.595	28.4579
β	-50	(1,1)	71.0183	2.89487	307.582	28.7735
	-25	(1,1)	66.6283	2.82119	313.139	28.6206
	25	(1,1)	55.8094	2.65068	325.678	28.2826
	50	(1,1)	48.856	2.54898	332.869	28.0899
γ	-50	(1,1)	0.000018	2.43737	304.363	32.3647
	-25	(1,1)	25.4241	2.43831	305.748	29.5386
	25	(1,1)	72.0177	2.9101	334.242	27.6102
	50	(1,1)	74.5207	3.02241	347.663	26.9873
p	-50	(1,1)	36.8533	3.21806	490.791	37.5369
	-25	(1,1)	52.6074	2.95839	398.934	31.8537
	25	(1,1)	66.2725	2.54763	247.339	26.0646
	50	(1,1)	67.4065	2.36958	181.551	24.1904
e	-50	(1,1)	52.5794	2.6027	295.81	27.8464
	-25	(1,1)	57.1936	2.67165	307.297	28.1554
	25	(1,1)	65.8742	2.8088	331.334	28.7536
	50	(1,1)	69.9421	2.87656	343.86	29.042
c	-50	(1,1)	68.3388	2.8591	328.044	29.0234
	-25	(1,1)	64.9253	2.79768	323.47	28.7319
	25	(1,1)	58.4254	2.68683	315.04	28.1994
	50	(1,1)	55.315	2.63646	311.149	27.9543
δ	-50	(1,1)	122.809	4.53269	1107.66	82.3103
	-25	(1,1)	87.96	3.37737	568.688	44.8095
	25	(1,1)	35.0461	2.27721	182.806	19.8036
	50	(2,1)	25.394	2	102	14.2583
M	-50	(1,1)	52.5818	2.57354	296.651	27.5207
	-25	(1,1)	57.5482	2.66488	308.944	28.0479
	25	(1,1)	64.9606	2.8023	328.033	28.776
	50	(1,1)	67.6719	2.85246	336.308	29.0218
μ	-50	(1,1)	4.02058	2.42589	239.165	22.2555
	-25	(1,1)	31.7382	2.48221	284.957	26.4512
	25	(1,1)	84.3337	2.97965	338.895	29.4687
	50	(1,1)	99.6242	3.16494	347.675	30.0458
$\mu < M$						
θ	-50	(1,2)	4.76382	2.96985	441.981	29.554
	-25	(1,2)	45.3103	2.96985	401.434	29.554
	25	(1,2)	96.3928	2.96985	350.351	29.554
	50	(1,2)	114.625	2.96985	332.119	29.554
β	-50	(1,2)	82.0654	3.11891	363.449	29.9006
	-25	(1,2)	78.3077	3.04752	367.886	29.7324
	25	(1,2)	69.2459	2.88447	377.851	29.3636
	50	(1,2)	63.6107	2.78927	383.526	29.158
γ	-50	(1,3)	0.000007	2.26713	354.085	29.4435
	-25	(1,2)	44.7627	2.66123	354.842	30.7079
	25	(1,2)	81.5611	3.15126	390.199	28.7046
	50	(1,2)	82.3349	3.27363	405.195	28.0894
p	-50	(1,2)	43.4975	3.3886	539.252	38.4927
	-25	(1,2)	62.1228	3.16205	450.54	32.8694
	25	(1,2)	81.9426	2.79983	301.731	27.2579
	50	(1,2)	86.8721	2.64475	235.838	25.5036

Table 1 continued

Parameter	%change	(i^*, j^*)	ξ^*	$\bar{T}^*$	ETP^*	$\bar{Q}^*$
e	-50	(1,2)	65.4551	2.81992	329.586	28.9084
	-25	(1,2)	69.8617	2.89515	350.949	29.2349
	25	(1,2)	78.1057	3.04375	394.721	29.865
	50	(1,2)	81.9457	3.1166	417.097	30.1674
c	-50	(1,2)	74.8063	2.98307	372.869	29.6102
	-25	(1,2)	74.438	2.97637	372.766	29.5818
	25	(1,2)	73.7276	2.96351	372.568	29.527
	50	(1,2)	73.3849	2.95734	372.473	29.5007
δ	-50	(1,2)	135.157	5.00167	1248.15	87.9354
	-25	(1,2)	100.597	3.69544	647.689	47.1826
	25	(1,3)	33.0657	2.25901	224.379	19.7854
	50	(1,3)	1.41192	2	136.901	15.868
M	-50	—	—	—	—	—
	-25	—	—	—	—	—
	25	(1,3)	60.9195	2.74543	406.995	28.5797
	50	(1,3)	65.8394	2.8264	448.942	28.9369
μ	-50	(1,3)	0.00007	2.31164	279.655	21.7397
	-25	(1,2)	43.1757	2.67959	331.144	16.9049
	25	(1,2)	99.3254	3.2674	399.748	17.3603
	50	—	—	—	—	—

