Influence of yielding base and rigid base on propagation of Rayleigh-type wave in a viscoelastic layer of Voigt type

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Abstract. The present study aims to study the propagation of Rayleigh-type wave in a layer, composed of isotropic viscoelastic material of Voigt type, with the effect of yielding base and rigid base in two distinct cases. With the aid of an analytical treatment, closed-form expressions of phase velocity and damped velocity for both the cases are deduced. As a special case of the problem it is found that obtained results are in good agreement with the established standard results existing in the literature. It is established through the study that volume-viscoelastic and shear-viscoelastic material parameter and yielding parameter have significant effect on phase and damped velocities of Rayleigh-type wave in both the cases. Numerical calculations and graphical illustration have been carried out for both the considered cases in the presence and the absence of viscoelasticity. A comparative study has been performed to analyse the effect of layer with yielding base, traction-free base and rigid base on the phase and damped velocities of Rayleigh-type wave.

Keywords. Rayleigh-type wave; isotropic; viscoelasticity; yielding; rigid.

1. Introduction

A Rayleigh wave is a seismic surface wave causing the ground to shake in an elliptical motion. Rayleigh waves are surface waves that travel near the surface of solids and include both longitudinal and transverse motions. The amplitude of this wave decreases exponentially with depth [1]. These waves were predicted in 1885 by Lord Rayleigh, after whom they were named. The study of Rayleigh-type waves is of great importance in several fields such as earthquake engineering, seismology, geophysics, acoustics, telecommunication industries and material sciences. In seismology, the study of Rayleigh waves is crucial for construction sector as it aids in understanding the cause of damage and also estimating the damage caused by the earthquake. They do not scatter into the depth as seismic body waves and hence their propagation is confined to the surface only.

Earth material behaves nearly elastically under the application of small transient forces but viscously under the application of long duration of forces [2]. Materials such as coal tar, salt, sediments, etc., which are buried beneath the earth surface can be modelled as viscoelastic materials. The property of materials of being viscoelastic represents attenuation in the propagation of elastic waves through them. The general theory of viscoelasticity describes the linear behaviour of both elastic and anelastic materials and

provides the basis for describing the attenuation of seismic waves due to anelasticity.

As seismic wave propagates, it attenuates due to energy dissipation [2, 3]. It is also established that this attenuation is mainly due to the imperfect elasticity of the crust and upper mantle. In addition to this, the lithosphere–asthenosphere is represented by a viscoelastic half-space or an elastic layer in welded contact with viscoelastic half-space below it or a viscoelastic layer free to slide over the material or else a layered viscoelastic half-space [4]. These viscoelastic models are relatively more successful in explaining the major features of seismic surface movements in seismically active regions.

The materials exhibiting both viscous and elastic behaviour are known as viscoelastic materials. The viscoelastic models are designed as different linear combinations of springs and dashpots and these different combinations lead to different models, which include the Maxwell model, the Kelvin–Voigt model and the Standard Linear Solid model. These models may be equivalently modelled as electrical circuits, which differ only in the arrangement of their springs and dashpots. In each of these models, the viscous property of the dashpot resembles the circuit's resistance, which is responsible for dissipation, whereas the elastic modulus of a spring may be compared to the circuit's capacitance, which stores energy. Therefore, a dashpot is responsible for resisting the change in length and thus for a shorter time period, it may be approximated as a rigid rod. The time

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derivative components of the stress–strain relationship dominate for shorter time periods. The series combination of a purely viscous damper and a purely elastic spring constitutes the Maxwell model. On the other hand, the parallel combination of a Newtonian damper and Hookean elastic spring forms the Kelvin–Voigt model, which is also known as the Voigt model.

The parallel combination of the Maxwell model (where the spring and a dashpot are in series with each other) with a lone spring constitutes the Standard Linear Solid model. Among all the aforementioned models, the Standard Linear Solid model is capable of predicting the material responses most accurately; however, the results are found to be mathematically inaccurate for strain under specific loading conditions and may also result in difficult calculations. In light of this it can be concluded that earth may be considered as a viscoelastic body and aforementioned viscoelastic models are found to be appropriate for the purpose of modelling but with their own limitations. Keeping in mind the solvability of the proposed mathematical model along with the applicability, Kelvin–Voigt models are chosen in our study.

In view of aforementioned facts, it becomes obligatory to study the Rayleigh wave propagation in viscoelastic medium distinctly. The anelastic characteristics and displacements of the Rayleigh waves considering a Voigt-type dissipation mechanism with small viscous damping and a Poisson solid were analysed [5]. The possibility of propagation of two types of Rayleigh wave on the surface of a linear viscoelastic medium, including energy balance, quality factors and energy velocities, was conferred [6]. The analysis of the geometry of propagation of Rayleigh waves along with the decay in amplitude in terms of the slowness and wave curves was established [7]. The surface wave propagation in general viscoelastic media of higher order involving time rate of strain was investigated [8]. A rigorous treatment is given to the wave propagation in an elastic continuum when the influence of initial stress is taken into account [9]. Propagation of Rayleigh waves in a medium under initial stresses has been discussed [10]. An attempt [11] has been made to study the effect of anisotropy on Rayleigh waves in layered medium. The generation of waves in an orthotropic elastic solid medium under the influence of initial stress and gravity field was investigated [12]. The propagation of shear waves in a transverse isotropic fluid-saturated porous plate was studied [13]. Propagation of Rayleigh-type wave on the surface of a viscoelastic solid in a linear theory of nonlocal elasticity is discussed [14]. In addition to these, a sufficient amount of knowledge about the Kelvin–Voigt model and hence the phenomena involving viscoelastic property of a material can be gained from the various disquisitions available in the literature [3, 15, 16].

Natural disasters like earthquakes are inevitable part of living on the planet like earth. We cannot avoid such types of disasters from happening. We can only make some attempt to minimize the effect produced by these disasters.

We can design our buildings in such a manner that they can absorb the power of sudden shocks to have minimum loss. Nowadays rocking structural systems are developed with yielding base, which reduces seismic responses of buildings by creating rocking vibration during strong earthquakes [17–19]. A yielding base is one that displaces sufficiently to minimize the devastating effect during strong earthquakes. Yielding base may be found in underground reservoir structures. Constructions over a suitable yielding base play a significant role in reducing seismic responses of buildings during earthquakes; therefore, research on the problems that include yield base is of great concern to the scientific community.

