Reduced order modelling and predictive control of multivariable nonlinear process

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Abstract. In this paper, an efficient model-predictive control strategy that can be applied to complex multivariable process is presented. A reduced order generalized predictive algorithm is proposed for online applications with reduction in complexity and time elapsed. The complex multivariable process considered in this work is a binary distillation column. The reduced order model is developed with a recently proposed hybrid algorithm known as Clustering Dominant Pole Algorithm and is able to compute the full set of dominant poles and their cluster centre efficiently. The controller calculates the optimal control action based on the future reference signals, current state and constraints on manipulated and controlled variables for a high-order dynamic simulated model of nonlinear multivariable binary distillation column process. The predictive control algorithm uses controlled auto-regressive integrated moving average model. The performance of constraint generalized predictive control scheme is found to be superior to that of the conventional PID controller in terms of overshoot, settling time and performance indices, mainly ISE, IAE and MSE.

Keywords. Predictive control; distillation column; reduced order model; dominant pole; clustering.

1. Introduction

Model-predictive control (MPC) is very frequently used and an important advanced control technique for difficult multivariable control problems with constrained parameters. Predictive algorithms are very popular techniques to solve optimal control problems in the discrete-time domain and suitable for non-minimum phase systems, process with delays, unstable systems and multivariable processes. It is designed to handle complex, constrained, high-dimensional processes with different numbers of controlled and manipulated variables even for non-linear processes. The importance of predictive algorithms in industrial applications includes explicit use of a dynamical process model for controlled variable prediction at a future time horizon and usage of an online optimization tool, which will generate optimal control actions required at every time instance by

minimizing an objective function based on predictions as the key features [1].

The various predictive algorithms differ amongst themselves in the model used to represent the process, the cost function to be minimized and optimization method. A conventional controller observes only the current and remembers the past process variables, whereas a predictive controller observes the current and also the future process variables and remembers the past variables [2]. It has features of optimal control, stochastic control, deterministic control, multivariable control, control of processes with dead time and future references when available. The main advantage of predictive algorithms is the receding strategy – at every sample time instant, the horizon is shifted towards the future and the whole procedure is repeated again [3].

The various predictive algorithms differ amongst themselves in the model used to represent the process and the cost function to be minimized [4]. In this paper, the Controlled Auto-Regressive Integrated Moving Average (CARIMA) model is considered and the key advantage of

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using Generalized Predictive Control (GPC) is that it utilizes input–output model without a requirement that state measurement is needed [5, 6]. The process model is used for the plant behaviour prediction and a receding horizon concept is applied to introduce the feedback.

With the increase in computational power, the MPC is not only limited to slow dynamics processes, where dynamical optimization is easily possible, but also there are new applications for faster systems; for example, MPC control techniques for trajectory tracking of mobile robots.

Specific eigenvectors and eigenvalues of the state matrix relate the dominant modes to corresponding dominant poles of the system transfer function. An eigenvalue method is required that computes the most dominant poles and corresponding modes [7, 8]. The Dominant Pole Algorithm (DPA) works only for stable systems as reported by Martins *et al* [9]. The recently proposed hybrid algorithm [10] combines the clustering method and DPA method. The Clustering Dominant Pole Algorithm (CDPA) method efficiently computes the poles that are more dominant in the system and retains the original characteristic behaviour [11]. The denominator polynomial of the reduced order model with respect to original model is determined by forming the clusters of the dominant poles and the coefficients of numerator polynomial with respect to original model are obtained by using the factor division algorithm. Comparative performance evaluations are analysed for both conventional PID and GPC as a class of MPC for first-principle model, linearized full order and reduced order models of a binary distillation column process.

2. Process description

Distillation column is the most commonly used separation method in the petroleum and chemical industries for purification of final products [12]. The design and efficient control of distillation column is a challenging task due to high non-linearity and dynamic behaviour. Binary distillation column consists of a vertical column, where plates or trays are used to increase the component separations. The portion of the column below the feed tray is called the stripping section and the trays above the feed tray are called rectification section. A condenser is used to cool or condense distillate vapour and the re-boiler is a heat exchanger, used to provide heat for the necessary vaporization from the bottom of the column. Condensed vapour is collected in a reflux drum and required amount of it is used as a reflux [13, 14]. The vertical column is designed for 14 trays plus condenser and re-boiler.

2.1 Mathematical modelling

2.1a Assumptions: The various assumptions considered for the distillation column modelling are given here:

- The relative volatility α is constant throughout the column. This means the vapor–liquid equilibrium relationship can be expressed by Eq. (10).
- The overhead vapour is totally condensed in a condenser.
- The liquid holdups on each tray, condenser and the re-boiler are constant and perfectly mixed.
- The holdup of vapour is negligible throughout the system.
- The molar flow rates of the vapour and liquid through the stripping and rectifying sections are constant.

The vertical column is designed for 14 trays plus condenser and re-boiler and is numbered from the bottom; 1 represents the re-boiler stage, $m = 2-7$ for the trays below the feed tray, $f = 8$ for the tray immediately below the feed tray, $f = 9$ for the tray immediately above the feed tray, $n = 10-14$ for the remaining trays above the feed tray, $N = 15$ for the tray below the condenser and $N + 1 = 16$ for the condenser stage.

2.1b Non-linear first-principle model: The plant nominal capacity is 130,000 t of raw condensate per year based on 24 operating hours per day and 350 working days per year [12]. The plant equipment is specified with a design margin of 10% above the nominal capacity and turndown ratio of 50%. Hence, the raw condensate feed rate for the plant is determined as follows:

$$\begin{aligned}\text{feed} &= 130,000\text{t}/((24\text{ h}) \times (350\text{ working days})) \\ &= 15.47619\text{ t/h}\end{aligned}$$

- the feed mass rate of the plant: $F_{\text{mass}} = 15.47619\text{ (t/h)}$;
- the gas percentage in the feed flow: $c_F = 38\%$;
- The internal vapour flow V_F is selected by the empirical relation $V_F = 28\%$;
- The feed stream (m^3/h) with the density $\rho_F = 0.670\text{ t/m}^3$ is calculated as follows:

$$F = F_{\text{mass}}/\rho_F = 15.47619/0.670 = 23.0988\text{ m}^3/\text{h}. \quad (1)$$

Composition X_F in the liquid and Y_F in the vapour phase of the feed are obtained by solving the flash equations

$$Fc_F = L_F X_F + V_F Y_F, \quad (2)$$

$$Y_F = \alpha X_F / (1 + (\alpha - 1) X_F). \quad (3)$$

Solving the flash equation with the relative volatility $\alpha = 5.68$, using Eqs. (1) and (2) gives $X_F = 0.26095$ and $Y_F = 0.66728$. Aschematic diagram of the multivariable nonlinear process considered is shown in figure 1.

The ordinary differential equation (ODE) representing the mathematical model of benchmark multivariable binary

Figure 1. Schematic diagram of binary distillation column process.

distillation column [12, 13, 15] and simplified expressions are given in Eqs. (4)–(13).

Condenser ($N + 1 = 16$):

$$M_D \dot{X}_D = (V + V_F)Y_N - LX_{N+1} - DX_D. \quad (4)$$

Tray n ($n = 10$ to $N = 15$):

$$M \dot{X}_n = (V + V_F)(Y_{n-1} - Y_n) + L(X_{n+1} - X_n). \quad (5)$$

Tray above the feed flow ($f + 1 = 9$):

$$M \dot{X}_{f+1} = V(Y_f - Y_{f+1}) + L(X_{f+2} - X_{f+1}) + V_F(Y_F - Y_{f+1}). \quad (6)$$

Tray below the feed flow ($f = 8$):

$$M \dot{X}_f = V(Y_{f-1} - Y_f) + L(X_{f+1} - X_f) + L_F(X_F - X_f). \quad (7)$$

Tray m ($m = 2$ – 7):

$$M \dot{X}_m = V(Y_{m-1} - Y_m) + (L + L_F)(X_{m+1} - X_m). \quad (8)$$

Re-boiler (Tray 1):

$$M_B \dot{X}_B = (L + L_F)X_2 - VY_1 - BX_B, \quad (9)$$

$$Y_n = \frac{\alpha X_n}{1 + (\alpha - 1)X_n}, \quad (10)$$

$$D + B = V_F + L_F, \quad (11)$$

$$V + V_F = D + L, \quad (12)$$

$$L + L_F = B + V. \quad (13)$$

The various stream data and nominal values for process parameters used in the mathematical modelling of binary distillation column process are listed in tables 1 and 2, respectively.