Figure 3. Concavity of profit function for uniformly distributed planning horizon.

Example 2 Let us consider that the planning horizon is uniformly distributed as: $f(x) = 1/5; 1 \leq x \leq 6$. The remaining data are same as of Example 1. The concavity of profit functions for given data set are shown in figure 5.

If $M = 1$ and $\mu = 2$, then Model I gives the optimal policy as: $(i^*, j^*) = (1, 1)$, $ETP_1^* = ETP_{(1,1)} = 2899.89$, $\xi^* = 132.885$, $T^* = 4.94609$, $Q^* = 172.431$.

If $M = 2.5$ and $\mu = 2$, then optimal solution from Model II is obtained as: $(i^*, j^*) = (1, 2)$, $ETP^* = 3152.06$, $\xi^* = 134.071$, $T^* = 5.11944$, $Q^* = 177.979$.

Sensitivity analysis. We change the parameters value same as of Example 1. The effects of changes on both the models are shown in table 2, figure 6 and figure 7. For both the models, p and μ are highly sensitive, and tendency of changes are same as of Example 1. The other observations are same as of Example 1, only magnitude changes. The magnitudes of Example 2 are more than of Example 1, because means of planning horizon of Example 1 is 1 and Example 2 is 3.5. However, both the

Figure 4. Effect on (a) ETP and (b) order quantity for exponentially distributed planning horizon with $M = 1, \mu = 2$.

Figure 5. Effect on (a) ETP and (b) order quantity for exponentially distributed planning horizon with $M = 2.5, \mu = 2$.

Table 2. Sensitivity analysis under variation of various parameter for uniform distribution.

<i>Parameter</i>	<i>%change</i>	(i^*, j^*)	ζ^*	$\bar{T}^*$	ETP^*	$\bar{Q}^*$
$M \leq \mu$						
θ	−50	(1,1)	63.5701	4.94609	3142.49	172.431
	−25	(1,1)	104.117	4.94609	3000.58	172.431
	25	(1,1)	155.199	4.94609	2821.79	172.431
	50	(1,1)	173.431	4.94609	2757.98	172.431
β	−50	(1,1)	136.522	5.04407	2879.54	174.041
	−25	(1,1)	134.807	4.9976	2889.22	173.266
	25	(1,1)	130.714	4.88861	2911.71	171.525
	50	(1,1)	128.24	4.82397	2924.87	170.541
γ	−50	(1,1)	127.729	4.5754	2567.75	189.216
	−25	(1,1)	139.82	4.83013	2761.95	178.59
	25	(1,1)	123.56	5.01296	3000.18	168.469
	50	(1,1)	114.704	5.05658	3076.24	165.715
p	−50	(1,1)	93.5289	5.37554	4005.99	206.472
	−25	(1,1)	117.554	5.16626	3427.72	185.893
	25	(1,1)	142.535	4.71305	2410.19	161.766
	50	(1,1)	147.839	4.46522	1953.61	152.296
e	−50	(1,1)	124.192	4.72021	2561.65	167.304
	−25	(1,1)	128.923	4.84172	2728.51	170.079
	25	(1,1)	136.217	5.03578	3074.91	174.428
	50	(1,1)	139.039	5.1131	3252.85	176.133
c	−50	(1,1)	135.794	5.10961	3058.23	177.068
	−25	(1,1)	134.419	5.02789	2977.71	174.739
	25	(1,1)	131.196	4.86432	2824.76	170.144
	50	(1,1)	129.36	4.78268	2752.27	167.877
M	−50	(1,1)	133.469	4.87463	2789.1	169.791
	−25	(1,1)	133.095	4.91118	2846.34	171.164
	25	(1,1)	132.823	4.97933	2949.71	173.594
	50	(1,1)	132.891	5.01086	2995.93	174.655
μ	−50	(1,1)	83.9983	5.1342	1790.94	100.819
	−25	(1,1)	111.693	4.95812	2383.68	143.674
	25	(1,1)	149.582	4.98317	3325.99	190.984
	50	(1,1)	163.109	5.04174	3665.03	201.676
$\mu < M$						
θ	−50	(1,2)	64.7568	5.11944	3394.66	177.979
	−25	(1,2)	105.303	5.11944	3252.75	177.979
	25	(1,2)	156.386	5.11944	3073.96	177.979
	50	(1,2)	174.618	5.11944	3010.15	177.979
β	−50	(1,2)	136.839	5.19942	3137.77	179.354
	−25	(1,2)	135.53	5.16142	3144.58	178.694
	25	(1,2)	132.436	5.07282	3160.32	177.205
	50	(1,2)	130.587	5.02066	3169.51	176.362
γ	−50	(1,2)	135.965	4.79124	2810.85	196.946
	−25	(1,2)	142.608	5.01553	3012.21	184.857
	25	(1,2)	124.084	5.1798	3252.99	173.576
	50	(1,2)	114.905	5.21932	3329.28	170.52
p	−50	(1,2)	91.5004	5.45893	4165.83	211.29
	−25	(1,2)	116.864	5.2927	3637.32	190.837
	25	(1,2)	146.126	4.9379	2697.47	168.244
	50	(1,2)	154.446	4.74711	2268.01	159.989
e	−50	(1,2)	127.202	4.9267	2787.5	173.629
	−25	(1,2)	130.953	5.03093	2968.09	175.994
	25	(1,2)	136.684	5.19491	3338.71	179.655
	50	(1,2)	138.892	5.25963	3527.49	181.079