Since underground structures can be categorized either as yielding or rigid unyielding, the study of seismic wave propagation in yielding as well as rigid boundary has a great importance in seismology [20]. Wave propagation in an isotropic surface stratum of the earth resting on an extremely rigid foundation has been investigated [21]. The study of the effect of yielding foundation in layer on propagation of elastic wave has always been of great concern to civil engineers and seismologists. The stresses in the earth crusts have been discovered under an axially symmetrical load assuming that the earth crusts rest on yielding foundation [22]. The propagation of Rayleigh waves in an isotropic layer resting on yielding medium was studied [23]. The ratio of the horizontal to vertical components of displacement for the propagation of Rayleigh waves on the free surface has been computed in some problems by various authors [21, 23]. To date, no attempt has been made to study the Rayleigh-type propagation in isotropic-viscoelastic layer with yielding and rigid base.

The present work deals with the propagation of Rayleigh-type wave in an isotropic viscoelastic material of Voigt type with yielding base and rigid base in two different cases. Closed-form expressions of phase velocity and damped velocity for both the cases are deduced analytically. It is found that obtained results are in good agreement with the established standard results (existing in the literature) in a special case of the problem. Volume-viscoelastic and shear-viscoelastic material parameter and yielding parameter of phase and damped velocities of Rayleigh-type wave in both the cases are traced out. Numerical calculations and graphical illustration have been carried out for both the considered cases in the presence and the absence of viscoelasticity. The influence of layer with yielding base, traction-free base and rigid base on the phase and damped velocities of Rayleigh-type wave is studied extensively in a comparative manner.

2. Formulation and solution of the problem

Let us consider the propagation of Rayleigh-type wave in a homogenous isotropic viscoelastic layer of thickness H . Introduce a co-ordinate system in such a way that x -axis

Figure 1. Geometry of the problem.

is along the direction of propagation of Rayleigh-type wave and y-axis is directed vertically downwards as shown in figure 1.

The equation governing propagation of small elastic disturbances in the absence of body forces is [24]

$$\tau_{ij,j} = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad (i, j = 1, 2, 3), \quad (1)$$

where τ_{ij} are stress components, $u_i = (u, v, w)$ are displacement components and ρ is the density. Plane strain deformation for the propagation of Rayleigh-type waves propagating in x-direction causing displacement along x (longitudinal) and y (transversal) directions may be characterized as

$$u = u(x, y, t), \quad v = v(x, y, t), \quad w = 0 \quad \text{and} \quad \frac{\partial(\cdot)}{\partial z} \equiv 0. \quad (2)$$

Stress-strain relation for isotropic Voigt viscoelastic medium is given as [25]

$$\tau_{ij} = \bar{\lambda} \Delta \delta_{ij} + 2\bar{\mu} e_{ij}, \quad (3)$$

with

$$\bar{\lambda} = \lambda \left(1 + \frac{\lambda'}{\lambda} \frac{\partial}{\partial t} \right) \quad \text{and} \quad \bar{\mu} = \mu \left(1 + \frac{\mu'}{\mu} \frac{\partial}{\partial t} \right),$$

where λ and μ are Lamé constants, $\frac{\lambda'}{\lambda}$ and $\frac{\mu'}{\mu}$ are parameters acting for volume-viscosity and shear viscosity, respectively.

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad (\text{cubical dilation}), \quad (4)$$

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (\text{strain components}). \quad (5)$$

With the aid of Eqs. (1)–(5), the non-vanishing equations of motion may be written as

$$\begin{aligned} \left[(\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial t} \right] \frac{\partial \Delta}{\partial x} + \left(\mu + \mu' \frac{\partial}{\partial t} \right) \nabla^2 u &= \rho \frac{\partial^2 u}{\partial t^2}, \\ \left[(\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial t} \right] \frac{\partial \Delta}{\partial y} + \left(\mu + \mu' \frac{\partial}{\partial t} \right) \nabla^2 v &= \rho \frac{\partial^2 v}{\partial t^2}, \end{aligned} \quad (6)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

Consider

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}, \quad (7)$$

where ϕ and ψ are the potential functions [26] of x, y and t .

In view of Eq. (7), Eq. (6) yields the following equations:

$$\begin{aligned} \frac{\partial}{\partial x} \left[\left\{ (\lambda + 2\mu) + (\lambda' + 2\mu') \frac{\partial}{\partial t} \right\} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) - \rho \frac{\partial^2 \phi}{\partial t^2} \right] \\ + \frac{\partial}{\partial y} \left[\left\{ \mu + \mu' \frac{\partial}{\partial t} \right\} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) - \rho \frac{\partial^2 \psi}{\partial t^2} \right] &= 0, \\ \frac{\partial}{\partial y} \left[\left\{ (\lambda + 2\mu) + (\lambda' + 2\mu') \frac{\partial}{\partial t} \right\} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) - \rho \frac{\partial^2 \phi}{\partial t^2} \right] \\ - \frac{\partial}{\partial x} \left[\left\{ \mu + \mu' \frac{\partial}{\partial t} \right\} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) - \rho \frac{\partial^2 \psi}{\partial t^2} \right] &= 0. \end{aligned} \quad (8)$$

Now, we have

$$\begin{aligned} \left[(\lambda + 2\mu) + (\lambda' + 2\mu') \frac{\partial}{\partial t} \right] \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) &= \rho \frac{\partial^2 \phi}{\partial t^2}, \\ \left[\mu + \mu' \frac{\partial}{\partial t} \right] \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) &= \rho \frac{\partial^2 \psi}{\partial t^2}. \end{aligned} \quad (9)$$

We look for solutions

$$\phi = \phi_0 \sin(\xi x) e^{ipt}, \quad \psi = \psi_0 \cos(\xi x) e^{ipt}. \quad (10)$$

Using Eq. (10), Eq. (9) yields

$$\begin{aligned} \frac{\partial^2 \phi_0}{\partial y^2} - \alpha'^2 \phi_0 &= 0, \\ \frac{\partial^2 \psi_0}{\partial y^2} - \beta'^2 \psi_0 &= 0, \end{aligned} \quad (11)$$

in view of following setting:

$$\xi^2 - h'^2 = \alpha'^2, \quad \xi^2 - k'^2 = \beta'^2,$$

where

$$h'^2 = \frac{\rho p^2}{\lambda(1 + i\alpha_1) + 2\mu(1 + i\alpha_2)}, \quad k'^2 = \frac{\rho p^2}{\mu(1 + i\alpha_2)},$$

$$\alpha_1 = \frac{p\lambda'}{\lambda}$$

is non-dimensional volume-viscosity and $\alpha_2 = \frac{p\mu'}{\mu}$ is non-dimensional shear viscosity.