2.2 Linearized model in deviation form

To obtain the linearized model in transfer function model form, the first-principle equations are linearized and converted into deviation form [16]. The linearization is carried

Table 1. Stream data for distillation column process.

Stream	Formula	%	Volume (m ³ /h)	Density (t/m ³)	Mass (t/h)	Molar (kg/kmol)	Molar flow (kmol/h)
Vapour rate in feed V_F	c_{F+4}	42	9.7015	0.591	5.7336	58.2	98.5152
Liquid rate in feed L_F	$100 - c_F$	58	13.3973	0.726	9.7264	93.3	104.2491
Internal vapour rate V	V_f	28	6.4677	0.598	3.8677	58.3	66.3407
Internal liquid rate L	V_{f+4}	32	7.3916	0.615	4.5458	60.1	75.6380
Distillate flow rate D	c_F	38	8.7775	0.576	5.0554	54.5	92.7597
Bottom flow rate B	$100 - c_F$	62	14.3213	0.727	10.405	93.8	110.9235

Table 2. Nominal values for process parameters of distillation column process.

Variable	Stream	Molar flow	Unit
M_B	Liquid holdup in the column base	31.11	kmol
M	Liquid holdup on a tray	5.8	kmol
M_D	Liquid holdup in the reflux drum	13.07	kmol

out for specific molar flow L and V concentration distributions in liquid and vapour phase at different column trays and the condenser. The full order linearized model in state space representation obtained from the first-principle equations in terms of deviation variables is given by the equation

$$\begin{aligned}\dot{\mathbf{X}}(t) &= \mathbf{A} \cdot \mathbf{X}(t) + \mathbf{B} \cdot \mathbf{u}(t) \\ \mathbf{Y}(t) &= \mathbf{C} \cdot \mathbf{X}(t) + \mathbf{D} \cdot \mathbf{u}(t)\end{aligned}\quad (14)$$

where

$$\mathbf{A} = \begin{bmatrix} \frac{-(K_1 \bar{V} + B)}{M_B} & \frac{\bar{L} + L_F}{M_B} & 0 & 0 & \dots & \dots & 0 & 0 \\ \frac{K_1 \bar{V}}{M} & \frac{-(K_1 \bar{V} + \bar{L} + L_F)}{M} & \frac{\bar{L} + L_F}{M} & 0 & 0 & \dots & \ddots & 0 \\ 0 & & & & & & & \vdots \\ 0 & & & & & & & \\ \vdots & & & & & & & \\ \vdots & & & & \frac{K_6 \bar{V}}{M} & \frac{-(K_7 \bar{V} + \bar{L} + L_F)}{M} & \frac{\bar{L} + L_F}{M} & 0 & 0 & 0 & 0 \\ 0 & \dots & 0 & \frac{K_7 \bar{V}}{M} & \frac{-(K_8 \bar{V} + \bar{L} + L_F)}{M} & \frac{\bar{L}}{M} & 0 & 0 & 0 & 0 \\ \vdots & & 0 & 0 & \frac{K_8 \bar{V}}{M} & \frac{-(K_9 \bar{V} + \bar{L} + K_9 V_F)}{M} & \frac{\bar{L}}{M} & 0 & 0 & \dots \\ 0 & & 0 & 0 & 0 & \frac{K_9 (\bar{V} + V_F)}{M} & \frac{-(K_{10} (\bar{V} + V_F) + \bar{L})}{M} & \frac{\bar{L}}{M} & 0 & \ddots \\ & & & & & & & & \vdots \\ \vdots & & & & & & \dots & 0 & \frac{K_{14} (\bar{V} + V_F)}{M} & \frac{K_{15} (\bar{V} + V_F) + \bar{L}}{M} & \frac{\bar{L}}{M} \\ 0 & \dots & & & & & \dots & 0 & \frac{K_{15} (\bar{V} + V_F)}{M_D} & \frac{-(\bar{L} + D)}{M_D} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \frac{\bar{X}_2}{M_B} & \frac{-\bar{Y}_1}{M_B} \\ \frac{\bar{X}_3 - \bar{X}_2}{M} & \frac{\bar{Y}_2 - \bar{Y}_1}{M} \\ \vdots & \vdots \\ \frac{\bar{X}_{16} - \bar{X}_{15}}{M} & \frac{\bar{Y}_{15} - \bar{Y}_{14}}{M} \\ \frac{-\bar{X}_{16}}{M_D} & \frac{\bar{Y}_{15}}{M_D} \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned}\mathbf{u}(t) &= \begin{bmatrix} \Delta L \\ \Delta V \end{bmatrix}, \quad \mathbf{Y}(t) = \begin{bmatrix} \Delta X_B \\ \Delta X_D \end{bmatrix}, \\ \mathbf{X}(t) &= \begin{bmatrix} X_1 - \bar{X}_1 \\ X_2 - \bar{X}_2 \\ \vdots \\ X_{15} - \bar{X}_{15} \\ X_{16} - \bar{X}_{16} \end{bmatrix} = \begin{bmatrix} \Delta X_1 \\ \Delta X_2 \\ \vdots \\ \Delta X_{15} \\ \Delta X_{16} \end{bmatrix}, \\ Y_n &= \bar{Y}_n + \frac{\alpha(X_n - \bar{X}_n)}{(1 + (\alpha - 1)\bar{X}_n)^2} \\ f(\bar{X}_n) &= \frac{\alpha \bar{X}_n}{1 + (\alpha - 1)\bar{X}_n}; \quad K_n = \frac{df}{dX_n}, \text{ at } \\ X_n = \bar{X}_n &\Rightarrow \quad K_n = \frac{\alpha}{(1 + (\alpha - 1)\bar{X}_n)^2}\end{aligned}$$

The $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ matrices of the linearized model are given as follows:

Substituting the values $\bar{L} = L = 75.6380$, $\bar{V} = V = 66.3407$, $L_F = 104.2491$, $V_F = 98.5152$, $D = 92.7597$, $B = 110.9235$, $M_D = 13.07$, $M_B = 31.11$, $M = 5.8$, $\alpha = 5.68$, $X_F = 0.2609$ and $Y_F = 0.66728$ the state space matrices **A–C** are obtained and steady-state values of composition on each trays are listed in table 3.

Full order transfer function matrix.

