On linear waveguides of square and triangular lattice strips: an application of Chebyshev polynomials

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Abstract. An analysis of the linear waves in infinitely-long square and triangular lattice strips of identical particles with nearest neighbour interactions for all combinations of fixed and free boundary conditions, as well as the periodic boundary, is presented. Expressions for the dispersion relations and the associated normal modes in these waveguides are provided in the paper; some of which are expressed implicitly in terms of certain linear combinations of the Chebyshev polynomials. The effect of next-nearest-neighbour interaction is also included for the square lattice waveguides. It is found that localized propagating waves, so called surface wave modes, occur in the triangular lattice strips, as well as square lattice strips with next-nearest-neighbour interactions, when either or both boundaries are free. In this paper, the even and odd modes are also discussed separately, wherever applicable. Graphical illustrations of the dispersion curves are included for all waveguides. The discrete waveguides analysed in the paper have broad applications in physics and engineering, including their merit in classical problems in elasticity, acoustics and electromagnetism, as well as recent technological issues involving various transport phenomena in quasi-one-dimensional nano-structures.

Keywords. Dispersion; normal modes; square; triangular; Chebyshev; tridiagonal.

1. Introduction

In classical elastic waveguides [1, 2] such as plates, rods, cylindrical shells, layered solids, etc., the ‘neighbouring’ parallel boundaries ‘guide’ the waves [3–5]. In fact, the research works on elastic waveguides can be traced to 1876 [6, 7], slightly before the well-known contributions due to Rayleigh [8, 9] and Lamb [10, 11]. Pochhammer transformed the three-dimensional equation of motion of a solid circular rod into cylindrical coordinates, and for (traction-) free surfaces, derived the dispersion relation [12] for each mode (see also the related works of [13–17]). The technological innovations and literature reviews involving the electromagnetic counterparts, which had a greater impact in communication technology, are historically more numerous [18, 19] (see also, for example, [20, 21], and [22, Chapter 10]). These analyses play a pivotal role in, the so called, mode-matching techniques for wave propagation in structured waveguides such as bifurcated waveguides, waveguides with discontinuities, etc. [23–27]. Indeed, any arbitrary time periodic motion of a given waveguide can be expressed in terms of the normal modes. In contrast with the continuous waveguides, the discrete waveguides allow an interpretation in terms of infinitely long mutually coupled chains of particles; this leads to a quasi-one-dimensional point of view. The wave propagation analysis for continuous

and discrete waveguides also finds many physical applications, such as pure crystals dynamics (simple, complex, etc.) in harmonic approximation [12, 28, 29] and also in the presence of impurities [30, 31], vibrations of macromolecules appearing in biological [32], chemical applications [33], etc. The technological interests, some of them quite recent, involving phononic [34, 35], photonic [35–37], electronic [38–41], phonon [35, 37, 42–45], mechanical [46–48] and magnetic-spin related transport [49–51], in nanostructures cannot be excluded as well; these apparently diverse applications involve the governing equations, which have similar mathematical structure, sometimes identical, to those analysed in this document.

The paper presents an analysis of simplified models of lattice waveguides with two types of structure: *square* and *triangular*, assuming nearest-neighbour interactions and identical nature of particles [52–54], from the point of view of *linear* propagating waves. A schematic illustration of the discrete structure in ‘classical’ waveguides is shown in figure 1. Both types of lattice models also occur in the research works on brittle fracture (see [55], and a large number of references provided in its bibliography), although the explicit nature of wave propagation, in square and triangular lattice waveguides, has not been analysed, so far, as part of these problems in dynamic fracture. Additionally, the present paper places the results, peculiar to

Figure 1. Illustration of lattice waveguide structure with upper–lower boundary: (a) fixed–fixed, (b) free–free and (c) fixed–free, along with a small snapshot of the mode shape across the waveguide ‘cross-section’ and (d) a ‘tube’ of square and triangular lattice waveguides undergoing a ‘small’ radial motion.

linear wave propagation, using the Chebyshev polynomials [56–58]. The latter are ubiquitous in numerical analysis, but they are also known to find applications in wave mechanics [59]. Several identities are available for the Chebyshev polynomials, owing to their historical importance and huge number of applications in science [58], some of which find applications in this paper. Due to a quasi-one-dimensional nature of the lattice waveguides, it can be anticipated that an important role is played by the structured matrices (tridiagonal, circulant, Toeplitz matrices, etc.) [60]. The paper presents an application of the analytical expression of the eigenvalues and eigenvectors of tridiagonal matrices that have two or four of their corners perturbed [61] (also see [60, 62, 63]). As an alternative method, which is in fact needed for triangular lattice waveguide with free boundary conditions, the manipulations elaborated in [55] and utilized in [64, 65] (and [66]) also find an application in the same context. The paper considers all combinations of fixed and free boundary conditions for the lattice waveguides, in addition to the periodic boundary condition (see figure 1). For each case of identical upper and lower boundaries, the even and odd modes are also discussed. The dispersion relations and normal modes for the propagating waves are derived in detail for each waveguide. The effect of a larger number of interactions is also discussed but to a limited extent. The square lattice waveguide is chosen for an illustration of the influence of the nearest-neighbour interactions. Incidentally, the analysis carries over rather easily as an analogue of that for the triangular lattice. Indeed, higher order of neighbour interactions in the lattice models leads to several difficulties from the analytical point of view [12, 54]. Besides this, the Chebyshev polynomials are anticipated to lose their relevance in this generalization due to the presence of banded matrices (with more than three ‘diagonals’, in place of the tridiagonal matrices, which are relatively well understood [60, 63, 67–69]); the corresponding study is relegated to another forum.

As one of the main results in this paper, it is found that the triangular lattice waveguides, as well as square

lattice strips with next-nearest-neighbour interactions, allow the presence of localized modes, also called *surface modes* [70–73]. The identification of the actual locations of surface wave modes has potential utility in several fields [74–76], as well as myriad physical applications in parallel, as mentioned earlier. It is found that surface (localized) wave modes exist for the triangular lattice¹ (even with the simplest form of interactions studied in this paper), as well as square lattice strips with next-nearest-neighbour interactions, whenever at least one of the boundary rows is free. Such surface wave modes do not exist for the simplified model of square lattice with only nearest-neighbour interactions, which is well known and expected due to its separable nature and similarity with the one dimensional lattice model having only nearest-neighbour interactions [12, 54]. The reader may note that all dispersion relations and normal modes listed in this paper have been verified against a numerical solution of the discrete Helmholtz equation, in context, with given lattice waveguide boundary conditions.

1.1 Notation

Let $\mathbb{Z}$ denote the set of integers, $\mathbb{Z}^2$ denote $\mathbb{Z} \times \mathbb{Z}$. Let $\mathbb{Z}_m^n$ denote the set of integers $\{m, m+1, \dots, n-1, n\}$. Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{C}$ denote the set of complex numbers. The symbol ξ is exclusively used throughout as a real variable that denotes a real wave number in the waveguide (horizontal) direction. Similarly, η denotes the equivalent wave number along the vertical direction, which may be real or complex depending on the

¹ The triangular lattice, according to the usage in this paper, is also known as hexagonal lattice in certain fields in engineering and sciences. Both lattice structures also appear in finite-difference-based numerical analysis [77], from the point of view of discretization of the traditional Laplacian.

context. Latin letters C_1, C_2 , etc., denote arbitrary constants in expressions. The notation for relevant physical entities is described in the main text.

2. Mechanical lattice model

Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be two orthonormal unit vectors in the two-dimensional Euclidean space $\mathbb{R}^2$. Suppose ρ is the three-dimensional mass density of an elastic material, μ is the linear elastic shear modulus and c_s is the macroscopic shear wave speed $\sqrt{\mu/\rho}$ [78]. Let a finite strip S_N in $\mathbb{R}^2$ be constructed of such elastic material. A lattice strip is defined as a subset of the infinite lattice of particles embedded in S_N . Assume there are N number of rows each containing infinite number of particles, arranged in the form of one-dimensional lattices. Both types of the lattice models, namely, square and triangular, contain identical particles, with each particle having mass M . In case of the first two models studied in this paper, it is assumed that only the in-plane nearest neighbours, separated by an equilibrium length b , are connected to each other by linearly elastic identical (massless) bonds to account for the relative displacement along anti-plane direction. The bonds have a spring constant K . In case of the third model, it is a square lattice with (linear) next-nearest-neighbour interactions having a spring constant γK ; other details of this model are provided later in the paper due to their resemblance with those for the first two, simpler, models. In addition to the natural interpretation of the lattice strips as two-dimensional layers undergoing out-of-plane motion, a three-dimensional interpretation of the same models is also possible; remarks of this order appear in [64, 65] for the first two models and are briefly included later as well.

2.1 Square lattice waveguide $\mathfrak{S}_N$

Let $b_1 = b_2 = b$ and $\mathfrak{S}_N = \{\sum_{i=1,2} v_i b_i \mathbf{e}_i \mid v_1 \in \mathbb{Z}, v_2 \in \mathbb{Z}_1^N\}$ denote a two-dimensional square lattice strip. The anti-plane displacement of a particle, indexed by its *lattice coordinates* $(x, y) \in \mathbb{Z}^2$, is denoted by $u_{x,y} \in \mathbb{C}$. Let $\mathbb{Z}_N^2$ denote the lattice coordinates of the particles in $\mathfrak{S}_N$, which have a degree of freedom to move in the anti-plane direction. The equation of motion of that portion of lattice waveguide $\mathfrak{S}_N$ which is away from the upper and lower boundary rows, while excluding the explicit dependence of u on time t , is

$$M \frac{d^2}{dt^2} u_{x,y} = K(u_{x+1,y} + u_{x-1,y} + u_{x,y+1} + u_{x,y-1} - 4u_{x,y}), \quad (x, y) \in \mathbb{Z}_N^2. \quad (1)$$

Motivated by a three-dimensional context, let $M = \rho b^3$, $K = \mu b$, $t = tc_s$. In the context of $\mathfrak{S}_N$, this implies that

Table 1. Variants of $\mathfrak{S}_N$.