Equation (11) yield

$$\begin{aligned} \phi_0 &= C_1 \cosh(\alpha' y) + C_2 \sinh(\alpha' y), \\ \psi_0 &= C_3 \cosh(\beta' y) + C_4 \sinh(\beta' y), \end{aligned} \quad (12)$$

where C_1 – C_4 are arbitrary constants.

With the help of Eq. (12), Eq. (10) give

$$\begin{aligned}\phi &= [C_1 \cosh(\alpha'y) + C_2 \sinh(\alpha'y)] \sin(\xi x) e^{ipt}, \\ \psi &= [C_3 \cosh(\beta'y) + C_4 \sinh(\beta'y)] \cos(\xi x) e^{ipt}.\end{aligned}\quad (13)$$

Therefore, the horizontal and vertical displacement for the layer medium due to propagation of Rayleigh-type wave can be written as

$$\begin{aligned}u &= [\xi \{C_1 \cosh(\alpha'y) + C_2 \sinh(\alpha'y)\} \\ &\quad + \beta' \{C_3 \sinh(\beta'y) + C_4 \cosh(\beta'y)\}] \cos(\xi x), \\ v &= [\alpha' \{C_1 \sinh(\alpha'y) + C_2 \cosh(\alpha'y)\} \\ &\quad + \xi \{C_3 \cosh(\beta'y) + C_4 \sinh(\beta'y)\}] \sin(\xi x),\end{aligned}\quad (14)$$

and the non-vanishing stress components are given by

$$\begin{aligned}\tau_{xy} &= \mu(1 + i\alpha_2) [(\xi^2 + \beta'^2) \{C_3 \cosh(\beta'y) + C_4 \sinh(\beta'y)\} \\ &\quad + 2\alpha' \xi \{C_1 \sinh(\alpha'y) + C_2 \cosh(\alpha'y)\}] \cos(\xi x), \\ \tau_{yy} &= \mu(1 + i\alpha_2) [(\xi^2 + \beta'^2) \{C_1 \cosh(\alpha'y) + C_2 \sinh(\alpha'y)\} \\ &\quad + 2\beta' \xi \{C_3 \sinh(\beta'y) + C_4 \cosh(\beta'y)\}] \sin(\xi x), \\ \tau_{xx} &= [(\lambda(1 + i\alpha_1)(\alpha'^2 - \xi^2) - 2\mu(1 + i\alpha_2)\xi^2) C_1 \cosh(\alpha'y) \\ &\quad - C_2 \sinh(\alpha'y)(\lambda(1 + i\alpha_1)(\alpha'^2 + \xi^2) + 2\mu(1 + i\alpha_2)\xi^2) \\ &\quad - 2\xi\beta' C_3 \sinh(\beta'y)(\lambda(1 + i\alpha_1) + \mu(1 + i\alpha_2)) \\ &\quad - 2\xi\beta' \mu(1 + i\alpha_2) C_4 \cosh(\beta'y)] \sin(\xi x), \\ \tau_{zz} &= \lambda(1 + i\alpha_1) [(\alpha'^2 - \xi^2) C_1 \cosh(\alpha'y) - (\alpha'^2 + \xi^2) \\ &\quad C_2 \sinh(\alpha'y) - 2\xi\beta' C_3 \sinh(\beta'y)] \sin(\xi x), \\ \tau_{xz} &= 0, \quad \tau_{yz} = 0.\end{aligned}\quad (15)$$

Now, we seek the dispersion and damping equations for the propagation of Rayleigh-type wave in isotropic viscoelastic layer with the effect of yielding base and rigid base in two distinct cases.

2.1 Case I

Here we consider the case of propagation of Rayleigh-type wave in homogenous isotropic viscoelastic layer with the effect of yielding base. The boundary condition for this case may be quoted as follows:

for the upper traction-free boundary surface of the layer

$$(i) \quad \tau_{xy} = 0 \text{ at } y = 0, \quad (16)$$

$$(ii) \quad \tau_{yy} = 0 \text{ at } y = 0, \quad (17)$$

for the lower boundary surface (with yielding effect) of the layer [23]

$$(iii) \quad \tau_{xy} = 0 \text{ at } y = H, \quad (18)$$

$$(iv) \quad \tau_{yy} = -Kv \text{ at } y = H, \quad (19)$$

where the parameter K is due to the yielding base.

With the help of Eqs. (14) and (15), boundary conditions (16)–(19) result in

$$(\xi^2 + \beta'^2)C_3 + 2\alpha'\xi C_2 = 0, \quad (20)$$

$$(\xi^2 + \beta'^2)C_1 + 2\xi\beta' C_4 = 0, \quad (21)$$

$$\begin{aligned}(\xi^2 + \beta'^2)(C_3 \cosh \beta'H + C_4 \sinh \beta'H) \\ + 2\alpha'\xi(C_1 \sinh \alpha'H + C_2 \cosh \alpha'H) = 0,\end{aligned}\quad (22)$$

$$\begin{aligned}\mu(1 + i\alpha_2) \{(\xi^2 + \beta'^2)(C_1 \cosh \alpha'H + C_2 \sinh \alpha'H) \\ + 2\beta'\xi(C_3 \sinh \beta'H + C_4 \cosh \beta'H)\} \\ = -K \{ \alpha'(C_1 \sinh \alpha'H + C_2 \cosh \alpha'H) \\ + \xi(C_3 \cosh \beta'H + C_4 \sinh \beta'H) \}.\end{aligned}\quad (23)$$