From input 1 to output...

$$\begin{aligned}
 & 0.0025s^{15} + 1.48s^{14} + 390.8s^{13} + 6.096e04s^{12} + 6.261e06s^{11} + 4.469e08s^{10} \\
 & + 2.281e10s^9 + 8.436e11s^8 + 2.266e13s^7 + 4.388e14s^6 + 6.022e15s^5 + 5.685e16s^4 \\
 & + 3.516e17s^3 + 1.31e18s^2 + 2.476e18s + 1.096e18, \\
 1 : & \frac{s^{16} + 580.3s^{15} + 1.504e05s^{14} + 2.304e07s^{13} + 2.327e09s^{12} + 1.635e11s^{11} + 8.233e12s^{10} + 3.012e14s^9 + 8.036e15s^8 + 1.555e17s^7 + 2.151e18s^6 + 2.072e19s^5 + 1.333e20s^4 + 5.334e20s^3 + 1.171e21s^2 + 1.089e21s + 2.784e20,}{s^{16} + 580.3s^{15} + 1.504e05s^{14} + 2.304e07s^{13} + 2.327e09s^{12} + 1.635e11s^{11} + 8.233e12s^{10} + 3.012e14s^9 + 8.036e15s^8 + 1.555e17s^7 + 2.151e18s^6 + 2.072e19s^5 + 1.333e20s^4 + 5.334e20s^3 + 1.171e21s^2 + 1.089e21s + 2.784e20,} \\
 & -0.0738s^{15} - 41.86s^{14} - 1.055e04s^{13} - 1.563e06s^{12} - 1.515e08s^{11} - 1.012e10s^{10} \\
 & - 4.785e11s^9 - 1.618e13s^8 - 3.906e14s^7 - 6.648e15s^6 - 7.768e16s^5 - 5.961e17s^4 \\
 & - 2.783e18s^3 - 6.851e18s^2 - 6.447e18s - 1.306e18, \\
 2 : & \frac{s^{16} + 580.3s^{15} + 1.504e05s^{14} + 2.304e07s^{13} + 2.327e09s^{12} + 1.635e11s^{11} + 8.233e12s^{10} + 3.012e14s^9 + 8.036e15s^8 + 1.555e17s^7 + 2.151e18s^6 + 2.072e19s^5 + 1.333e20s^4 + 5.334e20s^3 + 1.171e21s^2 + 1.089e21s + 2.784e20,}{s^{16} + 580.3s^{15} + 1.504e05s^{14} + 2.304e07s^{13} + 2.327e09s^{12} + 1.635e11s^{11} + 8.233e12s^{10} + 3.012e14s^9 + 8.036e15s^8 + 1.555e17s^7 + 2.151e18s^6 + 2.072e19s^5 + 1.333e20s^4 + 5.334e20s^3 + 1.171e21s^2 + 1.089e21s + 2.784e20,}
 \end{aligned}$$

From input 2 to output...

$$\begin{aligned}
 & -0.005s^{15} - 3.002s^{14} - 804.3s^{13} - 1.273e05s^{12} - 1.326e07s^{11} - 9.602e08s^{10} \\
 & - 4.974e10s^9 - 1.867e12s^8 - 5.093e13s^7 - 1.002e15s^6 - 1.395e16s^5 - 1.333e17s^4 \\
 & - 8.279e17s^3 - 3.032e18s^2 - 5.293e18s - 1.657e18, \\
 1 : & \frac{s^{16} + 580.3s^{15} + 1.504e05s^{14} + 2.304e07s^{13} + 2.327e09s^{12} + 1.635e11s^{11} + 8.233e12s^{10} + 3.012e14s^9 + 8.036e15s^8 + 1.555e17s^7 + 2.151e18s^6 + 2.072e19s^5 + 1.333e20s^4 + 5.334e20s^3 + 1.171e21s^2 + 1.089e21s + 2.784e20,}{s^{16} + 580.3s^{15} + 1.504e05s^{14} + 2.304e07s^{13} + 2.327e09s^{12} + 1.635e11s^{11} + 8.233e12s^{10} + 3.012e14s^9 + 8.036e15s^8 + 1.555e17s^7 + 2.151e18s^6 + 2.072e19s^5 + 1.333e20s^4 + 5.334e20s^3 + 1.171e21s^2 + 1.089e21s + 2.784e20,} \\
 & 0.0754s^{15} + 42.78s^{14} + 1.079e04s^{13} + 1.598e06s^{12} + 1.55e08s^{11} + 1.036e10s^{10} \\
 & + 4.902e11s^9 + 1.659e13s^8 + 4.012e14s^7 + 6.843e15s^6 + 8.022e16s^5 + 6.189e17s^4 \\
 & + 2.918e18s^3 + 7.352e18s^2 + 7.436e18s + 1.985e18, \\
 2 : & \frac{s^{16} + 580.3s^{15} + 1.504e05s^{14} + 2.304e07s^{13} + 2.327e09s^{12} + 1.635e11s^{11} + 8.233e12s^{10} + 3.012e14s^9 + 8.036e15s^8 + 1.555e17s^7 + 2.151e18s^6 + 2.072e19s^5 + 1.333e20s^4 + 5.334e20s^3 + 1.171e21s^2 + 1.089e21s + 2.784e20,}{s^{16} + 580.3s^{15} + 1.504e05s^{14} + 2.304e07s^{13} + 2.327e09s^{12} + 1.635e11s^{11} + 8.233e12s^{10} + 3.012e14s^9 + 8.036e15s^8 + 1.555e17s^7 + 2.151e18s^6 + 2.072e19s^5 + 1.333e20s^4 + 5.334e20s^3 + 1.171e21s^2 + 1.089e21s + 2.784e20,}
 \end{aligned}$$

Table 3. Steady-state values of x_n , y_n , K_n on each tray using ode45 solver.

Trays	16	15	14	13	12	11	10	9
x_n	0.9953	0.9739	0.9228	0.8168	0.6500	0.4746	0.3534	0.2920
y_n	0.9992	0.9953	0.9855	0.9620	0.9134	0.8369	0.7564	0.7008
K_n	0.1774	0.1839	0.2008	0.2442	0.3476	0.5475	0.8063	1.0143
Trays	8	7	6	5	4	3	2	1
x_n	0.2727	0.2696	0.2622	0.2451	0.2104	0.1539	0.0883	0.0365
y_n	0.6805	0.6771	0.6687	0.6484	0.6022	0.5082	0.3549	0.1770
K_n	1.0961	1.1102	1.1454	1.2320	1.4418	1.9192	2.8441	4.1441

3. Reduced order model formulation

3.1 DPA

DPA computes dominant poles of the transfer function $G(s)$ based on the Newton method [17]. A pole λ_i that corresponds to a residue $|R_i|$ with large magnitude $|R_i|$ is called a dominant pole. A dominant pole is well observable and controllable in the transfer function. This can also be observed from the corresponding Bode magnitude plot of $G(s)$, where peaks occur at frequencies close to the imaginary parts [18] of the dominant poles of $G(s)$.

$$G(s) = \mathbf{C}(\mathbf{sI} - \mathbf{A})^{-1}\mathbf{B} = \sum_{i=1}^n \frac{R_i}{s - \lambda_i}. \quad (15)$$

The transfer function $G(s)$ can be expressed as a sum of residue matrices given by Eq. (15), where residue $R_i = (\mathbf{C}x_i)(y_i\mathbf{B})$ and x_i , y_i , λ_i are eigen-triplets ($i = 1, 2, \dots, n$).

Consider a pole $\lambda = \alpha + j\beta$, with residue R ; then it is shown that

$$\begin{aligned} \lim_{\omega \rightarrow \beta} G(j\omega) &= \lim_{\omega \rightarrow \beta} \frac{R}{[j\omega - (\alpha + j\beta)]} \\ &+ \sum_{j=1}^{n-1} \frac{C}{[j\omega - \lambda_j]} \lim_{\omega \rightarrow \beta} G(j\omega) \\ &= \frac{R}{\alpha} + G_{n-1}(j\beta). \end{aligned} \quad (16)$$

Hence pole λ_j is dominant if $\left| \frac{R_j}{\text{Re}(\lambda_j)} \right|$ is large and causes a peak in the Bode plot.

3.2 Proposed hybrid algorithm

The recently proposed hybrid algorithm [10], CDPA, combines the features of clustering method and DPA, and effectively matches the original system characteristics. The CDPA is described in table 4.

Table 4. CDPA algorithm.