S. no.	Strip	Remark
1	$\mathfrak{S}_N^\bullet$	Even/odd modes
2	$\mathfrak{S}_N^\circ$	Even/odd modes
3	$\mathfrak{S}_N^{\bullet\circ}$	Asymmetric
4	$\mathfrak{S}_N^\odot$	–

the particles have a unit mass, the in-plane equilibrium spacing between the nearest neighbours is b and the nearest-neighbour interactions (for the relative displacement in anti-plane direction) have spring constant $1/b^2$. Evidently, the effect of boundary conditions on the force constants as well as lattice spacing is ignored in this simplified model. The square lattice dispersion relation [52, 64] is $\omega^2 = \frac{1}{b^2} \Omega_{\mathfrak{S}}^2(\xi, \eta)$, where

$$\Omega_{\mathfrak{S}}^2(\xi, \eta) = 4 \sin^2 \frac{1}{2} \xi + 4 \sin^2 \frac{1}{2} \eta, \quad (\xi, \eta) \in [-\pi, \pi]^2. \quad (2)$$

The representative variants of the square lattice waveguides along with boundary conditions are listed in table 1 (see also figure 2 and Remark 1). Using the lattice model formulation, the lattice waveguides of $\mathfrak{S}$ are decorated with the symbols $\bullet$ and $\circ$ as superscript (respectively subscript) to indicate the fixed and free upper (respectively lower) boundary condition, respectively, and $\odot$ to denote the periodic boundary condition.

2.2 Triangular lattice waveguide $\mathfrak{T}_N$

Let $b_1 = \frac{1}{2}b$, $b_2 = \frac{\sqrt{3}}{2}b$ and $\mathfrak{T}_N = \{\sum_{i=1,2} v_i b_i \mathbf{e}_i \mid v_1 \in 2\mathbb{Z} + \text{mod}(|v_2|, 2), v_2 \in \mathbb{Z}_1^N\}$ be a two-dimensional strip of a triangular lattice. Similar to the application by Sharma [65], a rectangular coordinate system is used here to allow certain convenience of labelling the slant arrangement of particles (alternative labels are used by Kantorovich and Krylov [77], Bilbao [79], Collatz [80] and others as examples). A replica of $\mathfrak{T}_N$, called a ‘replicated’ triangular (sub-)lattice strip and denoted by $\mathfrak{T}_N^R$, is juxtaposed with $\mathfrak{T}_N$, after imparting it a horizontal translation of magnitude b_1 . The union of both lattice strips is a rectangular lattice strip, denoted by $\mathfrak{R}_N$, with a period b_1 horizontally and b_2 vertically. Let the displacement of a particle in $\mathfrak{R}_N$, indexed by its *lattice coordinates* $(x, y) \in \mathbb{Z}^2$, be denoted by $u_{x,y} \in \mathbb{C}$. Let $\mathbb{Z}_N^2$ denote the lattice coordinates of the particles in $\mathfrak{T}_N$, which have a degree of freedom to move in the anti-plane direction. The equation of motion of the lattice, while suppressing the explicit dependence of u on time t , is

Figure 2. Illustration of (left) square and (right) triangular lattice waveguides, $\mathfrak{S}_N$ and $\mathfrak{T}_N$, respectively, with four different types of boundary. $\mathfrak{T}_N$ and $\mathfrak{T}_N^R$ are shown together. $\mathfrak{T}_N$ is also shown on the right.

$$M \frac{d^2}{dt^2} u_{x,y} = K(u_{x+1,y+1} + u_{x+1,y-1} + u_{x-1,y+1} + u_{x-1,y-1} + u_{x+2,y} + u_{x-2,y} - 6u_{x,y}), (x,y) \in \mathbb{Z}_N^2. \quad (3)$$

Motivated by a three-dimensional context, let $M = \rho b^3$, $K = \frac{2}{3}\mu b$ and $t = tc_s$. This implies that the particles have a unit mass, the equilibrium in-plane spacing between the nearest neighbours is b and the nearest neighbour (anti-plane) interactions have spring constant $2/(3b^2)$. The triangular lattice dispersion relation [52, 65, 81] is $\omega^2 = \frac{1}{b^2} \Omega_{\mathfrak{T}}^2(\xi, \eta)$, where

$$\Omega_{\mathfrak{T}}^2(\xi, \eta) = \frac{2}{3} \left(4 \sin^2 \xi + 4 \sin^2 \frac{1}{2}(\xi + \eta) + 4 \sin^2 \frac{1}{2}(\xi - \eta) \right), \quad (\xi, \eta) \in [-\pi, \pi]^2. \quad (4)$$

Table 2. Variants of $\mathfrak{T}_N$.

S. no.	Strip	Remark
1	$\mathfrak{T}_{\bullet}$	Even/odd modes
2	$\mathfrak{T}_{\circ}$	Even/odd modes
3	$\mathfrak{T}_{\circ}^{\bullet}$	Asymmetric
4	$\mathfrak{T}_{\odot}$	N is even

The representative variants of the triangular lattice waveguides along with boundary conditions are listed in table 2 (see also figure 2 and Remark 1). Using the lattice model formulation, the lattice waveguides of $\mathfrak{T}$ are decorated with the symbols $\bullet$ and $\circ$ as superscript (respectively, subscript) to indicate the fixed and free upper (respectively, lower) boundary condition, respectively, and $\odot$ to denote the periodic boundary condition.

Remark 1 Excluding the special cases of periodic boundary $\mathfrak{S}\odot$ and $\mathfrak{T}\odot$, it is easy to see that there is one combinatorially different variant that is absent (because it is related by reflection symmetry) in tables 1 and 2 for square and triangular lattice waveguides, respectively, out of the total four such possible variants.

Remark 2 u is dimensionless displacement, x is dimensionless horizontal position, y is dimensionless vertical position and t is the dimensionless time. For both lattice models, in the continuum limit, using appropriate continuous coordinates x, y, t , the usual wave equation can be obtained as $b/L \rightarrow 0$ (with reference to a macroscopic length scale L). The details are also available in standard texts in numerical analysis, for example, [77] (see also [64, 65]).

In the following three sections, several waveguides with three different structures are analyzed along with a provision of the detailed derivations of the dispersion relations and corresponding wave modes. First the square lattice waveguides are considered with nearest neighbor interactions only, followed by triangular lattice waveguides of the similar type, and the fourth section details the square lattice waveguides incorporating the next-nearest neighbor interactions as well. It is assumed throughout the paper that the time dependence of the propagating wave modes is described by the term $e^{-i\omega t}$ and the space dependence by $e^{i\mathbf{k}\cdot\mathbf{x}}$; moreover the former factor is suppressed. For convenience of notation (refer (2), (4)), a scaled frequency ω is defined by $\omega := b\omega$.

3. Waves in $\mathfrak{S}_N$

The equation for a particle in $\mathfrak{S}_N$ located at lattice site (x, y) , such that it does not belong to either of the strip boundary rows, i.e., $y \notin \{1, N\}$ but $y \in \mathbb{Z}_2^N - 1$, is

$$-\omega^2 u_{x,y} = \sum_{i \in \{-1, 1\}} (u_{x+i,y} + u_{x,y+i}) - 4u_{x,y}, \quad \forall x \in \mathbb{Z}. \quad (5)$$

Corresponding to the lattice waveguides listed in table 1, the boundary conditions at upper and lower rows studied in this paper $\forall x \in \mathbb{Z}$, are as follows: at $y = N$

$$-\omega^2 u_{x,y} = u_{x+1,y} + u_{x-1,y} + u_{x,y-1} - 4u_{x,y} \quad (\bullet) \quad (6a)$$

$$-\omega^2 u_{x,y} = u_{x+1,y} + u_{x-1,y} + u_{x,y-1} - 3u_{x,y} \quad (\circ) \quad (6b)$$

$$-\omega^2 u_{x,y} = u_{x+1,y} + u_{x-1,y} + u_{x,y-1} + u_{x,1} - 4u_{x,y} \quad (\odot), \quad (6c)$$

and at $y = 1$

$$-\omega^2 u_{x,y} = u_{x+1,y} + u_{x-1,y} + u_{x,y+1} - 4u_{x,y} \quad (\bullet) \quad (7a)$$

$$-\omega^2 u_{x,y} = u_{x+1,y} + u_{x-1,y} + u_{x,y+1} - 3u_{x,y} \quad (\circ) \quad (7b)$$

$$-\omega^2 u_{x,y} = u_{x+1,y} + u_{x-1,y} + u_{x,y+1} + u_{x,N} - 4u_{x,y} \quad (\odot). \quad (7c)$$

Remark 3 The expressions related to linear waves that have been derived for the square lattice waveguides are well known, though they are presented below in the language of the Chebyshev polynomials. A slightly generalized model with next-nearest-neighbour interactions is analysed later (section 5).

In the following, the four types of boundary conditions are considered for the lattice waveguides of square lattice structure (with nearest-neighbour interactions), namely, those with fixed boundary $\mathfrak{S}_\bullet^\bullet$, free boundary $\mathfrak{S}_\circ^\circ$, fixed-free boundary $\mathfrak{S}_\bullet^\circ$ and the periodic boundary $\mathfrak{S}\odot$ (see the left of figure 2 for illustrations).

3.1 Fixed boundary $\mathfrak{S}_\bullet^\bullet$

For the upper and lower boundary rows the conditions (6a) and (7a) hold, respectively. With the assumed space (x) dependence, i.e. $u_{x,y} = a_y e^{i\mathbf{k}_x \cdot \mathbf{x}}$, it follows that

$$-\omega^2 a_y = (1 - \delta_{y,N})a_{y+1} + (1 - \delta_{y,1})a_{y-1} + 2 \cos \xi a_y - 4a_y, \quad y \in \mathbb{Z}_1^N. \quad (8)$$

Let $\mathbf{A}_N \in \mathbb{C}^{N \times N}$, a matrix with four perturbed corners [63], be defined by

$$\mathbf{A}_N = \begin{bmatrix} \chi & \Upsilon & 0 & \cdots & 0 \\ \Upsilon & \chi & \Upsilon & 0 & \cdots \\ 0 & \Upsilon & \ddots & \ddots & 0 \\ \cdots & & \cdots & \chi & \Upsilon \\ 0 & \cdots & 0 & \Upsilon & \chi \end{bmatrix} + \begin{bmatrix} \delta_{NW} & 0 \cdots 0 & \delta_{NE} \\ 0 & 0 \cdots 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 \cdots 0 & 0 \\ \delta_{SW} & 0 \cdots 0 & \delta_{SE} \end{bmatrix}. \quad (9)$$

It is easy to see that u is a propagating wave in $\mathfrak{S}_\bullet^\bullet$, with $\mathbf{A}_N$ of the form (9), if $a \in \mathbb{C}^N$ satisfies

$$(\mathbf{A}_N - \Lambda \mathbf{I}_N)a = 0, \quad a \in \mathbb{C}^N, \text{ where} \quad (10)$$

$$\Lambda = \omega^2, \chi(\xi) = 4 - 2 \cos \xi, \Upsilon = -1, \quad (11)$$

and $\delta_{NE} = \delta_{SW} = 0, \delta_{NW} = \delta_{SE} = 0$. According to [61], the eigenvalue problem (10) is solved by the eigenpairs