Eliminating C_1 – C_4 from Eqs. (20)–(23) we obtain the following frequency equation:

$$\begin{aligned}\{ \mu(1 + i\alpha_2)(\xi^2 + \beta'^2)(\cosh \alpha'H - \cosh \beta'H) \\ + K \left(\alpha' \sinh \alpha'H - \frac{\xi^2 + \beta'^2}{2\beta'} \sinh \beta'H \right) \} \\ \times 2\alpha'\xi(\cosh \alpha'H - \cosh \beta'H) \\ = \left\{ \mu(1 + i\alpha_2) \left((\xi^2 + \beta'^2) \sinh \alpha'H - \frac{4\alpha'\beta'\xi^2}{\xi^2 + \beta'^2} \sinh \beta'H \right) \right. \\ \left. + K \left(\alpha' \cosh \alpha'H - \frac{2\alpha'\xi^2}{\xi^2 + \beta'^2} \cosh \beta'H \right) \right\} \\ \times \left\{ 2\alpha'\xi \sinh \alpha'H - \frac{(\xi^2 + \beta'^2)^2}{2\beta'\xi} \sinh \beta'H \right\}.\end{aligned}\quad (24)$$

Equation (24) presents the complex form of frequency equation for the propagation of Rayleigh-type wave in homogenous isotropic viscoelastic layer with a yielding base. It is worth noting that the frequency equation depends on the elastic parameters, volume viscosity, shear viscosity and yielding parameter. It is evident that frequency equation (24) is complex due to the dissipation of the system. We have to separate the real and imaginary part of frequency equation (24) to obtain expressions associated with phase velocity and damped velocity. Moreover, the frequency equation contains the complex quantities α' , β' , which may be written as follows:

$$\alpha' = E_1 + iF_1, \quad \beta' = E_2 + iF_2, \quad (25)$$

where expressions for E_1 , E_2 , F_1 and F_2 are provided in “Appendix A”.

Now, taking Eq. (25) into account, we separate the real and imaginary part of Eq. (24) as follows:

$$\begin{aligned}(S_1 + KS_3)S_5 - (S_2 + KS_4)S_6 - (S_7 + KS_9)S_{11} \\ + (S_8 + KS_{10})S_{12} = 0,\end{aligned}\quad (26)$$

$$\begin{aligned}(S_2 + KS_4)S_5 - (S_7 + KS_9)S_{12} - (S_8 + KS_{10})S_{11} \\ + (S_1 + KS_3)S_6 = 0,\end{aligned}\quad (27)$$

where $S_i (i = 1, 2, \dots, 12)$ are provided in Appendix A. Equations (26) and (27) represent the dispersion equation and damping equation, respectively, for the propagation of Rayleigh-type wave in an isotropic viscoelastic layer with a yielding base.

The ratio of the horizontal to vertical components of displacement for the propagation of Rayleigh waves on the free surface is of great importance [21]. Therefore, ratio of vertical to horizontal components of displacement on the free surface for Case I may be obtained as

$$\frac{v}{u} = \frac{4\alpha'\beta'\xi^2 \sinh(\alpha'H) - (\xi^2 + \beta'^2)^2 \sinh(\beta'H)}{2\beta'\xi(\xi^2 + \beta'^2)(\cosh(\alpha'H) - \cosh(\beta'H))}.$$

2.1a *Subcase I*: When the layer medium is isotropic without viscoelasticity, we consider $\mu' = 0$ and $\lambda' = 0$, which lead to the deduced dispersion relation [23]

$$\begin{aligned} & \left\{ \mu(\xi^2 + \beta^2)(\cosh \alpha H - \cosh \beta H) \right. \\ & \left. + K \left(\alpha \sinh \alpha H - \frac{\xi^2 + \beta^2}{2\beta} \sinh \beta H \right) \right\} \\ & \times 2\alpha\xi(\cosh \alpha H - \cosh \beta H) \\ & = \left\{ \mu \left((\xi^2 + \beta^2) \sinh \alpha H - \frac{4\alpha\beta\xi^2}{\xi^2 + \beta^2} \sinh \beta H \right) \right. \\ & \left. + K \left(\alpha \cosh \alpha H - \frac{2\alpha\xi^2}{\xi^2 + \beta^2} \cosh \beta H \right) \right\} \\ & \times \left\{ 2\alpha\xi \sinh \alpha H - \frac{(\xi^2 + \beta^2)^2}{2\beta\xi} \sinh \beta H \right\} \end{aligned} \quad (28)$$

and the damping equation vanishes identically.

2.2 Case II

In this case, we investigate the propagation of Rayleigh-type waves in a homogenous isotropic viscoelastic layer with the effect of rigid base. The boundary conditions of this proposed case are as follows:

for the upper traction-free boundary surface of the layer

$$(i) \tau_{xy} = 0 \text{ at } y = 0, \quad (29)$$

$$(ii) \tau_{yy} = 0 \text{ at } y = 0, \quad (30)$$

for the lower rigid boundary surface of the layer

$$(iii) u = 0 \text{ at } y = H, \quad (31)$$

$$(iv) v = 0 \text{ at } y = H. \quad (32)$$

Using these boundary conditions (29)–(32) with Eqs. (14) and (15), we have

$$(\xi^2 + \beta'^2)C_3 + 2\alpha'\xi C_2 = 0, \quad (33)$$

$$(\xi^2 + \beta'^2)C_1 + 2\beta'\xi C_4 = 0, \quad (34)$$

$$\begin{aligned} & \xi(C_1 \cosh \alpha'H + C_2 \sinh \alpha'H) \\ & + \beta'(C_3 \sinh \beta'H + C_4 \cosh \beta'H) = 0, \end{aligned} \quad (35)$$

$$\begin{aligned} & \alpha'(C_1 \sinh \alpha'H + C_2 \cosh \alpha'H) \\ & + \xi(C_3 \cosh \beta'H + C_4 \sinh \beta'H) = 0. \end{aligned} \quad (36)$$

Elimination of C_1 – C_4 from Eqs. (33)–(36) gives

$$\frac{\cosh \alpha'H - \frac{\xi^2 + \beta'^2}{2\xi^2} \cosh \beta'H}{\sinh \alpha'H - \frac{\xi^2 + \beta'^2}{2\alpha'\beta'} \sinh \beta'H} = \frac{\sinh \alpha'H - \frac{2\alpha'\beta'}{\xi^2 + \beta'^2} \sinh \beta'H}{\cosh \alpha'H - \frac{2\xi^2}{\xi^2 + \beta'^2} \cosh \beta'H}. \quad (37)$$