Step 1: Given a transfer function $G(s)$
Step 2: Initialization
Step 3: Provide an initial eigenvalue estimate s_k
Step 4: Specify the values for the output at all iterations as unity, $y(s_k) = 1$, where k is the iteration count
Step 5: Set $k = 0$
Step 6: Solve for x_k from $(s_k\mathbf{I} - \mathbf{A})x_k = \mathbf{B}$
Step 7: Solve for y_k from $(s_k\mathbf{I} - \mathbf{A})y_k = \mathbf{C}$
Step 8: Compute the new eigenvalue estimate s_{k+1}
Step 9: If the change in eigenvalue estimate is greater than convergence tolerance, then increase the iteration count ($k = k + 1$) and return to step 6
Step 10: Otherwise the algorithm has converged to the dominant pole
Step 11: These dominant poles are grouped into several clusters and then replaced by the corresponding cluster centres in the denominator polynomial
Step 12: Obtain the numerator polynomial using the factor division algorithm

In the CDPA method, the denominator polynomial of the reduced order model with respect to original model is determined by forming the clusters of the dominant poles and the coefficients of numerator polynomial with respect to original model are obtained using the factor division algorithm.

The poles of transfer function Eq. (15) are $\lambda \in C$ for which $\lim_{s \rightarrow \lambda} |G(s)| = \infty$.

Consider now the function $H(s)$ given by $H(s) = \frac{1}{G(s)}$ for $\lim_{s \rightarrow \lambda} |G(s)| = 0$. In other words, the poles are the roots of $H(s)$ and these roots can be computed by Newton's method. This is the basic idea behind DPA [19].

$$H'(s) = \frac{-G'(s)}{G^2(s)}. \quad (17)$$

Solve x_k from $(s_k \mathbf{I} - \mathbf{A})x_k = \mathbf{B}$ and y_k from $(s_k \mathbf{I} - \mathbf{A})y_k = \mathbf{C}$. Then compute the new pole estimate as follows:

$$s_{k+1} = s_k - \frac{\mathbf{C} x_k}{y_k \mathbf{I} x_k} = \frac{y_k \mathbf{A} x_k}{y_k \mathbf{I} x_k}. \quad (18)$$

In CDPA, the dominant poles are grouped into several clusters and then replaced by the corresponding cluster centres. By the factor division algorithm, the coefficients of the numerator polynomial are determined [20, 21]. Now, consider the n th-order linear dynamic system described by the transfer function as in Eq. 19:

$$G(s) = \frac{N(s)}{D(s)} = \frac{e_0 + e_1 s + e_2 s^2 + \dots + e_{n-1} s^{n-1}}{f_0 + f_1 s + f_2 s^2 + \dots + f_n s^n} \quad (19)$$

where e_i ; $0 \leq i \leq n-1$ and f_i ; $0 \leq i \leq n$ are scalar constants.

The corresponding k th ($k < n$)-order reduced model is synthesized as follows:

$$G_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{a_0 + a_1 s + a_2 s^2 + \dots + a_{k-1} s^{k-1}}{b_0 + b_1 s + b_2 s^2 + \dots + b_k s^k} \quad (20)$$

where a_i ; $0 \leq i \leq k-1$ and b_i ; $0 \leq i \leq k$ are scalar constants.

Let r real poles in one cluster be $(p_1, p_2, p_3, \dots, p_r)$; then the Inverse Distance Measure (IDM) criterion identifies the cluster centre, as in Eq. (21):

$$p_c = \left\{ \sum_{i=1}^r \frac{1}{p_i} \right\}^{-1} \quad (21)$$

where p_c is cluster centre from r real poles of the original system.

The power series of original n th-order system can be expanded about $s = 0$ as follows:

$$G(s) = C_0 + C_1 s + C_2 s^2 + \dots \quad (22)$$

The power series expansion coefficients are determined as follows:

$$C_0 = e_0/f_0, \quad (23)$$

$$C_i = \frac{1}{f_0} \left[e_i - \sum_{j=1}^i f_j C_{i-j} \right], i > 0, \quad (24)$$

$$e_i = 0, \quad i > n-1. \quad (25)$$

The reduced k th-order model is written as

$$G_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{\sum_{i=0}^{k-1} a_i s^i}{\sum_{i=0}^k b_i s^i}. \quad (26)$$

Reduced order transfer function matrix

From input 1 to output...

$$1 : \frac{0.002637s + 0.003613}{s^2 + 1.704s + 0.5211}$$

$$2 : \frac{0.0007901s + 0.00103}{s^2 + 1.704s + 0.5211}$$

From input 2 to output...

$$1 : \frac{-0.007541s - 0.004013}{s^2 + 1.704s + 0.5211}$$

$$2 : \frac{-0.0003125s - 0.000404}{s^2 + 1.704s + 0.5211}$$

4. MPC strategy

The model is governed by the first-principle differential equation of a multivariable nonlinear binary distillation column process. MPC can be directly used to control a great variety of simple or complex processes, with long delay times, non-minimum phase or unstable process. It can also be applied for multivariable processes with dead times [22, 23]. Knowing the future reference signals, constraints on input, state and output variable are part of the controller, and optimal control actions are calculated, for example quadratic programming (QP) [24].

The finite horizon quadratic criterion is considered and the task is to minimize a quadratic cost function of the control error and the manipulated variable increments during the corresponding prediction horizons as described by

$$J = \sum_{i=1}^{n_y} \sum_{j=N_1}^{N_1+N_2} p_i [\hat{y}_i(k+j) - r_i(k+j)]^2 + \sum_{i=1}^{n_u} \sum_{j=1}^{N_3} q_i [\Delta u_i(k+j-1)]^2 \quad (27)$$

where $\hat{y}_i(k+j)$ is an optimum j -step ahead prediction of i th output of the system, N_2 is the control error horizon, N_3 is the control horizon and $r_i(k+j)$ is the future set-point or reference for the i th controlled variable. The parameters p_i and q_i are the weighting coefficient for control errors and control increments, respectively; $\Delta u_i(k+j-1)$ is the control increment of the i th input; n_u and n_y are the number of inputs and number of outputs (manipulated and controlled variables).

The cost function consists of two parts, mainly costs due to control error in the control error horizon N_2 , which is also called the optimization or prediction horizon, and another is costs to penalize the control signal increments or decrement (control rate) in the manipulated variable horizon N_3 , which is also called the control horizon. For simplicity, in the following text, let $N_1 = 0$, $N_2 = N_3 = N$.

The basic structure of MPC algorithm [25] is shown in figure 2. At each time instant k :

Figure 2. Structure of MPC.

- 1) The future outputs on the prediction horizon N are predicted using the process model. Predicted outputs $y(k+i)$ for $i = 1, \dots, N$ depend on the past inputs and outputs and on the future control signals $u(k+i)$ for $i = 0, \dots, N-1$, which are to be calculated.
- 2) The set of future control signals is calculated by an optimizing criterion in order to keep the process as close

as possible to the reference trajectory. If the model is linear and there are no constraints, an explicit solution can be obtained; otherwise, an iterative optimization method has to be used.

- 3) The control signal $u(k)$ is given to the process while the next calculated control signals are rejected. At the next sampling instant, $y(k+1)$ is known and step 1 is repeated with this new value using the receding horizon concept.

4.1 GPC design for MIMO process

The most common transfer function model with MPC is CARIMA model. When a plant operates around a particular working point, after linearization, it can be described by a CARIMA model [26, 27]. The chosen model has importance when the uncertainty is included in a way that is a good representation of slow varying disturbances under non-zero steady-state condition.

This model is appropriate for applications in which disturbances are non-stationary and the resulting controller has an integrating behaviour. The reason a CARIMA model is used is that it incorporates a disturbance estimate and gives

Figure 3. Pole-zero mapping for full order binary distillation column; 'x' refers to poles and 'o' refers to zeros.

Figure 4. Bode plot for full order and reduced order model of a binary distillation column. In(1) refers to L , In(2) refers to V , Out(1) refers to X_B and Out(2) refers to X_D .

unbiased prediction in steady state, irrespective of some parameter uncertainty [28].

The control objective is to compute a future control sequence that the future plant output is driven similar to the reference trajectory. This is accomplished by minimizing quadratic cost function defined over the prediction horizon [29].