Figure 3. Dispersion relation $\Omega_\kappa, \kappa \in \mathbb{Z}_1^N$, with $N = 6$ for $\mathfrak{S}_1$: (a) $\mathfrak{S}_1^\bullet$, (b) $\mathfrak{S}_1^\circ$, (c) $\mathfrak{S}_1^\circ$ and (d) $\mathfrak{S}_1^\circ$. The pass band envelope of the infinite lattice is illustrated by grey-coloured shaded region.

described therein; in particular, each eigenvalue Λ is described by the general form

$$\Lambda = \chi + 2\Upsilon \cos \eta_\kappa, \quad \kappa = 1, \dots, N, \quad (12)$$

where η_κ is determined by a condition depending on $\mathbf{A}_N$, given by (3.10) of the same paper, i.e.

$$\begin{aligned} &\Upsilon^2 \sin(N+1)\eta_\kappa + (\delta_{NW}\delta_{SE} - \delta_{NE}\delta_{SW}) \sin(N-1)\eta_\kappa \\ &- \Upsilon(\delta_{NW} + \delta_{SE}) \sin N\eta_\kappa - \Upsilon(\delta_{NE} + \delta_{SW}) \sin \eta_\kappa = 0. \end{aligned} \quad (13)$$

Thus, in case of $\mathfrak{S}_1^\bullet$, the dispersion relations and normal modes are given by

$$\omega^2 = \Omega_\kappa^2(\xi) = 4 \sin^2 \frac{1}{2} \eta_\kappa + 4 \sin^2 \frac{1}{2} \xi = \Omega_\xi^2(\xi, \eta_\kappa), \quad (14a)$$

$$a_{(\kappa)v} = \sqrt{\frac{2}{N+1}} \sin v\eta_\kappa, \quad v \in \mathbb{Z}_1^N, \quad \eta_\kappa = \frac{\kappa\pi}{N+1}, \quad \kappa \in \mathbb{Z}_1^N. \quad (14b)$$

Remark 4 Since $\mathbf{A}_N$ is a Hermitian matrix, the vectors $\{a_{(\kappa)\cdot}\}_{\kappa \in \mathbb{Z}_1^N}$ form an orthonormal basis for $\mathbb{C}^N$ (with standard inner product).

Since the lattice rows adjacent to $\mathfrak{S}_1^\bullet$ are fixed, evidently $\sin 0\eta_\kappa = 0$ and $\sin(N+1)\eta_\kappa = 0$. In terms of the Chebyshev polynomials [56], the condition (13) can be expressed as

$$U_N(\vartheta) = 0 \quad \text{with } \vartheta = \cos \eta_\kappa. \quad (15)$$

Hence, using the well-known roots of the Chebyshev polynomials [58], it is found that $\vartheta = \cos \kappa\pi/(N+1)$, $\kappa \in \mathbb{Z}_1^N$, yielding (14b)₂. The expression (15) has been plotted and shown in figure 3a.

Further, it is easy to see that the modes with ‘odd’ symmetry, about the horizontal-centreline of $\mathfrak{S}_1^\bullet$, correspond to even values of κ , while the even modes correspond to odd values of κ . For instance, when $N = 2N$ and odd modes are considered, then using $\sin v\eta_\kappa = -\sin(2N+1-v)\eta_\kappa = (-1)^v \sin v\eta_\kappa$, it follows that κ is an even integer. Similar argument holds for even modes. There is also an alternative method to arrive at the same results (which is useful for more complicated lattices) and this is described in the following paragraph.

Suppose $N = 2N+1$ and consider the case of even modes. Then it is required that $a_v = a_{2N+2-v}$. The general solution [55, 64, 82] for (reduced) normal modes is given by

$$a_{N+v} = C_1 \lambda^v + C_2 \lambda^{-v}, \quad v \in \mathbb{Z}_1^{N+1}, \quad (16)$$

where, using (8), it is required that λ is a solution of

$$\lambda + \lambda^{-1} = 4 - 2 \cos \xi - \omega^2. \quad (17)$$

For convenience of obtaining the solution of this equation, it is assumed that $\Im\omega > 0$ (allowing the limit $\omega \rightarrow 0^+$ whenever required). Let

$$\mathcal{Q}(\xi) := 4 - 2\cos\xi - \omega^2, \iota^2 = \mathcal{Q} + 2, \ell^2 = \mathcal{Q} - 2, \quad (18)$$

and λ be defined by

$$\lambda = \frac{\iota - \ell}{\iota + \ell}, \quad (19)$$

and the branch cuts are chosen such that $|\lambda| < 1$ on the complex plane outside the branch cuts (see [55, 64] for details, for example). Using the definition of Chebyshev polynomial of second kind [58], it follows that

$$\lambda^{-n} - \lambda^n = (\lambda^{-1} - \lambda)U_{n-1}\left(\frac{1}{2}(\lambda + \lambda^{-1})\right), \quad 0 < n \in \mathbb{Z}. \quad (20)$$

Using (17) and (18), consider the definition

$$\vartheta := \vartheta(\xi) = \frac{1}{2}\mathcal{Q}(\xi), \xi \in [-\pi, \pi]. \quad (21)$$

Remark 5 Throughout the paper, the argument of Chebyshev polynomials ϑ is defined by (21). Further, let $\lambda = e^{i\eta}$, so that $\mathcal{Q} = 2\cos\eta$ by (17); also $\vartheta = \cos\eta$. It is easy to see that $\lambda^{-1} - \lambda = \iota\ell$. See [55] for many other relations and their applications.

Due to the fixed boundary condition at upper row, using (16) it is required that $C_1\lambda^{N+2} + C_2\lambda^{-N-2} = 0$. Taking into account the even symmetry, $C_1\lambda^2 + C_2\lambda^{-2} = C_1 + C_2$. For a non-trivial solution C_1, C_2 of the two (linear algebraic) equations, it is required that $(1 - \lambda^{-2})\lambda^{N+2} - (1 - \lambda^2)\lambda^{-N-2} = 0$. Using the Chebyshev polynomials U_n of second kind (20), it is found that

$$U_{N+1}(\vartheta) - U_{N-1}(\vartheta) = 0. \quad (22a)$$

Using the Chebyshev polynomials T_n of first kind, (22a) can be re-written as $T_{N+1}(\vartheta) = 0$, where $\vartheta = \cos\eta_\kappa$. Hence, using the well-known roots of the Chebyshev polynomials [58], $\cos\eta_\kappa = \cos(\kappa - \frac{1}{2})\pi/(N+1)$, $\kappa \in \mathbb{Z}_1^{N+1}$, same as those stated earlier (14b) for odd κ . Similarly, when $N = 2N$, for the even modes, $C_1\lambda^{N+1} + C_2\lambda^{-N-1} = 0$, and $C_1\lambda + C_2\lambda^{-1} = C_1 + C_2$. For a non-trivial solution C_1, C_2 of the two (linear algebraic) equations, it is required that $(1 - \lambda^{-1})\lambda^{N+1} - (1 - \lambda)\lambda^{-N-1} = 0$. Using the Chebyshev polynomials U_n of second kind (after the application of (20)), it is found that

$$U_N(\vartheta) - U_{N-1}(\vartheta) = 0. \quad (22b)$$

Using the Chebyshev polynomials V_n of third kind, this can be re-written as $V_N(\vartheta) = 0$, where $\vartheta = \cos\eta_\kappa$, which implies [58] that $\cos\eta_\kappa = \cos(\kappa - \frac{1}{2})\pi/(N + \frac{1}{2})$, $\kappa \in \mathbb{Z}_1^N$, same as those stated earlier (14b) for odd κ .

For the odd modes in case of $N = 2N + 1$, it is found that

$$U_N(\vartheta) = 0, \quad (23a)$$

which implies [58] that $\cos\eta_\kappa = \cos\kappa\pi/(N+1)$, $\kappa \in \mathbb{Z}_1^N$, same as those stated earlier (14b) for even κ . When $N = 2N$ and the odd modes are considered, then analogous to (22a), the condition can be found as

$$U_N(\vartheta) + U_{N-1}(\vartheta) = 0, \quad (23b)$$

where ϑ is given by (21). Using the Chebyshev polynomials W_n of fourth kind, this can be re-written as $W_N(\vartheta) = 0$, $\vartheta = \cos\eta_\kappa$ yielding $\cos\eta_\kappa = \cos\kappa\pi/(N + \frac{1}{2})$, $\kappa \in \mathbb{Z}_1^N$, same as those stated earlier in (14b) for even values of κ .

3.2 Free boundary $\mathfrak{S}_\circ^\circ$

For $\mathfrak{S}_\circ^\circ$, i.e., $\mathfrak{S}_N$ with a free upper and lower boundary, the equations (6b) and (7b) hold, respectively. Again, based on an equation analogous to (8), it is found that (10) holds for the normal modes, due to the same reasons as cited for $\mathfrak{S}_\bullet^\circ$, where $\mathbf{A}_N$ has the form (9) with (11) and $\delta_{NE} = \delta_{SW} = 0$, $\delta_{NW} = \delta_{SE} = -1$. Thus, in case the boundary rows of $\mathfrak{S}_N$ are free, the dispersion relations (14a) are obtained with the corresponding normal modes and η_κ given by

$$a_{(\kappa)v} = \sqrt{\frac{2}{N}} \begin{cases} \cos\left(v - \frac{1}{2}\right)\eta_\kappa, & \kappa \neq 1 \\ \frac{1}{\sqrt{2}}, & \kappa = 1 \end{cases}, v \in \mathbb{Z}_1^N, \quad \eta_\kappa = \frac{(\kappa-1)\pi}{N}, \kappa \in \mathbb{Z}_1^N. \quad (24)$$

Indeed, using (13), $\sin(N+1)\eta_\kappa + \sin(N-1)\eta_\kappa - 2\sin N\eta_\kappa = 0$, i.e., in terms of the Chebyshev polynomials of second kind (20),

$$2(\vartheta - 1)U_{N-1}(\vartheta) = 0, \quad (25)$$

which implies $\vartheta = \cos(\kappa-1)\pi/N$, $\kappa \in \mathbb{Z}_1^N$, the same as stated in (24)₂. The expression (25) has been plotted and shown in figure 3b.

The odd modes correspond to even values of κ , while the even modes correspond to odd values of κ . For instance, when $N = 2N$ and odd modes are considered, then (since $\kappa \neq 1$) using $\cos(v - \frac{1}{2})\eta_\kappa = -\cos(2N+1 - v - \frac{1}{2})\eta_\kappa = (-1)^\kappa \cos(v - \frac{1}{2})\eta_\kappa$, it is found that κ is even. Similar reasoning holds for even modes.