Now, like Eq. (24), Eq. (37) presents the complex form of frequency equation for the propagation of Rayleigh-type wave in homogenous isotropic viscoelastic layer with a rigid base, containing the complex quantities α' , β' . Therefore, the real and imaginary parts of frequency equation (37) need to be separated so as to obtain the expressions associated with phase velocity and damped velocity:

$$(R_1 R_3 - R_2 R_4 - R_5 R_7 + R_6 R_8) = 0, \quad (38)$$

and

$$(R_2 R_3 - R_6 R_7 - R_8 R_5 + R_1 R_4) = 0, \quad (39)$$

respectively. Expressions for newly introduced symbols $R_i (i = 1, 2, \dots, 12)$ are provided in Appendix A. Equations (38) and (39) represent the dispersion equation and damping equation, respectively, for the propagation of Rayleigh-type wave in an isotropic viscoelastic layer with a yielding base.

The ratio of vertical to horizontal components of displacement on the free surface for Case II may be obtained as

$$\frac{v}{u} = \frac{\alpha'(2\xi^2 \cosh \alpha'H - (\xi^2 + \beta'^2) \cosh \beta'H)}{\xi((\xi^2 + \beta'^2) \sinh \alpha'H - 2\alpha'\beta' \sinh \beta'H)}.$$

2.2a *Subcase I*: When viscoelasticity is absent in the layer medium, i.e., the layer medium is isotropic and elastic, $\mu' = 0$ and $\lambda' = 0$. In this case the obtained dispersion relation (38) gives

$$\frac{\cosh \alpha H - \frac{\xi^2 + \beta^2}{2\xi^2} \cosh \beta H}{\sinh \alpha H - \frac{\xi^2 + \beta^2}{2\alpha\beta} \sinh \beta H} = \frac{\sinh \alpha H - \frac{2\alpha\beta}{\xi^2 + \beta^2} \sinh \beta H}{\cosh \alpha H - \frac{2\xi^2}{\xi^2 + \beta^2} \cosh \beta H}, \quad (40)$$

as given in [23] and damping equation (39) vanishes identically.

3. Numerical results and discussion

Numerical calculation and graphical illustration have been carried out with derived dispersion and damping equations of Rayleigh-type wave for two distinct cases: (i) when the

layer is composed of isotropic viscoelastic material with yielding base (Case I) and (ii) when the layer is composed of isotropic viscoelastic material with rigid base (Case II). Influences of non-dimensional volume viscosity ($\alpha_1 = \frac{\rho \lambda'}{\lambda}$), non-dimensional shear viscosity ($\alpha_2 = \frac{\rho \mu'}{\mu}$) and non-dimensional yielding parameter ($\frac{KH}{\mu}$) on phase and damped velocities of Rayleigh-type wave for both the cases have been analysed and their pictorial delineation has been accomplished in figures 2–8.

For the sake of numerical computation, following data [27] for isotropic viscoelastic layer medium have been taken into account:

$$\lambda = 2.51 \times 10^{10} \text{ Nm}^{-2}, \quad \mu = 1.987 \times 10^{10} \text{ Nm}^{-2}, \\ \rho = 4705 \text{ kg m}^{-3}.$$

Figures 2–4 correspond to the case where the layer is in the effect of yielding base (Case I) whereas figures 5 and 6 correspond to the case when the layer is in the effect of rigid base (Case II). For the sake of comparative study of both the cases (Cases I and II) with and without viscoelasticity, figure 7 is shown.

It is evident from the figures that the phase velocity and damped velocity both increase with the increase of dimensionless wavelength ($\frac{2\pi}{\xi H}$). The study of such effects in different types of media, within the earth, is useful for determining the type and state of rock and minerals composing the earth. Further, the increase in phase velocity with increase in wavelength (i.e., the longer wavelength wave propagates faster than the shorter wavelength wave) is termed as dispersion. Violation of the said phenomenon leads to some anomaly, which may result in an unusual dispersion of the propagating wave that is not at all observed in the graphical results of the study.

The effect of non-dimensional volume-viscosity (α_1) on the phase velocity and damped velocity of Rayleigh-type wave (Case I) has been depicted in figure 2a and b, respectively. Meticulous examination of figure 2a establishes that for a fixed magnitude of wavelength (i.e., $\frac{2\pi}{\xi H} = 1.0$) the increase in phase velocity corresponding to $\alpha_1 = 3$ (curve 2) and $\alpha_1 = 6$ (curve 3) with respect to $\alpha_1 = 0.01$ (curve 1) is nearly 1.13% and 1.8%, respectively; however, in figure 2b, for the same magnitude of wave number, the increase in damped velocity corresponding to $\alpha_1 = 3$ (curve 2) and $\alpha_1 = 6$ (curve 3) is 20.97% and 16.98%, respectively, with respect to $\alpha_1 = 0.01$ (curve 1). Hence, it is clear that the effect of volume viscosity on the damped velocity is more prominent as compared with that of phase velocity. The effect of volume viscosity on phase velocity is significant in lower range of wavelength but insignificant for higher range of wavelength. On the other hand, the effect of volume viscosity on damped velocity is substantial in lower as well as higher range of wavelength.

Figure 2. Variation of (a) non-dimensional phase velocity and (b) non-dimensional damped velocity against non-dimensional wavelength for different values of non-dimensional volume viscosity (α_1) when the layer is in the effect of yielding base.