4.1a *GPC with transfer function model:* CARIMA model for a Two Input Two Output (TITO) [3, 25] multivariable process can be expressed as

$$A_1(z^{-1}) y_1(k) = B_{11}(z^{-1}) u_1(k-1) + B_{12}(z^{-1}) u_2(k-1) + \frac{C_1(z^{-1})}{\Delta} e_1(k) \quad (28)$$

$$A_2(z^{-1}) y_2(k) = B_{21}(z^{-1}) u_1(k-1) + B_{22}(z^{-1}) u_2(k-1) + \frac{C_2(z^{-1})}{\Delta} e_2(k) \quad (29)$$

where $A_1(z^{-1})$, $A_2(z^{-1})$, $B_{11}(z^{-1})$, $B_{12}(z^{-1})$, $B_{21}(z^{-1})$, $B_{22}(z^{-1})$, $C_1(z^{-1})$ and $C_2(z^{-1})$ are polynomials defined as follows:

$$\begin{aligned} A_1(z^{-1}) &= 1 + a_{11}z^{-1} + a_{12}z^{-2} + \dots + a_{1n}z^{-n} \\ A_2(z^{-1}) &= 1 + a_{21}z^{-1} + a_{22}z^{-2} + \dots + a_{2n}z^{-n} \\ B_{11}(z^{-1}) &= b_{111} + b_{112}z^{-1} + b_{113}z^{-2} + \dots + b_{11n}z^{-n+1} \\ B_{12}(z^{-1}) &= b_{121} + b_{122}z^{-1} + b_{123}z^{-2} + \dots + b_{12n}z^{-n+1} \\ B_{21}(z^{-1}) &= b_{211} + b_{212}z^{-1} + b_{213}z^{-2} + \dots + b_{21n}z^{-n+1} \\ B_{22}(z^{-1}) &= b_{221} + b_{222}z^{-1} + b_{223}z^{-2} + \dots + b_{22n}z^{-n+1} \\ C_1(z^{-1}) &= 1 + c_{11}z^{-1} + c_{12}z^{-2} + \dots + c_{1n_c}z^{-n_c} \\ C_2(z^{-1}) &= 1 + c_{21}z^{-1} + c_{22}z^{-2} + \dots + c_{2n_c}z^{-n_c} \end{aligned}$$

The operator Δ is defined as $\Delta = 1 - z^{-1}$. The variables $y_1(k)$, $y_2(k)$, $u_1(k)$, $u_2(k)$, $e_1(k)$ and $e_2(k)$ are output, input and noise, respectively. The noise is supposed to be white noise random signal with zero mean.

Polynomials $A_1(z^{-1})$, $A_2(z^{-1})$ and $B_{11}(z^{-1})$, $B_{12}(z^{-1})$, $B_{21}(z^{-1})$, $B_{22}(z^{-1})$ are denominator and numerator of transfer function, respectively. Polynomials $C_1(z^{-1})$ and $C_2(z^{-1})$ are disturbance model polynomials or can be seen as a filtering polynomial. Rearranging the terms from the relations

Figure 5. Comparative output responses of a binary distillation column with PID controller tuned using various models for (a) bottom product and (b) top product.

$$\begin{aligned}
 \Delta A_1(z^{-1}) y_1(k) &= B_{11}(z^{-1}) \Delta u_1(k-1) \\
 &+ B_{12}(z^{-1}) \Delta u_2(k-1) + C_1(z^{-1}) e_1(k) \\
 (1 - z^{-1}) A_1(z^{-1}) y_1(k) &= B_{11}(z^{-1}) \Delta u_1(k-1) \\
 &+ B_{12}(z^{-1}) \Delta u_2(k-1) + C_1(z^{-1}) e_1(k) \\
 (1 - z^{-1})(1 + a_{11}z^{-1} + a_{12}z^{-2} + \dots + a_{1n}z^{-n}) y_1(k) \\
 &= (b_{111} + b_{112}z^{-1} + b_{113}z^{-2} \\
 &+ \dots + b_{11n}z^{-n+1}) \Delta u_1(k-1) \\
 &+ (b_{121} + b_{122}z^{-1} + b_{123}z^{-2} + \dots + b_{12n}z^{-n+1}) \\
 &\Delta u_2(k-1) + (1 + c_{11}z^{-1} + c_{12}z^{-2} \\
 &+ \dots + c_{1n_c}z^{-n_c}) e_1(k)
 \end{aligned}$$

the difference equation for step $k + 1$ for the first output is obtained as

$$\begin{aligned}
 \hat{y}_1(k+1) &+ (a_{11} - 1)y_1(k) + (a_{12} - a_{11})y_1(k-1) \\
 &+ \dots + (-a_{1n})y_1(k-n) \\
 &= b_{111}\Delta u_1(k) + b_{112}\Delta u_1(k-1) \\
 &+ \dots + b_{11n}\Delta u_1(k-n+1) \\
 &+ b_{121}\Delta u_2(k) + b_{122}\Delta u_2(k-1) \\
 &+ \dots + b_{12n}\Delta u_2(k-n+1) \\
 &+ e_1(k+1) + c_{11}e_1(k) + \dots + c_{1n_c}e_1(k-n_c+1)
 \end{aligned} \quad (30)$$

The difference equation for step $k + 2$ is obtained as

$$\begin{aligned}
 \hat{y}_1(k+2) &+ (a_{11} - 1)\hat{y}_1(k+1) + (a_{12} - a_{11})y_1(k) \\
 &+ \dots + (-a_{1n})y_1(k-n+1) \\
 &= b_{111}\Delta u_1(k+1) + b_{112}\Delta u_1(k) \\
 &+ \dots + b_{11n}\Delta u_1(k-n+2) \\
 &+ b_{121}\Delta u_2(k+1) + b_{122}\Delta u_2(k) \\
 &+ \dots + b_{12n}\Delta u_2(k-n+2) \\
 &+ e_1(k+2) + c_{11}e_1(k+1) \\
 &+ \dots + c_{1n_c}e_1(k-n_c+2)
 \end{aligned} \quad (31)$$

Similarly, the difference equation for step $k + N$ for the first output y_1 is obtained as

$$\begin{aligned}
 \hat{y}_1(k+N) &+ (a_{11} - 1)\hat{y}_1(k+N-1) \\
 &+ (a_{12} - a_{11})\hat{y}_1(k+N-2) + \dots \\
 &+ (-a_{1n})\hat{y}_1(k+N-n-1) = b_{111}\Delta u_1(k+N-1) \\
 &+ b_{112}\Delta u_1(k+N-2) + \dots + b_{11n}\Delta u_1(k+N-n) \\
 &+ b_{121}\Delta u_2(k+N-1) + b_{122}\Delta u_2(k+N-2) \\
 &+ \dots + b_{12n}\Delta u_2(k+N-n) \\
 &+ e_1(k+N) + c_{11}e_1(k+N-1) + c_{12}e_1(k+N-2) \\
 &+ \dots + c_{1n_c}e_1(k+N-n_c)
 \end{aligned} \quad (32)$$