By an application of the second method, as described in section 3.1, the same expressions, as stated earlier, can be

also derived. For even modes in $\mathfrak{S}_0^\circ$ with $N = 2N + 1$, it is required that $a_v = a_{2N+2-v}$. Using (16), taking into account the symmetry and free boundary condition, for non-trivial C_1, C_2 , it is easily found that $(1 - \lambda^{-2})(\lambda^{N+2} - \lambda^{N+1}) - (1 - \lambda^2)(\lambda^{-N-2} - \lambda^{-N-1}) = 0$. Using the Chebyshev polynomials U_n of second kind (after an application of (20))

$$U_{N+1}(\vartheta) - U_{N-1}(\vartheta) - (U_N(\vartheta) - U_{N-2}(\vartheta)) = 0, \quad (26a)$$

where ϑ is given by (21). Using (102a)₁ and the Chebyshev polynomials W_n of fourth kind, (26a) is expressed as $2(\vartheta - 1)W_N(\vartheta) = 0$, with $\vartheta = \cos \eta_\kappa$. Hence, it is found that $\eta_\kappa = \kappa\pi/(N + \frac{1}{2})$, $\kappa \in \mathbb{Z}_0^N$, and η_κ is an even multiple of π/N , i.e., the same values of η_κ as earlier above in (24) for odd values of κ . Similarly, for $N = 2N$, using the general solution

$$a_{N+v} = C_1 \lambda^v + C_2 \lambda^{-v}, v \in \mathbb{Z}_1^N, \quad (26b)$$

for the even modes, it is found that $(1 - \lambda^{-1})(\lambda^{N+1} - \lambda^N) - (1 - \lambda)(\lambda^{-N-1} - \lambda^{-N}) = 0$, i.e., $2U_{N-1}(\vartheta) - U_N(\vartheta) - U_{N-2}(\vartheta) = 0$, which can be written as

$$2(\vartheta - 1)U_{N-1}(\vartheta) = 0. \quad (26c)$$

This implies that $\eta_\kappa = \kappa\pi/N$, $\kappa \in \mathbb{Z}_0^N$. Hence, η_κ is an even multiple of π/N , i.e. the same values of η_κ as stated earlier in (24) for odd values of κ . When $N = 2N$, for the odd modes, analogous to (26a),

$$U_N(\vartheta) - U_{N-2}(\vartheta) = 0, \quad (27a)$$

where ϑ is given by (21). Using the Chebyshev polynomials T_n of first kind, this can be re-written as $T_N(\vartheta) = 0$, $\vartheta = \cos \eta_\kappa$, yielding $\cos \eta_\kappa = \cos(\kappa - \frac{1}{2})\pi/N$, $\kappa \in \mathbb{Z}_1^N$, the same as those stated above in (24) for even κ . For the odd modes in case of $N = 2N + 1$, taking into account the symmetry and free boundary condition, it is easily found that (20) $(\lambda^{N+1} - \lambda^N) - (\lambda^{-N-1} - \lambda^{-N}) = 0$, i.e.

$$U_N(\vartheta) - U_{N-1}(\vartheta) = 0. \quad (27b)$$

Using the Chebyshev polynomials V_n of third kind, $V_N(\vartheta) = 0$, $\vartheta = \cos \eta_\kappa$, which implies that $\cos \eta_\kappa = \cos(\kappa - \frac{1}{2})\pi/(N + \frac{1}{2})$, $\kappa \in \mathbb{Z}_1^N$. Hence, η_κ is an odd multiple of π/N , i.e., the same values of η_κ as stated earlier in (24) for even values of κ .

3.3 Fixed-free boundary $\mathfrak{S}_\bullet^\circ$

Consider the waveguide $\mathfrak{S}_\bullet^\circ$, as shown schematically in the left side of figure 2. The equation of motion for the lattice row at $y = 1$ is (6a), corresponding to a fixed boundary, while at $y = N$ it is (7b), corresponding to a free boundary. Hence, based on an equation analogous to (8), it is found

that (10) holds for the normal modes, where $\mathbf{A}_N$ has the form (9) with (11) and $\delta_{NE} = \delta_{SW} = 0$, $\delta_{NW} = 0$, $\delta_{SE} = -1$. Thus, in the present case when the boundary rows are fixed-free, using (13), $\sin(N + 1)\eta_\kappa - \sin N\eta_\kappa = 0$, as a consequence of (20) this yields

$$U_N(\vartheta) - U_{N-1}(\vartheta) = 0. \quad (28)$$

Using the Chebyshev polynomials V_n of third kind, this implies that $V_N(\vartheta) = 0$, $\vartheta = \cos \eta_\kappa$, so that $\vartheta = \cos(\kappa - \frac{1}{2})\pi/(N + \frac{1}{2})$, $\kappa \in \mathbb{Z}_1^N$. Thus, the dispersion relation is (14a) and the corresponding normal modes are

$$a_{(\kappa)v} = \sqrt{\frac{4}{2N+1}} \cos\left(v - \frac{1}{2}\right)\eta_\kappa, v \in \mathbb{Z}_1^N, \quad (29)$$

$$\eta_\kappa = \frac{(2\kappa - 1)\pi}{2N + 1}, \kappa \in \mathbb{Z}_1^N.$$

In the free-fixed case, i.e. $\mathfrak{S}_\bullet^\circ$, it is easy to find that $a_{(\kappa)v} = \sqrt{4/(2N+1)} \sin v\eta_\kappa$, $v \in \mathbb{Z}_1^N$. The expression (28) has been plotted and shown in figure 3c. An application of the second method, as described in sections 3.1 and 3.2, yields the same expressions as obtained earlier.

3.4 Periodic boundary $\mathfrak{S}_\odot$

Due to the periodic upper and lower boundary rows, (6c) and (7c) hold. Based on an equation analogous to (8), it follows that (10) holds, where $\mathbf{A}_N$ has the form (9) with (11) and $\delta_{NE} = \delta_{SW} = \mathcal{T}$, $\delta_{NW} = \delta_{SE} = 0$; in particular, it is a circulant matrix for which the eigenpairs are straightforward to find [60]. The normal modes satisfy the dispersion relation (14a) with

$$a_{(\kappa)v} = \frac{1}{\sqrt{N}} e^{iv\eta_\kappa}, v \in \mathbb{Z}_1^N, \quad \eta_\kappa = \frac{2\kappa\pi}{N}, \kappa \in \mathbb{Z}_1^N. \quad (30)$$

An alternative derivation based on [61] is the following. From (13), the dispersion relation is given by $\omega^2 = \Omega_\kappa^2$ with (12) and η_κ such that $\mathcal{T}^2 \sin(N + 1)\eta_\kappa - \mathcal{T}^2 \sin(N - 1)\eta_\kappa - 2\mathcal{T}^2 \sin \eta_\kappa = 0$, i.e.

$$U_N(\vartheta) - U_{N-2}(\vartheta) - 2 = 0. \quad (31)$$

It is easy to see that (30)₂ describes the roots that satisfy this condition. By an application of the results of [61], the eigenvectors $a_{(\kappa)v}$ can be also expressed as linear combinations of $\sin v\eta_\kappa$ and $\cos v\eta_\kappa$ for $v \in \mathbb{Z}_1^N$, equivalent to (30). The expression (31) has been plotted and shown in figure 3d by marking the doubly repeated dispersion relations (except for the uppermost and lowermost). An application of the second method (i.e., using λ), as described in sections 3.1 and 3.2, yields the same expressions.

Remark 6 Note that there are no surface modes, i.e., localized propagating waves, in any of the square lattice waveguides listed in table 1. Also, the Remark 4 holds for the normal modes of all square lattice waveguides stated in table 1. See Appendix 2.1 for the proof of normalization that appears in (14b), (24) and (29).

The following section presents the analysis, for the entire collection of four types of boundary conditions, similar to the earlier one for the lattice waveguides with triangular structure.

4. Waves in $\mathfrak{T}_N$

The equation for particles in $\mathfrak{T}_N$, away from the boundary rows, is

$$-\frac{3}{2}\omega^2 u_{x,y} = u_{x+2,y} + u_{x-2,y} + u_{x-1,y+1} + u_{x-1,y-1} + u_{x+1,y-1} + u_{x-1,y-1} - 6u_{x,y}, \quad \forall x \in \mathbb{Z}, y \in \mathbb{Z}_2^{N-1}. \quad (32)$$

The different pairs of boundary conditions leading to the symbolic arrangements listed in table 2 involve the following: $\forall x \in \mathbb{Z}, y = N$

$$-\frac{3}{2}\omega^2 u_{x,y} = u_{x+2,y} + u_{x-2,y} + u_{x+1,y-1} + u_{x-1,y-1} - 6u_{x,y} \quad (\bullet) \quad (33a)$$

$$-\frac{3}{2}\omega^2 u_{x,y} = u_{x+2,y} + u_{x-2,y} + u_{x+1,y-1} + u_{x-1,y-1} - 4u_{x,y} \quad (\circ) \quad (33b)$$

$$-\frac{3}{2}\omega^2 u_{x,y} = u_{x+2,y} + u_{x-2,y} + u_{x+1,y-1} + u_{x-1,y-1} + u_{x+1,y-N+1} + u_{x-1,y-N+1} - 6u_{x,y} \quad (\odot), \quad (33c)$$

and $\forall x \in \mathbb{Z}, y = 1$

$$-\frac{3}{2}\omega^2 u_{x,y} = u_{x+2,y} + u_{x-2,y} + u_{x+1,y+1} + u_{x-1,y+1} - 6u_{x,y} \quad (\bullet), \quad (34a)$$

$$-\frac{3}{2}\omega^2 u_{x,y} = u_{x+2,y} + u_{x-2,y} + u_{x+1,y+1} + u_{x-1,y+1} - 4u_{x,y} \quad (\circ), \quad (34b)$$

$$-\frac{3}{2}\omega^2 u_{x,y} = u_{x+2,y} + u_{x-2,y} + u_{x+1,y+1} + u_{x-1,y+1} + u_{x+1,y+N-1} + u_{x-1,y+N-1} - 6u_{x,y} \quad (\odot). \quad (34c)$$

Remark 7 The expressions derived later for the triangular lattice waveguides have not all been studied earlier,

especially the case of waveguide with one or both boundary conditions free; see an application of the fixed boundary condition by Sharma [24].

In the following, the four types of boundary conditions, namely fixed boundary $\mathfrak{T}_\bullet^\bullet$, free boundary $\mathfrak{T}_\circ^\circ$, fixed-free boundary $\mathfrak{T}_\bullet^\circ$ and the periodic boundary $\mathfrak{T}_\odot^\odot$, are considered (see the right of figure 2 for illustrations).