Table 2 continued

Parameter	%change	(i^*, j^*)	ζ^*	$\bar{T}^*$	ETP^*	$\bar{Q}^*$
c	-50	(1,2)	136.103	5.19562	3200.6	179.876
	-25	(1,2)	135.098	5.15744	3176.04	178.925
	25	(1,2)	133.025	5.08166	3128.66	177.041
	50	(1,2)	131.961	5.04412	3105.83	176.109
M	-50	—	—	—	—	—
	-25	—	—	—	—	—
	25	(1,2)	135.169	5.17276	3240.94	179.421
	50	(1,2)	136.263	5.21488	3339.7	180.472
μ	-50	(1,2)	85.7317	5.34315	1943.03	102.891
	-25	(1,2)	113.289	5.14221	2585.42	147.843
	25	(1,2)	150.35	5.1511	3626.2	197.467
	50	—	—	—	—	—

Figure 6. Effect on (a) ETP and (b) order quantity for uniformly distributed planning horizon with $M = 1, \mu = 2$.

examples proof the robustness of the mathematical formulation.

8. Conclusions

In this paper we developed a deteriorating item inventory model in the stochastic framework by integrating the concepts of preservation technology, time varying ramp-type demand and trade-credit finance in the form of permissible delay in payment. Shortage of item is allowed and is the mixture of partial backlog and lost sales. An initial investment in PT is made to reduce the deterioration rate.

The planning horizon for ramp-type demand is taken as a random variable. As per our knowledge, no inventory model under such scenario has been developed in a stochastic framework. Furthermore, the mathematical model is illustrated with some numerical examples. In the numerical experiment, random planning horizon is validated by exponential distribution and uniform distribution. The robustness of the model is tested through sensitivity analysis by changing the value of key parameters. Sensitivity analysis is also used to delineate the managerial insights.

The planning horizon of many seasonal and fashionable items randomly varies in real life business transaction.

Figure 7. Effect on (a) ETP and (b) order quantity for uniformly distributed planning horizon with $M = 2.5$, $\mu = 2$.

Hence, this model can be applied while dealing with seasonal or fashionable items. In this paper, we have considered that deterioration rate is constant. But we know that deterioration rate of many items increases with time. Thus, the present model can be extended by considering non-constant deterioration rate. The model further can be extended by considering one of the most important key parameter ‘demand’, as a non-deterministic variable such as random, fuzzy or fuzzy stochastic.

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