Assume $e_1(k+1) = e_1(k+2) = \dots = e_1(k+N) = 0$. Separating variables in vectors that create forced and free response of the system and rewriting Eqs. (30)–(32) in matrix form, Eq. (33) is obtained:

$$\begin{aligned}
 \Delta \mathbf{A}_1 \underbrace{\begin{bmatrix} \hat{y}_1(k+1) \\ \hat{y}_1(k+2) \\ \vdots \\ \hat{y}_1(k+N) \end{bmatrix}}_{\mathbf{\hat{Y}}_1} &= \mathbf{B}_{11} \underbrace{\begin{bmatrix} \Delta u_1(k) \\ \Delta u_1(k+1) \\ \vdots \\ \Delta u_1(k+N-1) \end{bmatrix}}_{\Delta \mathbf{U}_1} \\
 &+ \mathbf{B}_{12} \underbrace{\begin{bmatrix} \Delta u_2(k) \\ \Delta u_2(k+1) \\ \vdots \\ \Delta u_2(k+N-1) \end{bmatrix}}_{\Delta \mathbf{U}_2} + \mathbf{A}_{p1} \underbrace{\begin{bmatrix} y_1(k) \\ y_1(k-1) \\ \vdots \\ y_1(k-n) \end{bmatrix}}_{\mathbf{Y}_{p1}} \\
 &+ \mathbf{B}_{p11} \underbrace{\begin{bmatrix} \Delta u_1(k-1) \\ \Delta u_1(k-2) \\ \vdots \\ \Delta u_1(k-n+1) \end{bmatrix}}_{\Delta \mathbf{U}_{p1}} + \mathbf{B}_{p12} \underbrace{\begin{bmatrix} \Delta u_2(k-1) \\ \Delta u_2(k-2) \\ \vdots \\ \Delta u_2(k-n+1) \end{bmatrix}}_{\Delta \mathbf{U}_{p2}} \\
 &\mathbf{C}_{p1} \underbrace{\begin{bmatrix} e_1(k) \\ e_1(k-1) \\ \vdots \\ e_1(k-n_c+1) \end{bmatrix}}_{\mathbf{E}_1}
 \end{aligned} \quad (33)$$

Similarly, difference equation for step $k+N$ for the second output y_2 is obtained as

$$\begin{aligned}
 &\hat{y}_2(k+N) + (a_{21} - 1)\hat{y}_2(k+N-1) \\
 &+ (a_{22} - a_{21})\hat{y}_2(k+N-2) + \dots \\
 &+ (-a_{2n})\hat{y}_2(k+N-n-1) = b_{211}\Delta u_1(k+N-1) \\
 &+ b_{212}\Delta u_1(k+N-2) + \dots + b_{21n}\Delta u_1(k+N-n) \\
 &+ b_{221}\Delta u_2(k+N-1) + b_{222}\Delta u_2(k+N-2) \\
 &+ \dots + b_{22n}\Delta u_2(k+N-n) \\
 &+ e_2(k+N) + c_{21}e(k+N-1) + c_{22}e_2 \\
 &(k+N-2) + \dots + c_{2n_c}e_2(k+N-n_c)
 \end{aligned} \quad (34)$$

Assume $e_2(k+1) = e_2(k+2) = \dots = e_2(k+N) = 0$. Separating variables in vectors that create forced and free response of the system and rewriting in matrix form gives the following relation:

$$\begin{aligned}
 \Delta \mathbf{A}_2 \underbrace{\begin{bmatrix} \hat{y}_2(k+1) \\ \hat{y}_2(k+2) \\ \vdots \\ \hat{y}_2(k+N) \end{bmatrix}}_{\mathbf{\hat{Y}}_2} &= \mathbf{B}_{21} \underbrace{\begin{bmatrix} \Delta u_1(k) \\ \Delta u_1(k+1) \\ \vdots \\ \Delta u_1(k+N-1) \end{bmatrix}}_{\Delta \mathbf{U}_1} \\
 &+ \mathbf{B}_{22} \underbrace{\begin{bmatrix} \Delta u_2(k) \\ \Delta u_2(k+1) \\ \vdots \\ \Delta u_2(k+N-1) \end{bmatrix}}_{\Delta \mathbf{U}_2} + \mathbf{A}_{p2} \underbrace{\begin{bmatrix} y_2(k) \\ y_2(k-1) \\ \vdots \\ y_2(k-n) \end{bmatrix}}_{\mathbf{Y}_{p2}} \\
 &+ \mathbf{B}_{p21} \underbrace{\begin{bmatrix} \Delta u_1(k-1) \\ \Delta u_1(k-2) \\ \vdots \\ \Delta u_1(k-n+1) \end{bmatrix}}_{\Delta \mathbf{U}_{p1}} + \mathbf{B}_{p22} \underbrace{\begin{bmatrix} \Delta u_2(k-1) \\ \Delta u_2(k-2) \\ \vdots \\ \Delta u_2(k-n+1) \end{bmatrix}}_{\Delta \mathbf{U}_{p2}} \\
 &\mathbf{C}_{p2} \underbrace{\begin{bmatrix} e_1(k) \\ e_1(k-1) \\ \vdots \\ e_1(k-n_c+1) \end{bmatrix}}_{\mathbf{E}_2}
 \end{aligned} \quad (35)$$

Rearranging first and second output variables into a single matrix, the resultant matrix is as described in Eq. (36):

$$\mathbf{A}\hat{\mathbf{Y}} = \mathbf{B}\mathbf{U} + \mathbf{A}_p\mathbf{Y}_p + \mathbf{C}_p\mathbf{E}_p + \mathbf{B}_p\mathbf{U}_p \quad (36)$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{A}_p$, $\mathbf{C}_p$ and $\mathbf{B}_p$ matrices are filled with corresponding coefficients from Eq. (35):

$$\begin{aligned}
 \mathbf{A} \underbrace{\begin{bmatrix} \hat{y}_1(k+1) \\ \hat{y}_2(k+1) \\ \hat{y}_1(k+2) \\ \hat{y}_2(k+2) \\ \vdots \\ \hat{y}_1(k+N) \\ \hat{y}_2(k+N) \end{bmatrix}}_{\mathbf{\hat{Y}}} &= \mathbf{B} \underbrace{\begin{bmatrix} \Delta u_1(k) \\ \Delta u_2(k) \\ \Delta u_1(k+1) \\ \Delta u_2(k+1) \\ \vdots \\ \Delta u_2(k+N-1) \\ \Delta u_2(k+N-1) \end{bmatrix}}_{\mathbf{U}} \\
 &+ \mathbf{A}_p \underbrace{\begin{bmatrix} y_1(k) \\ y_2(k) \\ y_1(k-1) \\ y_2(k-1) \\ \vdots \\ y_1(k-n) \\ y_2(k-n) \end{bmatrix}}_{\mathbf{Y}_p} + \mathbf{B}_p \underbrace{\begin{bmatrix} \Delta u_1(k-1) \\ \Delta u_2(k-1) \\ \Delta u_1(k-2) \\ \Delta u_2(k-2) \\ \vdots \\ \Delta u_1(k-n+1) \\ \Delta u_2(k-n+1) \end{bmatrix}}_{\mathbf{U}_p}
 \end{aligned}$$

Figure 6. 5% change in feed composition variation as a disturbance in a distillation column.

Figure 7. Regulatory response of the simulated model based controller for top product quality.

Table 5. PID controller design.

Compositions	Linearized full order model			Linearized reduced order model			First-principle model		
	K_P	K_I	K_D	K_P	K_I	K_D	K_P	K_I	K_D
Bottom product	108.5052	207.1671	–	182.3268	220.3698	–	139.5425	170.1675	–
Top product	223.0076	159.0710	–74.6747	212.9030	126.1064	–135.5175	110.1067	200.1211	5.6700

$$+ \mathbf{C}_p \begin{bmatrix} e_1(k) \\ e_2(k) \\ e(k-1) \\ e(k-1) \\ \vdots \\ e_1(k-n_c+1) \\ e_2(k-n_c+1) \end{bmatrix}. \quad (37)$$

$\mathbf{E}_p$

Multiplying $\mathbf{A}^{-1}$ on both sides in Eq. (36), $\hat{\mathbf{Y}}$ is given by Eq. (38):

$$\hat{\mathbf{Y}} = \underbrace{\mathbf{A}^{-1}\mathbf{B}\mathbf{U}}_{\text{Forced response}} + \underbrace{\mathbf{A}^{-1}\mathbf{A}_p\mathbf{Y}_p + \mathbf{A}^{-1}\mathbf{B}_p\mathbf{U}_p + \mathbf{A}^{-1}\mathbf{C}_p\mathbf{E}_p}_{\text{Free response}}. \quad (38)$$