4.1 Fixed boundary $\mathfrak{T}_\bullet^\bullet$

For this case, at $y = N$, (33a) holds while for $y = 1$, (34a) holds corresponding to either of the fixed boundaries of $\mathfrak{T}_\bullet^\bullet$. With the representation of a propagating wave in $\mathfrak{T}_\bullet^\bullet$ as the ansatz $u_{x,y} = a_y e^{i\kappa x}$, it is found that

$$-\frac{3}{2}\omega^2 a_y = (1 - \delta_{y,N})2 \cos \xi a_{y+1} + (1 - \delta_{y,1})2 \cos \xi a_{y-1} + 2 \cos 2\xi a_y - 6a_y, \quad y \in \mathbb{Z}_1^N. \quad (35)$$

Equivalently, $a \in \mathbb{C}^N$ satisfies (10) where $\mathbf{A}_N$ has the form (9) with

$$A = \frac{3}{2}\omega^2, \chi(\xi) = 6 - 2 \cos 2\xi, \mathcal{T}(\xi) = -2 \cos \xi, \quad (36)$$

and $\delta_{NE} = \delta_{SW} = 0, \delta_{NW} = \delta_{SE} = 0$. Applying (13), $\sin(N+1)\eta_\kappa = 0$ or $U_N(\vartheta) = 0, \vartheta = \cos \eta_\kappa$. Thus, in this case when the boundary rows are fixed (i.e., $a_0 = 0, a_{N+1} = 0$), for $\kappa \in \mathbb{Z}_1^N$, the dispersion relations and the corresponding eigemodes are given by

$$\frac{3}{2}\omega^2 = \frac{3}{2}\Omega_\kappa^2(\xi) = 6 - 2 \cos 2\xi - 4 \cos \xi \cos \eta_\kappa = \frac{3}{2}\Omega_\mathfrak{T}^2(\xi, \eta_\kappa), \quad (37)$$

$$a_{(\kappa)v} = \sqrt{\frac{2}{N+1}} \sin v\eta_\kappa, v \in \mathbb{Z}_1^N, \quad \eta_\kappa = \frac{\kappa\pi}{N+1}, \kappa \in \mathbb{Z}_1^N. \quad (38)$$

The odd modes correspond to even values of κ , while the even modes correspond to odd values of κ ; the detailed reasoning is the same as that for $\mathfrak{S}_\bullet^\bullet$ (section 3.1). The relations (37) have been graphically illustrated in figure 4a.

Remark 8 Let $N = 2N$. Notice that, with $\tilde{\xi} = \pi - \xi$,

$$\frac{3}{2}\Omega_\kappa^2(\tilde{\xi}) = 6 - 2 \cos 2\tilde{\xi} - 4 \cos \tilde{\xi} \cos \eta_\kappa = 6 - 2 \cos 2\xi - 4 \cos \xi \cos \frac{2N+1-\kappa}{2N+1}\pi,$$

and when κ is odd (respectively even), $2N+1-\kappa$ is even (respectively odd). This implies that the dispersion curves for the odd and even modes do not have symmetry about the point corresponding to $\xi = \pm \frac{1}{2}\pi$. However, with $N = 2N+1$, since κ odd (respectively even) is equivalent to $2N+2-\kappa$ odd (respectively even), the dispersion

Figure 4. Dispersion relation Ω_κ , $\kappa \in \mathbb{Z}_1^N$, with $N = 6$ for $\mathfrak{T}_N$: (a) $\mathfrak{T}_N^\bullet$, (b) $\mathfrak{T}_N^\circ$, (c) $\mathfrak{T}_N^\circ$ and (d) $\mathfrak{T}_N^\circ$. The dashed curves, connecting the black dots, correspond to the surface wave dispersion curves in (b) and (c). The pass band envelope of the infinite lattice is illustrated by grey-coloured shaded region.

curves for the odd and even modes are symmetric about $\xi = \pm \frac{1}{2}\pi$.

4.2 Free boundary $\mathfrak{T}_N^\circ$

In this lattice waveguide, both boundaries are free so that for $y = N$, (33b) holds while for $y = 1$, (34b) holds. Based on an equation analogous to (35), it is easy to see that the normal modes are required to satisfy (10), where $\mathbf{A}_N$ has the form (9) with definitions provided in (36) and $\delta_{NE} = \delta_{SW} = 0$, $\delta_{NW} = \delta_{SE} = -2$. Applying the results of [61] and using (13), in case of $\mathfrak{T}_N^\circ$, it is found that the dispersion relations are given by (37) where η_κ is such that for $\kappa \in \mathbb{Z}_1^N$, $\cos^2 \xi \sin(N+1)\eta_\kappa + \sin(N-1)\eta_\kappa - 2 \cos \xi \sin N\eta_\kappa = 0$. Indeed, using the Chebyshev polynomials U_n of second kind, the latter is re-written as

$$\cos^2 \xi U_N(\vartheta) + U_{N-2}(\vartheta) - 2 \cos \xi U_{N-1}(\vartheta) = 0. \quad (39)$$

Using (37), it is clear that η_κ is a function of ξ such that

$$\cos \eta_\kappa = \vartheta = \frac{1}{2} \frac{6 - 2 \cos 2\xi - \frac{3}{2}\omega^2}{2 \cos \xi}. \quad (40)$$

By the nature of the expression (40), it is noted that η_κ is allowed to assume complex values. Indeed, $\eta_\kappa(\xi)$ is real provided it does not correspond to a surface mode (localized wave along y) as described in section 4.2c. In general, it is assumed that $\xi \neq \pm \frac{1}{2}\pi$, while in the special case of $\xi = \pm \frac{1}{2}\pi$, (39) can be used directly to calculate η_κ . Notice that

unlike the expression (24) for the analogous case of $\mathfrak{T}_N^\circ$, (39) does not admit a closed-form expression for η_κ . Once η_κ is determined (numerically in this case) that satisfies Eq. (39) for given ξ and $\kappa \in \mathbb{Z}_1^N$, the eigenvalue Λ of $\mathbf{A}_N$ is given by (12) (and (36)) and thus the dispersion relations are determined with $\Lambda = \frac{3}{2}\Omega_\kappa^2(\xi) = \frac{3}{2}\Omega_\kappa^2(\xi, \eta_\kappa)$. Applying the results of [61], the eigenvectors may be found for $\kappa \in \mathbb{Z}_1^N$, as a nontrivial constant multiple of the vector with components $-\cos \xi \sin v\eta_\kappa + \sin(v-1)\eta_\kappa$. The same can be normalized (as described in Appendix 2.2) to obtain, for $\kappa \in \mathbb{Z}_1^N$,

$$a_{(\kappa)v} = \frac{2(-\cos \xi \sin v\eta_\kappa + \sin(v-1)\eta_\kappa)}{\sqrt{2N(1 + \cos^2 \xi - 2\vartheta \cos \xi) - 2 + 2 \cos^2 \xi}}, \quad v \in \mathbb{Z}_1^N. \quad (41)$$

Instead of solving for η_κ , alternatively, Ω_κ^2 can be also obtained directly from (39). In particular, the function $\omega^2 = \Omega_\kappa^2(\xi)$ is implicitly given by

$$\begin{aligned} & \frac{1}{4} \prod_{j=1}^N \left(6 - 2 \cos 2\xi - \frac{3}{2}\Omega_\kappa^2 - 4 \cos \xi \cos \frac{j\pi}{N+1} \right) \\ & + \prod_{j=1}^{N-2} \left(6 - 2 \cos 2\xi - \frac{3}{2}\Omega_\kappa^2 - 4 \cos \xi \cos \frac{j\pi}{N-1} \right) \\ & - \prod_{j=1}^{N-1} \left(6 - 2 \cos 2\xi - \frac{3}{2}\Omega_\kappa^2 - 4 \cos \xi \cos \frac{j\pi}{N} \right) = 0. \end{aligned} \quad (42)$$

Figure 5. Contour plot of the triangular lattice dispersion relation $\Omega_{\mathfrak{T}}(\xi, \eta)$ and $\Omega_{\kappa}, \kappa \in \mathbb{Z}_1^N$, for $\mathfrak{T}^\bullet$ (left) and $\mathfrak{T}^\circ$ (right) with thick curves corresponding to the curves shown in figure 4b and c, respectively. Notice the gaps (marked by dashed segments) due to the surface wave dispersion curves.

The index κ is placed in such a way that $\Omega_{\kappa}(0)$ is an increasing sequence with respect to κ . figure 4b and the left of figure 5 illustrate the dispersion curves for this case.

4.2a Odd modes: For $N = 2N$, by definition of an odd mode, it is required that $a_v = -a_{2N+1-v}$, i.e., the problem is equivalent to finding the eigenvectors of (10) where $\mathbf{A}_N$ has the form (9) with definitions provided in (36) and $\delta_{NE} = \delta_{SW} = 0, \delta_{NW} = -2, \delta_{SE} = 2 \cos \xi$. This results in the condition

$$\cos \xi (U_N(\vartheta) + U_{N-1}(\vartheta)) - U_{N-1}(\vartheta) - U_{N-2}(\vartheta) = 0. \quad (43)$$

This can be also expressed in terms of the Chebyshev polynomials [58] of fourth kind W , leading to $\cos \xi W_N(\vartheta) - W_{N-1}(\vartheta) = 0, \vartheta = \cos \eta_{\kappa}$. The eigenvectors are given by (41) with suitable modification for a normalized odd mode. Circumventing the results stated by [61], an alternative and direct derivation of (43) is also possible using ‘the second method’ (as elaborated earlier in the case of $\mathfrak{S}_N$); this is presented later.

Using the equation analogous to (35), it is easy to see that the general solution [65, 82] for normal modes is given by (26b), where λ satisfies

$$\lambda + \lambda^{-1} = \mathcal{Q}(\xi) := (2 \cos \xi)^{-1} \left(6 - 2 \cos 2\xi - \frac{3}{2} \omega^2 \right). \quad (44)$$

Note that (21) also holds due to (44) and (40). Again the expressions $\lambda^2 = \mathcal{Q} + 2, \lambda^{-2} = \mathcal{Q} - 2$ hold and λ is given by (19). Due to free boundary condition, it is required that $2C_1\lambda^N + 2C_2\lambda^{-N} = 2 \cos \xi (C_1\lambda^{N+1} + C_2\lambda^{-N-1})$. Taking into account the symmetry, it is also required that $C_1\lambda + C_2\lambda^{-1} = -C_1 - C_2$. For a non-trivial solution C_1, C_2 of both equations, it is required that $(1 - \lambda \cos \xi)(1 + \lambda^{-1})\lambda^N - (1 - \lambda^{-1} \cos \xi)(1 + \lambda)\lambda^{-N} = 0$. Using the Chebyshev polynomials U_n of second kind (after an application of (20)), it is easy to see that (43) follows, where ϑ is given by (21) and (44). The eigenvectors are also, consequently, described by the expression

$$a_{N+v} = -a_{(\kappa)N-v+1} = C_N \cos \left(v - \frac{1}{2} \right) \eta_{\kappa}, v \in \mathbb{Z}_1^N, \quad \text{where} \\ C_N^{-2} = N + (\vartheta + 1) \frac{\sin^2 \frac{1}{2} \xi}{\cos^2 \frac{1}{2} \xi} U_{N-1}^2(\vartheta), \quad (45)$$

η_{κ}, ϑ satisfy (43), and ϑ is given by (21). The normalization in (45) is detailed in Appendix 2.2.