In figure 3a and b the effects of non-dimensional shear viscosity (α_2) on phase and damped velocity of Rayleigh-type wave for Case I have been elucidated, respectively. The effect is found to be considerable for both the velocities in lower as well as higher range of wavelength. In particular, figure 3a shows that for $\frac{2\pi}{\xi H} = 1.0$, there is 1.04% increment in phase velocity when $\alpha_2 = 0.1$ (curve 2) and 3.7% increment in phase velocity when $\alpha_2 = 0.2$ (curve 3), compared with the case when $\alpha_2 = 0.01$ (curve 1). On the other hand, figure 3b elucidates that the increase in damped velocity for $\alpha_2 = 0.1$ (curve 2) and $\alpha_2 = 0.2$ (curve 3) is 0.59% and 1.22%, respectively, with respect to $\alpha_2 = 0.01$ (curve 1). Also, it may be noticed from figure 3b that for $\frac{2\pi}{\xi H} = 0.8$, is a point of inversion, after which the nature of the curves gets reversed, i.e., with increase in shear viscosity, the

Figure 3. Variation of (a) non-dimensional phase velocity and (b) non-dimensional damped velocity against non-dimensional wavelength for different values of non-dimensional shear viscosity (α_2) when the layer is in the effect of yielding base.

damped velocity decreases for $\frac{2\pi}{\xi H} < 0.8$, whereas it increases for $\frac{2\pi}{\xi H} > 0.8$.

Influence of yielding base of the layer on phase velocity and damped velocity of Rayleigh-type wave (Case I) is exhibited in figure 4a and b, respectively. In figure 4a and b, curves 1–3 are associated with the case where the layer medium is isotropic with viscoelasticity whereas curves 4–6 in figure 4a are associated with the case when the layer medium is isotropic without viscoelasticity. Curve 1 in both figure 4a and b describes the situation where the layer has a traction-free foundation and rest of the curves are concerned with the case where the layer has a yielding foundation. In figure 4a, when the layer medium is isotropic with viscoelasticity, for $\frac{2\pi}{\xi H} = 1.0$ there is 1.39% and 2.78% increase, respectively, in the phase velocity for curve 2 (with $\frac{KH}{\mu} = 2$) and curve 3 (with $\frac{KH}{\mu} = 4$) as compared

Figure 4. Variation of (a) non-dimensional phase velocity and (b) non-dimensional damped velocity against non-dimensional wavelength for different values of non-dimensional yielding parameter ($\frac{KH}{\mu}$) when the layer is in the effect of yielding base.

with curve 1 (with $\frac{KH}{\mu} = 0$), which corresponds to the traction-free foundation. Now, if the layer medium is isotropic without viscoelasticity then for the same magnitude of wavelength (i.e., $\frac{2\pi}{\xi H} = 1.0$), the increase in phase velocity corresponding to curve 5 (with $\frac{KH}{\mu} = 2$) and curve 6 (with $\frac{KH}{\mu} = 4$) is 1.64% and 3.29%, respectively, with respect to curve 4 (with $\frac{KH}{\mu} = 0$), which corresponds to the traction-free foundation.

Hence it can be inferred that effect on phase velocity is more prominent for the isotropic case rather than the isotropic viscoelastic case. Similarly, in figure 4b, it may be seen that the increment in the damped velocity corresponding to $\frac{KH}{\mu} = 2$ (curve 2) and $\frac{KH}{\mu} = 4$ (curve 3) is 1.57% and 3%, respectively, with respect to the case where $\frac{KH}{\mu} = 0$

Figure 5. Variation of (a) non-dimensional phase velocity and (b) non-dimensional damped velocity against non-dimensional wavelength for different values of non-dimensional volume viscosity (α_1) when the layer is in the effect of rigid base.

(curve 1). Therefore, the effect of yielding base on the damped velocity is found to be much more significant as compared with its effect on the phase velocity. The study of both figures establishes that the layer with yielding foundation supports the phase and damped velocities of Rayleigh-type wave more as compared with the layer with traction-free foundation. Moreover, as viscoelasticity prevails in the layer medium, phase and damped velocities of Rayleigh-type wave increase. This result may be useful for civil engineers in their analysis for designing a stable structure.

In the case where the layer medium is in the effect of rigid base, figure 5a and b analyses the effect of non-dimensional volume viscosity on the phase velocity and damped velocity of Rayleigh-type wave, respectively, whereas figure 6a and b examines the influence of non-dimensional shear viscosity on the phase velocity and damped velocity of Rayleigh-type wave, respectively. In figure 5a and b, when the layer is in the effect of rigid base, it may be obtained that

Figure 6. Variation of (a) non-dimensional phase velocity and (b) non-dimensional damped velocity against non-dimensional wavelength for different values of non-dimensional shear viscosity (α_2) when the layer is in the effect of rigid base.

corresponding to volume-viscosity $\alpha_1 = 2$ (curve 2) and $\alpha_1 = 4$ (curve 3), the increment in phase velocity is 1.7% and 3.6%, respectively; however, the increment in damped velocity is 3.91% and 5.15%, respectively, with respect to $\alpha_1 = 0.01$ (curve 1). Hence, it is clear that in the case of rigid base the effect of volume viscosity on the damped velocity is more as compared with that of phase velocity, but small with respect to the case of yielding base. For figure 6a and b, the increment in phase velocity is 3.24% and 1.24% for $\alpha_2 = 0.2$ (curve 2) and $\alpha_2 = 0.3$ (curve 3), respectively, with respect to phase velocity for $\alpha_2 = 0.1$ (curve 1) whereas the increment in damped velocity is 15.51% and 3.9% for $\alpha_2 = 0.2$ (curve 2) and $\alpha_2 = 0.3$ (curve 3), respectively, with respect to damped velocity for $\alpha_2 = 0.1$ (curve 1). Moreover, the inversion point in figure 6a and b is found at the point $\frac{2\pi}{\xi H} = 0.7$ and 0.5 , respectively. It is revealed from these four figures that phase and damped velocities of Rayleigh-type wave

Figure 7. Variation of (a) non-dimensional phase velocity and (b) non-dimensional damped velocity against non-dimensional wavelength for both Cases I and II when the layer with or without viscoelasticity is in the effect of yielding base and rigid base.

are encouragingly affected by volume and shear viscosity of layer medium in the effect of rigid base.