The prediction of the output is separated into two parts – free response and forced response. The simplified representation of the combined predicted output is as given in Eq. (39):

$$\hat{\mathbf{Y}} = \mathbf{G}\mathbf{U} + \underbrace{[\mathbf{F}_y \quad \mathbf{F}_u \quad \mathbf{F}_e]}_{\mathbf{f}} \underbrace{\begin{bmatrix} \mathbf{Y}_p \\ \mathbf{U}_p \\ \mathbf{E}_p \end{bmatrix}}_{\mathbf{x}} = \mathbf{G}\mathbf{U} + \underbrace{\mathbf{f}\mathbf{x}}_{\mathbf{f}} \quad (39)$$

4.1b *Cost function:* The cost function in Eq. (27) can be represented in matrix format as

$$\mathbf{J} = (\hat{\mathbf{Y}} - \mathbf{r})^T \mathbf{P} (\hat{\mathbf{Y}} - \mathbf{r}) + \mathbf{U}^T \mathbf{Q} \mathbf{U} \quad (40)$$

where $\mathbf{P}$ and $\mathbf{Q}$ are diagonal matrix with diagonal elements p and q , respectively.

$$\mathbf{r} = \begin{bmatrix} r_1(k+1) \\ r_2(k+1) \\ r_1(k+2) \\ r_2(k+2) \\ \vdots \\ r_1(k+N) \\ r_2(k+N) \end{bmatrix}.$$

Substituting $\hat{\mathbf{Y}}$ from Eq. (39) gives

$$\mathbf{J} = (\mathbf{G}\mathbf{U} + \mathbf{f} - \mathbf{r})^T \mathbf{P} (\mathbf{G}\mathbf{U} + \mathbf{f} - \mathbf{r}) + \mathbf{U}^T \mathbf{Q} \mathbf{U} \quad (41)$$

$$\mathbf{J} = \mathbf{U}^T \mathbf{M} \mathbf{U} + 2\mathbf{m}^T \mathbf{U} + \mathbf{k} \quad (42)$$

where

$$\mathbf{M} = \mathbf{G}^T \mathbf{P} \mathbf{G} + \mathbf{Q}, \quad \mathbf{m}^T = (\mathbf{f} - \mathbf{r})^T \mathbf{P}^T \mathbf{G},$$

$$\mathbf{k} = \mathbf{f}^T \mathbf{P} \mathbf{f} - 2\mathbf{f}^T \mathbf{P} \mathbf{r} + \mathbf{r}^T \mathbf{P} \mathbf{r}.$$

The control law for unconstrained MPC cost function minimization is

$$\mathbf{U} = -\mathbf{M}^{-1} \mathbf{m} = \underbrace{(\mathbf{G}^T \mathbf{P} \mathbf{G} + \mathbf{Q})^{-1} \mathbf{G}^T \mathbf{P}}_{\mathbf{L}} (\mathbf{r} - \mathbf{f}). \quad (43)$$

First two rows $\mathbf{K}_1$ and $\mathbf{K}_2$ in $\mathbf{L}$ correspond to control action at time k :

$$\begin{bmatrix} \Delta u_1(k) \\ \Delta u_2(k) \end{bmatrix} = \mathbf{K}(\mathbf{r} - \mathbf{f}), \quad (44)$$

$$\begin{bmatrix} \Delta u_1(k) \\ \Delta u_2(k) \\ \Delta u_1(k+1) \\ \Delta u_2(k+1) \\ \vdots \\ \Delta u_1(k+N-1) \\ \Delta u_2(k+N-1) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{bmatrix}}_{\mathbf{L}} (\mathbf{r} - \mathbf{f}). \quad (45)$$

4.1c *Constraints:* In a long-range predictive control, the controller has to anticipate constraint violation and correct them in an appropriate way. The output constraints are mainly due to safety reasons and must be controlled in advance, because output variables are affected by process dynamics [30]. Input/manipulated variables can always be kept bounded, by limiting the control action to an appropriate value satisfying the constraints [31]. The TITO constraints are described in Eqs. (46) and (47):

$$u_{1\min} \leq u_1(k+i-1) \leq u_{1\max}; \quad y_{1\min} \leq y_1(k+i) \leq y_{1\max};$$

$$i = \{1, N\}, \quad (46)$$

$$u_{2\min} \leq u_2(k+i-1) \leq u_{2\max};$$

$$y_{2\min} \leq y_2(k+i) \leq y_{2\max}; \quad i = \{1, N\}. \quad (47)$$

The implementation of MPC with constraints involves the minimization of a quadratic cost function subject to linear

Figure 8. Regulatory response of the simulated model based controller for bottom product quality.

Table 6. PID controller performance comparison of binary distillation column.

Process	Compositions	PID tuned using full order model			PID tuned using reduced order model		
		ISE	IAE	MSE	ISE	IAE	MSE
Linear	X_B	0.4141	20.8925	0.0836	0.3780	18.9139	0.0757
	X_D	306.13	894.06	1.0358	407.57	937.24	1.0375
Non-linear	X_B	44.69	24.1392	0.0841	32.80	27.584	0.0746
	X_D	206.12	985.75	1.0422	387.65	845.42	1.0472

inequalities: QP – in MATLAB *quadprog* function from optimization toolbox. The Hessian matrix $\mathbf{M}$ and gradient vector $\mathbf{m}$ are obtained from Eq. (42). The inequality constraints and lower and upper bounds [31] are given by

$$\mathbf{A}\mathbf{U} \leq \mathbf{b}; \mathbf{A}_{eq}\mathbf{U} = \mathbf{b}_{eq}; \quad \mathbf{lb} \leq \mathbf{U} \leq \mathbf{ub}.$$

Step by step implementation procedure for constrained GPC with transfer function model is described as follows:

- 1) Define system continuous time process model and initialize sampling time, constraints, polynomial filter coefficients, weighting coefficient for control error and control increment.
- 2) Discretize the system transfer function model.
- 3) Compute prediction matrices ($\mathbf{G}$, $\mathbf{F}$).
- 4) Compute $\mathbf{H}$, $\mathbf{g}$ and other matrices for cost function by quadratic programming. Update $\mathbf{g}$ and $\mathbf{b}$ inside control cycle.
- 5) MPC iteration, at each time instant k recursively

- a) compute $y_1(k)$ and $y_2(k)$ (simulated output or measured output);
- b) find prediction error $e_1(k) = y_1(k) - \hat{y}_1(k)$ and $e_2(k) = y_2(k) - \hat{y}_2(k)$;
- c) find Δu_1 and Δu_2 from cost function by considering constraints using QP;
- d) determine predicted outputs $\hat{y}_1(k+1)$ and $\hat{y}_2(k+1)$.

5. Results and discussion

5.1 Location of poles

The pole-zero spectrum of the full order (16th) original transfer function is shown in figure 3. The plant poles are located at $p_1 = -99.8905$, $p_2 = -80.0087$, $p_3 = -69.7103$, $p_4 = -62.3181$, $p_5 = -50.1214$, $p_6 = -42.2187$, $p_7 = -34.012$, $p_8 = -29.9989$, $p_9 = -24.4728$, $p_{10} = -17.9833$, $p_{11} = -15.0707$, $p_{12} = -10.7188$, $p_{13} = -7.8889$, $p_{14} = -5.5823$,

$p_{15} = -1.5563$ and $p_{16} = -0.4098$. From figure 3, it is evident that the given original system has all the poles lying on the left side of the s -plane and having no conjugate poles. Using the proposed CDPA, the reduced order system poles are calculated as $p_{c1} = -0.3996$ and $p_{c2} = -1.3040$.

Using CDPA, the cluster centres are efficiently determined based on the dominant pole concept of the characteristic polynomial. However, in CPDA result only one transfer function pole is mentioned because the original and reduced transfer function matrices have a common denominator polynomial representing the characteristic polynomial. Hence a 16th-order full order model is converted to a 2nd-order reduced model.