For $N = 2N + 1$ and odd modes, it is required that $a_v = a_{2N+2-v}$. Using the ansatz (16), the equations, based on free boundary condition and the symmetry, admit a non-trivial C_1, C_2 , provided $(1 - \lambda \cos \xi)(1 + \lambda^{-2})\lambda^{N+1} - (1 - \lambda^{-1} \cos \xi)(1 + \lambda^2)\lambda^{-N-1} = 0$. Using the Chebyshev polynomials U_n of second kind (after an application of (20)), it follows that

$$\cos \xi U_N(\vartheta) - U_{N-1}(\vartheta) = 0, \quad (46)$$

where ϑ is given by (21). Referring to the expressions derived in section 4.3, (46) is the same as that in the case of $\mathfrak{T}^\circ$ of width N , as intuitively expected². The eigenvectors corresponding to the dispersion relations, based on the value of η_{κ} satisfying (46), are given by a nontrivial constant multiple of the vector with components (41) or, in simpler form, for $\kappa \in \mathbb{Z}_1^N$,

$$a_{(\kappa)N+1+v} = C_N \sin v \eta_{\kappa}, a_{(\kappa)v+1} = -a_{(\kappa)2N+1-v}, v \in \mathbb{Z}_0^N, \quad \text{where} \\ C_N^{-2} = N + \frac{1}{2} - \frac{1}{2} \sin^2 \xi U_N^2(\vartheta). \quad (47)$$

See the derivation of the normalization in Appendix 2.2.

4.2b Even modes: For even modes in $\mathfrak{T}^\circ$ with $N = 2N$, it is required that $a_v = a_{2N+1-v}$. The problem of finding propagating waves is equivalent to finding the eigenvectors of (10) where $\mathbf{A}_N$ has the form (9) with definitions provided in (36) and $\delta_{NE} = \delta_{SW} = 0, \delta_{NW} = -2, \delta_{SE} = -2 \cos \xi$. After the application of the results stated in [61] (recalled as (12) and (13)), it is found that

² Alternatively, the same expression results using (10) where $\mathbf{A}_{N+1}$ has the form (9) with definitions provided in (36) and $\delta_{NE} = \delta_{SW} = 0, \delta_{NW} = -2, \delta_{SE} = 0$.

$$\cos \xi (U_N(\vartheta) - U_{N-1}(\vartheta)) - U_{N-1}(\vartheta) + U_{N-2}(\vartheta) = 0. \quad (48)$$

Using the Chebyshev polynomials V_n of third kind, the condition (48) can also be expressed as $\cos \xi V_N(\vartheta) - V_{N-1}(\vartheta) = 0$, $\vartheta = \cos \eta_\kappa$. The eigenvectors are given by a nontrivial constant multiple of the vector with components (41). An alternative derivation of the dispersion relation using the second method is included here as well. Using the definition of λ in (20) and (44), the general solution for normal modes is given by (26b). Due to free-boundary condition, it is required that $2C_1\lambda^N + 2C_2\lambda^{-N} = 2\cos \xi (C_1\lambda^{N+1} + C_2\lambda^{-N-1})$. Taking into account the symmetry, it is also required that $C_1\lambda + C_2\lambda^{-1} = C_1 + C_2$. For a non-trivial solution C_1, C_2 of both equations, it is necessary that $(1 - \lambda \cos \xi)(1 - \lambda^{-1})\lambda^N - (1 - \lambda^{-1} \cos \xi)(1 - \lambda)\lambda^{-N} = 0$. Using the Chebyshev polynomials U_n of second kind (after an application of (20)), it is found that (48) follows. The eigenvectors are described by the expression

$$a_{N+v} = a_{(\kappa)N-v+1} = C_N \cos\left(v - \frac{1}{2}\right) \eta_\kappa, v \in \mathbb{Z}_1^N, \\ \text{where } C_N^{-2} = N + (\vartheta - 1) \frac{\cos^2 \frac{1}{2} \xi}{\sin^2 \frac{1}{2} \xi} U_{N-1}^2(\vartheta), \quad (49)$$

η_κ, ϑ satisfy (48). The details for the normalization are provided in Appendix 2.2. For even modes in $\mathfrak{T}_\circ^\circ$ with $N = 2N + 1$, it is required that $a_v = a_{2N+2-v}$. Using (16), the free-boundary condition and the symmetry, non-trivial C_1, C_2 exist provided $(1 - \lambda \cos \xi)(1 - \lambda^{-2})\lambda^{N+1} - (1 - \lambda^{-1} \cos \xi)(1 - \lambda^2)\lambda^{-N-1} = 0$. Using the Chebyshev polynomials U_n of second kind (20)

$$\cos \xi (U_{N+1}(\vartheta) - U_{N-1}(\vartheta)) - U_N(\vartheta) + U_{N-2}(\vartheta) = 0. \quad (50)$$

Using the Chebyshev polynomials T_n of first kind, (48) can be also expressed as $\cos \xi T_{N+1}(\vartheta) - T_N(\vartheta) = 0$, $\vartheta = \cos \eta_\kappa$. The eigenvectors are given by a nontrivial constant multiple of the vector with components (41) or, in simpler form analogous to (47), for $\kappa \in \mathbb{Z}_1^N$,

$$a_{(\kappa)N+1+v} = C_N \cos v \eta_\kappa, a_{(\kappa)v+1} = a_{(\kappa)2N+1-v}, v \in \mathbb{Z}_0^N, \text{ where} \\ C_N^{-2} = N + \frac{1}{2} + \frac{1}{2}(\vartheta^2 - 1) \frac{\sin^2 \xi}{(\vartheta \cos \xi - 1)^2} U_{N-1}^2(\vartheta). \quad (51)$$

The normalization represented by the symbol C_N in (51) is derived in Appendix 2.2.

4.2c Surface modes: It is well known that the surface wave may exist below the pass band envelope for these lattice

models with nearest-neighbour interactions in the presence of perturbations [12, 54]. Indeed, in the case of $\mathfrak{T}_\circ^\circ$, a critical value of ξ , a ‘threshold’ in the surface wave dispersion relation, is given by $\vartheta = \cos \eta_\kappa = \pm 1$. Recall the well-known identity (102b)₃ [58]; its substitution in (39) implies that either $\cos \xi = 1$ or $\cos \xi = \frac{N-1}{N+1}$, and either $\cos \xi = -1$ or $\cos \xi = -\frac{N-1}{N+1}$. Note that both such critical points ξ approach either 0 or π separately, as $N \rightarrow \infty$ (implying a surface wave dispersion relation for the semi-infinite case). Simplifying the corresponding value Ω_κ^2 obtained from (37), it is found to be equal to $\frac{2}{3}(12 - ((3N+2)/(N+1))^2)$. Let

$$\xi_s = \cos^{-1} \frac{N-1}{N+1} \in [0, \pi]. \quad (52)$$

Then one of the surface modes exists for $\xi \in (\xi_s, \pi] \cup [-\pi, -\xi_s)$ while another surface mode exists for $\xi \in (\xi_s - \pi, \pi - \xi_s)$. For the odd modes in case of $\mathfrak{T}_\circ^\circ$ with $N = 2N$, using the identities (2.29, 2.30) in [58], it is found that $W_N(+1) = (2N+1)$, $W_N(-1) = (-1)^N$; again it is easy to see that either $\cos \xi = (2N-1)/(2N+1)$ or $\cos \xi = -1$, so that one of the surface modes is odd while the other is even. See the region between abruptly ending dispersion curves on the contour in figure 5 (left) and corresponding dashed portions in dispersion curves in figure 4b. Notice that the surface mode dispersion curves in figure 4b lie entirely in the stop band, as expected [12, 54].

4.3 Fixed-free boundary $\mathfrak{T}_\bullet^\circ$

Consider $\mathfrak{T}_\bullet^\circ$ as schematically illustrated in the right side of figure 2. At $y = N$, the boundary condition is described by (33a), while at the lattice row $y = 1$, (34b) holds. For the determination of the propagating waves, based on an equation analogous to (35), it is found that (10) holds for the normal modes, where $\mathbf{A}_N$ has the form (9) with definitions provided in (36) and $\delta_{NE} = \delta_{SW} = 0$, $\delta_{NW} = 0$, $\delta_{SE} = -2$. Applying the results of [61], i.e., using (13), in this case for $\kappa \in \mathbb{Z}_1^N$, η_κ satisfies $\cos \xi \sin(N+1)\eta_\kappa - \sin N\eta_\kappa = 0$. Using the Chebyshev polynomials U_n of second kind, the condition is re-written as

$$\cos \xi U_N(\vartheta) - U_{N-1}(\vartheta) = 0. \quad (53)$$

Similar to the case of $\mathfrak{T}_\circ^\circ$, (53) also does not allow the determination of $\vartheta(\xi)$ or $\eta_\kappa(\xi)$ in a closed form [62]. After the value of η_κ is obtained numerically that satisfies (53) for given ξ , an eigenvalue Λ of $\mathbf{A}_N$ is obtained by (12). The dispersion relations are determined with $\Lambda = \frac{3}{2}\Omega_\kappa^2(\xi) = \frac{3}{2}\Omega_\kappa^2(\xi, \eta_\kappa)$, which are illustrated by figure 4c, along with figure 5 (right). The index κ is placed in such a way that $\Omega_\kappa(0)$ is an increasing sequence with respect to κ . The eigenvectors may be found in the same manner as before. In particular, the eigenvector is a constant multiple of the

vector with components $\sin v\eta_k$. Normalizing the same (as shown in the Appendix 2.2) it is found that, for $\kappa \in \mathbb{Z}_1^N$,

$$a_{(\kappa)v} = C_N \sin v\eta_k, v \in \mathbb{Z}_1^N, \text{ where } C_N^{-2} = \frac{1}{4}(2N+1 - \sin^2 \xi U_N^2(\vartheta)). \quad (54)$$

Instead of solving for η_k , alternatively, Ω_k^2 can be determined directly such that (53) holds. In particular, analogous to (42), the function $\omega^2 = \Omega_k^2(\xi)$ is implicitly given by $\frac{1}{2} \prod_{j=1}^N (6 - 2 \cos 2\xi - \frac{3}{2} \Omega_k^2 - 4 \cos \xi \cos \frac{j}{N+1} \pi) - \prod_{j=1}^{N-1} (6 - 2 \cos 2\xi - \frac{3}{2} \Omega_k^2 - 4 \cos \xi \cos \frac{j}{N} \pi) = 0$. Using the same equation (44), when $\cos \xi \neq 0$ (i.e. $\xi \neq \pm \frac{1}{2}\pi$), η_k can be obtained from $\cos \eta_k(\xi) = (6 - 2 \cos 2\xi - \frac{3}{2} \Omega_k^2(\xi)) / (4 \cos \xi)$. Indeed, η_k is real provided it does not correspond to a surface mode (localized wave along y). For $\xi = \pm \frac{1}{2}$, (53) can be used directly to calculate η_k . Also, note that an application of the second method, as described in section 4.2, yields the same expressions.