In figure 7a, curves 1 and 3 correspond to the case where the layer medium is isotropic viscoelastic with yielding base and traction-free base, respectively. Curves 2 and 4 are concerned with the case where the layer medium is isotropic without viscoelasticity with yielding base and traction-free base, respectively; curves 5 and 6 denote the case where the layer is with a rigid base composed of isotropic material with viscoelasticity and without viscoelasticity, respectively. The comparative study suggests that irrespective of the presence or absence of viscoelasticity in the layered medium, phase velocity of Rayleigh-type wave is maximum when the layer has a yielding base, minimum when the layer has a rigid base and moderate when the layer has a traction-free base. Figure 7b establishes that damped velocity of Rayleigh-type wave is maximum when

Figure 8. Surface plot depicting variation of non-dimensional wavelength and non-dimensional yielding parameter when non-dimensional volume viscosity $\alpha_1 = 0.01$ and non-dimensional shear viscosity $\alpha_2 = 0.01$ against (a) phase velocity and (b) damped velocity.

the isotropic viscoelastic layer is with a yielding base, minimum when it is with a rigid base and moderate when it is with a traction-free base.

Variation of dimensionless phase velocity against dimensionless wavelength and dimensionless yielding parameter has been shown by means of surface plots in figure 8a. Surface plots in figure 8b exhibit the variation of dimensionless damped velocity against dimensionless wavelength and dimensionless yielding parameter.

Thus it may be adduced through the comparative study undertaken in figures 7a, b and 8a, b that both phase and damped velocities are affected significantly in the sense of

Figure 9. Variation of amplitude ratio against non-dimensional wave number when non-dimensional volume viscosity $\alpha_1 = 0.3$ and non-dimensional shear-viscosity $\alpha_2 = 0.1$ and the layer is in the effect of (a) yielding base and (b) rigid base.

increase in the situation where the layer has a yielding base as compared with traction-free base or rigid base.

Figure 9a and b pictures the variation of amplitude ratio $\frac{u}{u_0}$ against dimensionless wave number ξH for Cases I and II, respectively. Although the amplitude ratio finally gets diminished in both the figures, a comparative study of the figures manifests that it is significant up to a greater range of wave number for Case II as compared with that of Case I.

Figure 10a and b exhibits the variation of amplitude ratio $\frac{u}{u_0}$ against dimensionless wavelength $\frac{L}{H}$ for Cases I and II, respectively. In figure 10a, curve 1 reflects an isotropic elastic layer with small viscoelasticity ($\alpha_1 = 0.3$, $\alpha_2 = 0.1$) resting on a yielding base whereas curve 2 corresponds to the established result [23]. It is apparently observed in this figure that the curve

Figure 10. Variation of amplitude ratio against dimensionless wavelength when the layer is in the effect of (a) yielding base and (b) rigid base.

representing the result of the present problem for yielding base is very close to the curve reflecting the established result [23]. It is worth mentioning that if the viscoelasticity is made even smaller, both the curves will coincide. Thus, it can be concluded that the obtained result for yielding base in the present problem is in good agreement with the established result [23].

In figure 10b, curve 1 corresponds to an isotropic elastic layer with small viscoelasticity ($\alpha_1 = 0.3$, $\alpha_2 = 0.1$) resting on a rigid base whereas curve 2 reflects the established result [21]. It can be observed that in this figure the curve representing the result of the present problem for rigid base is very close to the curve reflecting the established result [21]. It is worth mentioning that if the viscoelasticity is made even smaller, both the curves coincide. Thus, it can be concluded that the obtained result for rigid base in the

present problem is in good agreement with the established result [21].

4. Conclusion

The current study deals with the propagation of Rayleigh-type wave in a layer composed of isotropic viscoelastic material, with the effect of a yielding base (Case I) and a rigid base (Case II) in two distinct cases. A closed form of dispersion equation and damping equation has been deduced for both the cases. Deduced results are found to be in good agreement with the established standard results existing in the literature. The substantial effect of wavelength, shear viscosity, volume viscosity and yielding parameter on the phase velocity and damped velocity is established through the study. Numerical computation has been carried out for the derived results. The outcome of the present study can be encapsulated as follows:

- I. Phase and damped velocities of Rayleigh-type wave substantially increase with the increase of wavelength.
- II. Shear viscosity and volume viscosity significantly affect phase and damped velocities of Rayleigh-type wave in both the cases where the layer is with a yielding base and a rigid base. Phase and damped velocities are increased by the increase in both of the viscoelastic parameters.
- III. Irrespective of the presence or absence of viscoelasticity in the layer medium, yielding parameter concerned with yielding base of the layer affects phase velocity as well as damped velocity of Rayleigh-type wave considerably in the sense of increase.
- IV. If viscoelasticity prevails in the layer medium, phase and damped velocities of Rayleigh-type wave are increased.
- V. The phase and damped velocities of Rayleigh-type wave are maximum in the case where the layer is in the effect of yielding base and minimum in the case when the layer is in the effect of rigid base. In case of traction-free base, they are found to be moderate.

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Appendix A

$$S_1 = (A_1 - A_2)\{\mu(\xi^2 + E_2^2 - F_2^2) - 2p\mu'E_2F_2\} \\ - (B_1 - B_2)\{p\mu'(\xi^2 + E_2^2 - F_2^2) + 2E_2F_2\mu\},$$

$$S_2 = (B_1 - B_2)\{\mu(\xi^2 + E_2^2 - F_2^2) - 2p\mu'E_2F_2\} \\ + (A_1 - A_2)\{p\mu'(\xi^2 + E_2^2 - F_2^2) + 2E_2F_2\mu\},$$

$$S_3 = E_1C_1 - F_1D_1 - M_1C_2 + N_1D_2,$$

$$S_4 = F_1C_1 - N_1C_2 - M_1D_2 + D_1E_1,$$

$$S_5 = 2\xi\{E_1(A_1 - A_2) - F_1(B_1 - B_2)\},$$

$$S_6 = 2\xi\{E_1(B_1 - B_2) + F_1(A_1 - A_2)\},$$

$$S_7 = \mu\{(\xi^2 + E_2^2 - F_2^2)C_1 - 2D_1E_2F_2\} \\ - p\mu'\{2C_1E_2F_2 + D_1(\xi^2 + E_2^2 - F_2^2)\} \\ - M_2C_2 + N_2D_2,$$

$$S_8 = p\mu'\{(\xi^2 + E_2^2 - F_2^2)C_1 - 2D_1E_2F_2\} \\ + \mu\{2C_1E_2F_2 + D_1(\xi^2 + E_2^2 - F_2^2)\} \\ - N_2C_2 - D_2M_2,$$