It is seen that the poles from p_1 to p_{12} lie far away from the s -plane origin and cause fast response in decaying with less effect on the system characteristics. The constant coefficients of the denominator polynomial used in the various techniques have an important role to play in stability and performance of an LTI system, while the numerator coefficients also have an influence on the system's response to applied inputs [16]. Model-based control schemes are effectively used in industrial applications, namely cement industry, coal mill industry, etc., where accurate model coefficient estimation plays a crucial role in improving the closed loop system performances.

The gain characteristics of full order model are compared to the model equivalent using CDPA as shown in figure 4. In the Bode diagram, solid red and blue lines represent full order and reduced order model, respectively, of a binary distillation column process. It is observed that the essential dynamics of the system lie in the frequency range of 0.001–1 radian/s from the response. The magnitude drops in both the very low and the high-frequency ranges. It is observed that the reference model has good tracking capabilities in each region of operation with multiple stepping reference inputs.

5.2 PID controller tuning

Controller tuning is an essential preliminary procedure for almost all the industrial process control systems. Every process has to be controlled, so that the expected response can be obtained. In industries, PID controllers are commonly used as the controllers. Despite the significant developments in advanced process control schemes such as predictive control, internal model control and sliding mode control, PID controllers are still widely used in industrial control applications, because of their structural simplicity, reputation and easy implementation. In this work, tuning of conventional PID controller uses the Ziegler–Nichols closed loop method for controller design. The controller parameters obtained are summarized in table 5.

Comparative closed loop controller responses of PID controller for linearized full order, linearized reduced order and first-principle model are shown in figure 5. In figure 5,

the setpoints for bottom product X_B and top product X_D are plotted as red solid lines at various time intervals, black dotted and dashed lines are the output responses for linearized reduced order and full order models, respectively, and blue dotted line represents the output responses using the first-principle model. The result obtained shows that the reference is followed satisfactorily but with an overshoot. Output responses of a binary distillation column with PID controller tuned using first-principle model show that both outputs take more settling time to track the setpoints when compared with the responses of linearized reduced order and full order model of a binary distillation column in figure 5.

Table 6 summarizes comparative studies of PID controller performance indices, mainly ISE, IAE and MSE of the binary distillation column. The study shows that the bottom product error values are minimum for linearized reduced order model and top product error values are minimum for linearized full order model when compared with the first-principle model.

PID controller tuned using reduced order model is able to perform on par with PID controller tuned using full order model. The PID controller tuned using reduced order model exhibits good performance for the simulated non-linear process.

5.3 Analysis of controller

The linearized and simulated models were analysed for 5% feed composition variation as a disturbance in a distillation column. The model responses are shown in figure 6.

The simulation experiment was carried by increasing feed flow rate by 3% from its nominal value. The responses obtained for the decoupled system with simulated model are shown in figures 7 and 8.

It is observed from figures 7 and 8 that the designed controller is able to reject the disturbance with smaller overshoot and lesser settling time.

5.4 GPC as a class of MPC

In this section, the response is obtained for multiple step changes in the reference in a GPC with prediction horizon

Table 7. Generalized Predictive Controller parameters for full order and reduced order binary distillation column.

Controller parameter	Value
Sampling time	1 s
Start of costing horizon	1
End of costing horizon	40
End of control horizon	40
Weighting coefficient for control error	1×10^5
Weighting coefficient for control increments	1

Table 8. MPC performance comparison of full order with reduced order linear and nonlinear model of binary distillation column.

Process		GPC tuned using full order model			GPC tuned using reduced order model		
		ISE	IAE	MSE	ISE	IAE	MSE
Linear	X_B	0.33297	9.1201	0.0013	0.33296	9.1209	0.0013
	X_D	247.66	248.82	0.9906	247.65	248.83	0.9906
Simulated non-linear	X_B	0.33315	9.1200	0.0013	0.33323	9.1204	0.0013
	X_D	247.65	248.83	0.9906	247.66	248.83	0.9906

Figure 9. Comparative output responses of binary distillation column with constrained GPC design using various models for (a) predictive control actions, (b) bottom product and (c) top product.

of 40, a control horizon of 40 and weighting factors of 1×10^5 and 1 are applied to the system for control error and control increments, respectively. The controller parameters for full order and reduced order binary distillation column are summarized in table 7.

Multiple positive and negative set-point steps were applied for bottom product X_B (output 1) at time $t = 25$ h, 175 h and $t = 100$ h, respectively. Similarly, multiple positive and negative set-point steps were also applied for top product X_D (output 2) at time $t = 135$ h, 220 h and $t = 50$ h, respectively.

Figure 9a shows the behaviour of the multivariable process in the presence of changes in L and V with solid black and red lines, which represents the nonlinear model-predictive controller action, respectively, and remaining lines in the subplot indicate linearized reduced order and full order model-predictive controller actions. It is observed that the output follows the reference by means of the contribution of the two manipulated variables L and V . It is also seen that any change affects all the variables with a slight deviation from its reference during the transient stage.

Table 9. Comparison of time elapsed for full order with reduced order linear and nonlinear model of binary distillation column.

Process	Time elapsed (s)	
	Full order system	Reduced order system
Linear	5.2713	4.3387
Simulated non-linear	5.8200	4.5376

Table 10. Comparison of matrix dimensions and memory used for full order to reduced order constrained MPC of binary distillation column.

Vector	Full order system		Reduced order system	
	Matrix dimension	Memory (bytes)	Matrix dimension	Memory (bytes)
F	80×132	84480	40×20	12800
X	132×1	1056	20×1	160

In figure 9b and c, the setpoints for bottom product X_B and top product X_D are plotted as red solid lines at various time intervals; black and blue dotted lines are the output responses for linearized reduced order and full order models, respectively; solid black line represents the output responses using first-principle model. Output responses of the controller with the transfer function for linear and nonlinear model are shown in figure 9. It is observed that similar and better results are obtained when compared with control response using first-principle model of binary distillation column.

The simulation results of constrained GPC controller with linearized and nonlinear models for the simulated binary distillation column are presented. Satisfactory closed loop responses are obtained for the non-linear process with the controller designed using the linear model.

GPC performance comparisons of full order to reduced order linear and nonlinear models are tabulated in table 8. Studies shows that performance indices with linearized reduced and full order are minimum when compared with first-principle model. The performance indices such as ISE, IAE and MSE are significantly lower in constrained GPC controller for the chosen multivariable binary distillation column. It is also investigated that time elapsed for simulating the reduced order model is lesser for a reduced order model when compared with that of a full order model and is summarised in table 9.

Table 10 gives a comparative study of matrix dimension and memory utilized for simulation of full order and reduced order constrained GPC of a binary distillation column. The reduced order model requires less memory storage and provides similar dynamics to represent a

complex nonlinear process; hence computational complexity is low and it is simple to design a multivariable binary distillation column process.

6. Conclusions

A standard predictive control scheme using a model order reduction method based on a new hybrid algorithm has been presented. In the proposed CDPA method, the denominator of the reduced model is obtained using the cluster technique in which the dominant poles are grouped into several clusters and replaced by the corresponding cluster centres. The GPC as a class of MPC employs three different model structures for a binary distillation column, namely (i) linearized full order model, (ii) linearized reduced order model and (iii) nonlinear first-principle model using MATLAB. The output responses of GPC based on the linearized reduced order model match the system dynamics and show characteristics close to that of the linearized full order model. The proposed method can be used for many complex nonlinear multivariable processes. The efficacy of the proposed algorithm is demonstrated with the help of a simulated complex non-linear multivariable binary distillation column process; it is observed that it shows better responses than those from conventional techniques in terms of error minimization for processes with constraints on inputs and outputs. The simulated numerical results show that the proposed method is simple and efficient to compute dominant poles that match the original system properties to give minimum performance indices. Hence, the computational complexity, computer memory and time elapsed for simulation can be reduced using a reduced order model for a distillation column process.

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