4.3a Surface modes: A surface wave exists below the pass band envelope (illustrated by grey-coloured shaded region in figure 4(c)). Clearly, a critical value of ξ that signals the threshold for surface wave dispersion relation satisfies $\cos \eta_k(\xi) = \pm 1$. Using (102b)₃, either $\cos \xi = N/(N+1)$ or $\cos \xi = -N/(N+1)$. Analogous to the case of $\mathfrak{T}_o^\circ$, both critical values of ξ approach $0, \pi$ as $N \rightarrow \infty$. The values of the frequency ω corresponding to the end-points of the surface wave dispersion curves are easy to find as $\frac{2}{3}(12 - ((3N+2)(N+1))^2)$. Let

$$\xi_s = \cos^{-1} \frac{N}{N+1} \in [0, \pi]. \quad (55)$$

Then only one surface mode exists for fixed-free boundary when $\xi \in (\xi_s, \pi - \xi_s) \cup (-\pi + \xi_s, -\xi_s)$. See the region between abruptly ending dispersion curves on the contour in figure 5 (right) and corresponding dashed portions in dispersion curves in figure 4c. Notice that the surface mode dispersion curves in figure 4c lie entirely in the stop band as expected [12, 54].

4.4 Periodic boundary $\mathfrak{T}_\odot$

At the upper boundary row of $\mathfrak{T}_\odot$ (see the illustration in figure 2), i.e., $y = N$, the condition (33c) holds, while for $y = 1$, (34c) holds. Hence, based on an equation analogous to (35), (10) holds for the normal modes, where $\mathbf{A}_N$ has the form (9) with definitions provided in (36) and $\delta_{NE} = \delta_{SW} = \mathcal{Y}$, $\delta_{NW} = \delta_{SE} = 0$. In particular, it is observed that $\mathbf{A}_N$ is a circulant matrix; consequently, the eigenpairs are straightforward to find as (30). An alternative derivation of this case is identical to that for the square lattice (except for replacing the expression of λ by that provided for triangular lattice); hence it is not

repeated here along with (31) provided in section 3.4. The expression (31) has been plotted and shown in figure 4d marking the doubly repeated dispersion relations (except for the uppermost and lowermost). An application of the second method, as described in section 4.2, yields the same expressions. Notice that the periodic boundary condition for $\mathfrak{T}_N$ is possible only when N is an even number (see Remark 10). For $N = 2N$, a possible case to investigate is that of ‘odd’ and ‘even’ modes with respect to the rectangular lattice strip $\mathfrak{R}_N$ (containing $\mathfrak{T}_N$); in this context, see also Remark 10.

4.4a ‘Odd’ modes: For odd modes, (10) holds where $\mathbf{A}_N$ has the form (9) with definitions provided in (36) and $\delta_{NE} = \delta_{SW} = 0$, $\delta_{NW} = -\mathcal{Y}$, $\delta_{SE} = -\mathcal{Y}$. This results in the condition $\mathcal{Y}^2 \sin(N+1)\eta_k + \mathcal{Y}^2 \sin(N-1)\eta_k + 2\mathcal{Y}^2 \sin N\eta_k = 0$, which is, in terms of the Chebyshev polynomials of second kind,

$$(\vartheta + 1)U_{N-1}(\vartheta) = 0, \vartheta = \cos \eta_k. \quad (56)$$

Hence, $\eta_k = \kappa\pi/N$, κ lies in $\mathbb{Z}_1^N$ and $a_{(\kappa)N+v} = \frac{1}{\sqrt{N}} \sin(v - \frac{1}{2})\eta_k$, $v \in \mathbb{Z}_1^N$.

4.4b ‘Even’ modes: In a manner similar to the odd modes, consider the even modes. Hence (10) holds, where $\mathbf{A}_N$ has the form (9) with definitions provided in (36) and $\delta_{NE} = \delta_{SW} = 0$, $\delta_{NW} = \mathcal{Y}$, $\delta_{SE} = \mathcal{Y}$. This results in the condition $\mathcal{Y}^2 \sin(N+1)\eta_k + \mathcal{Y}^2 \sin(N-1)\eta_k - 2\mathcal{Y}^2 \sin N\eta_k = 0$. Using the Chebyshev polynomials of second kind,

$$(\vartheta - 1)U_{N-1}(\vartheta) = 0, \vartheta = \cos \eta_k. \quad (57)$$

Hence, $\eta_k = \kappa\pi/N$, κ lies in $\mathbb{Z}_0^{N-1}$ and $a_{(\kappa)N+v} = \left(\left(\frac{1}{\sqrt{2}} - 1\right)\delta_{\kappa,0} + 1\right)\frac{1}{\sqrt{N}} \cos(v - \frac{1}{2})\eta_k$, $v \in \mathbb{Z}_1^N$.

Remark 9 The reasoning analogous to Remark 8 holds for the periodic boundary condition also. Note that Remark 4 holds for the normal modes (including the surface modes wherever they exist) of all triangular lattice waveguides stated in table 2.

Remark 10 The case $N = 2N + 1$ for periodic boundary condition on $\mathfrak{R}_N$ can be interpreted as corresponding to a ‘doubly’ rolled triangular lattice strip; but this appears to be less relevant. Appendix 2.2 contains a detailed proof of the normalization that appears earlier, for example in (38), (41) and (54).

5. Square lattice model $\hat{\mathfrak{E}}_N$ with next-nearest-neighbour interactions

As an indication of the generalization of the analysis, presented so far in this paper, to the case of more

Figure 6. Illustration of the square lattice waveguides $\hat{\mathfrak{S}}_N$ (with NNN interactions) corresponding to figure 2.

complicated interactions [12, 54], the square lattice of section 3 is extended to include the effect of four next-nearest-neighbour interactions (schematically shown in figure 6). The equation of motion for a particle in $\hat{\mathfrak{S}}_N$ located at lattice site (x, y) , such that it does not belong to either of the strip boundary rows, is

$$M \frac{d^2}{dt^2} u_{x,y} = K \sum_{i \in \{-1,1\}} ((u_{x+i,y} + u_{x,y+i}) + \gamma(u_{x+i,y+1} + u_{x+i,y-1})) - 4K(1 + \gamma)u_{x,y}, \quad \forall x \in \mathbb{Z}, y \in \mathbb{Z}_2^{N-1}, \quad (58)$$

where $\gamma \in \mathbb{R}$ is the ratio of the next-nearest-neighbour force constant $\hat{K}$ to the nearest-neighbor force constant K (i.e. $\hat{K} = \gamma K$)³. In the context of Remark 2, it is a straightforward exercise to obtain the traditional continuum model as a continuum limit of the square lattice model with next-nearest-neighbour interactions; in particular, it is easy to see that the shear wave speed (c_s) is $\sqrt{1 + 2\gamma} \sqrt{Kb^2/M}$. Hence, it is assumed that $\gamma > -1/2$. For convenience of notation, let

$$\Gamma_0 := 1 + 2\gamma. \quad (59)$$

Let $M = \rho b^3, K = \Gamma_0^{-1} \mu b, t = tc_s$. The square lattice dispersion relation is $\omega^2 = \frac{1}{b^2} \Omega_{\hat{\mathfrak{S}}}^2(\xi, \eta)$, where

$$\Omega_{\hat{\mathfrak{S}}}^2(\xi, \eta) = \Gamma_0^{-1} \left(4 \sin^2 \frac{1}{2} \xi + 4 \sin^2 \frac{1}{2} \eta + 4\gamma \left(\sin^2 \frac{1}{2} (\xi + \eta) + \sin^2 \frac{1}{2} (\xi - \eta) \right) \right), \quad (\xi, \eta) \in [-\pi, \pi]^2. \quad (60)$$

Corresponding to the lattice waveguides listed in table 1, the boundary conditions at upper and lower rows studied in this paper are $\forall x \in \mathbb{Z}$, as follows: at $y = N$

$$-\Gamma_0 \omega^2 u_{x,y} = u_{x+1,y} + u_{x-1,y} + u_{x,y-1} + \gamma(u_{x+1,y-1} + u_{x-1,y-1}) - 4(1 + \gamma)u_{x,y} \quad (\bullet) \quad (61a)$$

$$-\Gamma_0 \omega^2 u_{x,y} = u_{x+1,y} + u_{x-1,y} + u_{x,y-1} + \gamma(u_{x+1,y-1} + u_{x-1,y-1}) - (3 + 2\gamma)u_{x,y} \quad (\circ) \quad (61b)$$

$$-\Gamma_0 \omega^2 u_{x,y} = u_{x+1,y} + u_{x-1,y} + u_{x,y-1} + u_{x,1} \quad (61c)$$

$$+ \gamma(u_{x+1,y-1} + u_{x-1,y-1}) + \gamma(u_{x+1,1} + u_{x-1,1}) - 4(1 + \gamma)u_{x,y} \quad (\odot), \quad (61d)$$

and at $y = 1$

$$-\Gamma_0 \omega^2 u_{x,y} = u_{x+1,y} + u_{x-1,y} + u_{x,y+1} + \gamma(u_{x+1,y+1} + u_{x-1,y+1}) - 4(1 + \gamma)u_{x,y} \quad (\bullet) \quad (62a)$$

³ It is well known that the sign of γ depends on the structural details for a specific material [12, 29, 54, 83, 84].

$$\begin{aligned}
-\Gamma_0 \omega^2 u_{x,y} &= u_{x+1,y} + u_{x-1,y} + u_{x,y+1} \\
&\quad + \gamma(u_{x+1,y+1} + u_{x-1,y+1}) - (3 + 2\gamma)u_{x,y} \quad (\circ) \\
\end{aligned} \tag{62b}$$

$$-\Gamma_0 \omega^2 u_{x,y} = u_{x+1,y} + u_{x-1,y} + u_{x,y+1} + u_{x,N} \tag{62c}$$

$$\begin{aligned}
&+ \gamma(u_{x+1,y-1} + u_{x-1,y-1}) + \gamma(u_{x+1,N} + u_{x-1,N}) \\
&- 4(1 + \gamma)u_{x,y} \quad (\odot).
\end{aligned} \tag{62d}$$

Clearly, $\gamma = 0$ corresponds to the model *without* the next-nearest neighbour interactions as analysed in section 3.