$$S_9 = E_1A_1 - F_1B_1 - A_2M_3 + B_2N_3,$$

$$S_{10} = F_1A_1 - B_2M_3 - A_2N_3 + B_1E_1,$$

$$S_{11} = 2\xi(E_1C_1 - D_1F_1) - M_4C_2 + N_4D_2,$$

$$S_{12} = 2\xi(F_1C_1 + D_1E_1) - N_4C_2 - M_4D_2,$$

$$M_1 = \frac{E_2(\xi^2 + E_2^2 - F_2^2) + 2E_2F_2^2}{2(E_2^2 + F_2^2)},$$

$$N_1 = \frac{2E_2^2F_2 - F_2(\xi^2 + E_2^2 - F_2^2)}{2(E_2^2 + F_2^2)},$$

$$M_2$$

$$= 4\xi^2 \frac{(E_1E_2 - F_1F_2) \cdot (\xi^2 + E_2^2 - F_2^2) + 2E_2F_2(F_1E_2 + F_2E_1)}{(\xi^2 + E_2^2 - F_2^2)^2 + 4E_2^2F_2^2},$$

$$N_2$$

$$= 4\xi^2 \frac{(F_1E_2 + F_2E_1)(\xi^2 + E_2^2 - F_2^2) - 2E_2F_2(E_1E_2 - F_1F_2)}{(\xi^2 + E_2^2 - F_2^2)^2 + 4E_2^2F_2^2},$$

$$M_3 = 2\xi^2 \frac{E_1(\xi^2 + E_2^2 - F_2^2) + 2E_2F_2F_1}{(\xi^2 + E_2^2 - F_2^2)^2 + 4E_2^2F_2^2},$$

$$N_3 = 2\xi^2 \frac{F_1(\xi^2 + E_2^2 - F_2^2) - 2E_2F_2E_1}{(\xi^2 + E_2^2 - F_2^2)^2 + 4E_2^2F_2^2},$$

M_4

$$= \frac{E_2 \left((\xi^2 + E_2^2 - F_2^2)^2 - 4E_2^2F_2^2 \right) + 4(\xi^2 + E_2^2 - F_2^2)E_2F_2^2}{2\xi^2(E_2^2 + F_2^2)},$$

N_4

$$= \frac{F_2 \left((\xi^2 + E_2^2 - F_2^2)^2 - 4E_2^2F_2^2 \right) + 4(\xi^2 + E_2^2 - F_2^2)E_2^2F_2}{2\xi^2(E_2^2 + F_2^2)},$$

$$A_1 = \cosh E_1 H \cos F_1 H, \quad B_1 = \sinh E_1 H \sin F_1 H,$$

$$A_2 = \cosh E_2 H \cos F_2 H, \quad B_2 = \sinh E_2 H \sin F_2 H,$$

$$C_1 = \sinh E_1 H \cos F_1 H, \quad D_1 = \cosh E_1 H \sin F_1 H,$$

$$C_2 = \sinh E_2 H \cos F_2 H, \quad D_2 = \cosh E_2 H \sin F_2 H,$$

$$E_1 = (a_1^2 + b_1^2)^{\frac{1}{4}} \cos\left(\frac{\theta_1}{2}\right), \quad F_1 = (a_1^2 + b_1^2)^{\frac{1}{4}} \sin\left(\frac{\theta_1}{2}\right),$$

$$E_2 = (a_2^2 + b_2^2)^{\frac{1}{4}} \cos\left(\frac{\theta_2}{2}\right), \quad F_2 = (a_2^2 + b_2^2)^{\frac{1}{4}} \sin\left(\frac{\theta_2}{2}\right),$$

$$a_1 = \xi^2 - \frac{\rho p^2(\lambda + 2\mu)}{(\lambda + 2\mu)^2 + p^2(\lambda' + 2\mu')^2},$$

$$a_2 = \xi^2 - \frac{\rho p^2 \mu}{\mu^2 + p^2 \mu'^2},$$

$$b_1 = \frac{\rho p^3(\lambda' + 2\mu')}{(\lambda + 2\mu)^2 + p^2(\lambda' + 2\mu')^2}, \quad b_2 = \frac{\rho p^3 \mu'}{\mu^2 + p^2 \mu'^2},$$

$$\theta_1 = \tan^{-1}\left(\frac{b_1}{a_1}\right), \quad \theta_2 = \tan^{-1}\left(\frac{b_2}{a_2}\right).$$

$$R_1 = P_1 A_1 - Q_1 B_1 - A_2(P_1^2 - Q_1^2) + 2B_2 P_1 Q_1,$$

$$R_2 = Q_1 A_1 + P_1 B_1 - B_2(P_1^2 - Q_1^2) - 2A_2 P_1 Q_1,$$

$$R_3 = P_1 A_1 - Q_1 B_1 - A_2, R_4 = Q_1 A_1 + P_1 B_1 - B_2,$$

$$R_5 = P_2 C_1 - Q_2 D_1 - C_2, R_6 = P_2 D_1 + Q_2 C_1 - D_2,$$

$$R_7 = P_2 C_1 - Q_2 D_1 - C_2(P_2^2 - Q_2^2) + 2P_2 Q_2 C_2,$$

$$R_8 = P_2 D_1 + Q_2 C_1 - D_2(P_2^2 - Q_2^2) - 2P_2 Q_2 C_2,$$

$$P_1 = \frac{\xi^2 + E_2^2 - F_2^2}{\xi^2},$$

$$P_2 = \left\{ \frac{2(E_1 E_2 - F_1 F_2)(\xi^2 + E_2^2 - F_2^2) + 4E_2 F_2(E_2 F_1 + E_1 F_2)}{(\xi^2 + E_2^2 - F_2^2)^2 + 4E_2^2 F_2^2} \right\},$$

$$Q_1 = \frac{E_2 F_2}{\xi^2},$$

$$Q_2 = \left\{ \frac{4E_2 F_2(E_1 E_2 - F_1 F_2) + 2(E_2 F_1 + E_1 F_2)(\xi^2 + E_2^2 - F_2^2)}{(\xi^2 + E_2^2 - F_2^2)^2 + 4E_2^2 F_2^2} \right\}.$$

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