5.1 Fixed boundary $\hat{\mathfrak{E}}_\bullet$

For the upper and lower boundary rows the conditions (61a) and (62a) hold, respectively. With the ansatz $u_{x,y} = a_y e^{ik_x x}$, it follows that

$$\begin{aligned}
-\Gamma_0 \omega^2 a_y &= (1 + 2\gamma \cos \xi)((1 - \delta_{y,N})a_{y+1} \\
&\quad + (1 - \delta_{y,1})a_{y-1}) + 2 \cos \xi a_y \\
&\quad - 4(1 + \gamma)a_y, y \in \mathbb{Z}_1^N.
\end{aligned} \tag{63}$$

It is easy to see that u is a propagating wave in $\hat{\mathfrak{E}}_\bullet$, with $\mathbf{A}_N$ of the form (9), if $a \in \mathbb{C}^N$ satisfies (10), where

$$\Lambda = \Gamma_0 \omega^2, \chi(\xi) = 4(1 + \gamma) - 2 \cos \xi, \Upsilon = -1 - 2\gamma \cos \xi, \tag{64}$$

and $\delta_{NE} = \delta_{SW} = 0, \delta_{NW} = \delta_{SE} = 0$. For convenience of notation, let

$$\Gamma(\xi) := 1 + 2\gamma \cos \xi, \tag{65}$$

so that $\Upsilon = -\Gamma$ (as defined in (64)). Notice that $\Gamma(0) = 1 + 2\gamma = \Gamma_0$. According to [61], the eigenvalue problem (10) is solved by the eigenpairs described therein; in particular, each eigenvalue Λ is described by the general form (12), where η_κ are determined by a specific condition depending on $\mathbf{A}_N$ (13). Thus, the dispersion relations and normal modes are given by

$$\begin{aligned}
\omega^2 &= \Omega_\kappa^2(\xi) = \Gamma_0^{-1} \left(4 \sin^2 \frac{1}{2} \eta_\kappa + 4 \sin^2 \frac{1}{2} \xi + 4\gamma \left(\sin^2 \frac{1}{2} (\xi + \eta_\kappa) \right. \right. \\
&\quad \left. \left. + \sin^2 \frac{1}{2} (\xi - \eta_\kappa) \right) \right) = \Omega_{\hat{\mathfrak{E}}}^2(\xi, \eta_\kappa),
\end{aligned} \tag{66a}$$

$$a_{(\kappa)v} = \sqrt{\frac{2}{N+1}} \sin v \eta_\kappa, v \in \mathbb{Z}_1^N, \quad \eta_\kappa = \frac{\kappa \pi}{N+1}, \kappa \in \mathbb{Z}_1^N. \tag{66b}$$

The expression (66) for the dispersion relations has been plotted and shown in figure 7a. The odd modes correspond to even values of κ , while the even modes correspond to

odd values of κ . Similar to the case of triangular lattice $\hat{\mathfrak{T}}_\bullet$ presented in section 4.1, the detailed reasoning for $\hat{\mathfrak{E}}_\bullet$ is also the same as that for the square lattice waveguide $\hat{\mathfrak{S}}_\bullet$ (provided in section 3.1).

Remark 11 Let $\xi_* \in [0, \pi]$ such that $\Gamma(\xi_*) = 1 + 2\gamma \cos \xi_* = 0$; clearly for the existence of ξ_* , it is required that $\gamma > 1/2$ (since $\gamma > -1/2$). When the latter holds, it is evident from the four choices of γ in figure 7 that the character of the pass band for $\hat{\mathfrak{E}}_N$ changes and resembles more of that for the triangular lattice. Note that when $\gamma > \frac{1}{2}$, it is possible that there exists a *flat band* (zero group velocity); in general, there is a possibility of a flat band whenever $\cos \eta_\kappa = -\frac{1}{2\gamma}$ for $\gamma > \frac{1}{2}$, i.e., $\frac{N+1}{\pi} \arccos(-\frac{1}{2\gamma})$ is an integer.

5.2 Free boundary $\hat{\mathfrak{E}}_\circ$

For $\hat{\mathfrak{E}}_\circ$ (figure 6), i.e., $\hat{\mathfrak{E}}_N$ with a free upper and lower boundary, Eqs. (61b) and (62b) hold, respectively. Again, based on an equation analogous to (63), it is found that (10) holds for the normal modes, where $\mathbf{A}_N$ has the form (9) with (64) and $\delta_{NE} = \delta_{SW} = 0, \delta_{NW} = \delta_{SE} = -1 - 2\gamma$. In this context, it is also useful to recollect the analysis of $\hat{\mathfrak{T}}_\circ$ in section 4.2. Using (13), it is found that the dispersion relations are given by (66b), where η_κ is such that for $\kappa \in \mathbb{Z}_1^N$, $\Gamma(\xi)^2 \sin(N+1)\eta_\kappa + \Gamma_0^2 \sin(N-1)\eta_\kappa - 2\Gamma_0 \Gamma(\xi) \sin N\eta_\kappa = 0$. Indeed, using the Chebyshev polynomials U_n of second kind, the latter is re-written as

$$\Gamma(\xi)^2 U_N(\vartheta) + \Gamma_0^2 U_{N-2}(\vartheta) - 2\Gamma_0 \Gamma(\xi) U_{N-1}(\vartheta) = 0. \tag{67}$$

Using (66b), it is clear that η_κ is a function of ξ such that

$$\cos \eta_\kappa = \vartheta = \frac{1}{2} \frac{4(1 + \gamma) - 2 \cos \xi - \Gamma_0 \omega^2}{\Gamma(\xi)}. \tag{68}$$

As remarked before, $\eta_\kappa(\xi)$ in (68) is real provided it does not correspond to a surface mode (discussed separately later). Also, it is assumed that $\Gamma(\xi) \neq 0$, while in the special case of $\Gamma(\xi) = 0$, (67) can be used directly to calculate η_κ . Once η_κ is determined (numerically in this case) that satisfies Eq. (67) for given ξ and $\kappa \in \mathbb{Z}_1^N$, the eigenvalue Λ of $\mathbf{A}_N$ is given by (12) (and (64)) and thus the dispersion relations are determined with $\Lambda = \Omega_\kappa^2(\xi) = \Omega_{\hat{\mathfrak{E}}}^2(\xi, \eta_\kappa)$. Applying the results of [61], the eigenvectors may be found for $\kappa \in \mathbb{Z}_1^N$, as a nontrivial constant multiple of the vector with components $-\Gamma(\xi) \sin v \eta_\kappa + \Gamma_0 \sin(v-1)\eta_\kappa$; the same can be normalized to obtain, for $\kappa \in \mathbb{Z}_1^N$,

$$a_{(\kappa)v} = \frac{-2\Gamma(\xi) \sin v \eta_\kappa + 2\Gamma_0 \sin(v-1)\eta_\kappa}{\sqrt{2N(1 + \Gamma(\xi)^2 - 2\vartheta \Gamma_0 \Gamma(\xi)) - 2 + 2\Gamma(\xi)^2}}, v \in \mathbb{Z}_1^N. \tag{69}$$

Figure 7. Dispersion relation $\Omega_\kappa, \kappa \in \mathbb{Z}_1^N$, with $N = 6$ for $\hat{\mathfrak{E}}_N$ with next-nearest-neighbour interactions (first row: $\gamma = -0.25$, second row: $\gamma = 0.25$, third row: $\gamma = 1$, fourth row: $\gamma = 7.25$): (a) $\hat{\mathfrak{E}}_\bullet^\bullet$, (b) $\hat{\mathfrak{E}}_\circ^\circ$, (c) $\hat{\mathfrak{E}}_\bullet^\circ$ and (d) $\hat{\mathfrak{E}}_\circ^\bullet$. The pass band envelope of the infinite lattice is illustrated by grey-coloured shaded region.

Instead of solving for η_κ , alternatively, Ω_κ^2 can be solved for such that (67) holds. In particular, the function $\omega^2 = \Omega_\kappa^2(\xi)$ is implicitly given by

$$\begin{aligned} & \prod_{j=1}^N \left(4(1+\gamma) - 2\cos\xi - \Omega_\kappa^2 - \Gamma(\xi) \cos \frac{j\pi}{N+1} \right) \\ & + \Gamma_0^2 \prod_{j=1}^{N-2} \left(4(1+\gamma) - 2\cos\xi - \Omega_\kappa^2 - \Gamma(\xi) \cos \frac{j\pi}{N-1} \right) \\ & - 2\Gamma_0 \prod_{j=1}^{N-1} \left(4(1+\gamma) - 2\cos\xi - \Omega_\kappa^2 - \Gamma(\xi) \cos \frac{j\pi}{N} \right) = 0. \end{aligned} \quad (70)$$

The index κ is placed in such a way that $\Omega_\kappa(0)$ is an increasing sequence with respect to κ . figure 7b and the left of figure 8 illustrate the dispersion curves for this case. Recall Remark 11 as well in the present context of the waveguide $\hat{\mathfrak{E}}_\bullet^\bullet$.

5.2a Odd modes: For $N = 2N$, by definition of an odd mode, it is required that $a_v = -a_{2N+1-v}$, i.e., the problem is equivalent to finding the eigenvectors of (10) where $\mathbf{A}_N$ has the form (9) with definitions provided in (64) and $\delta_{NE} =$

$\delta_{SW} = 0, \delta_{NW} = -1 - 2\gamma, \delta_{SE} = \Gamma(\xi)$. Using (13), this leads to the condition

$$\Gamma(\xi)(U_N(\vartheta) + U_{N-1}(\vartheta)) - \Gamma_0(U_{N-1}(\vartheta) + U_{N-2}(\vartheta)) = 0. \quad (71)$$

The relation (71) can be also expressed in terms of the Chebyshev polynomials [58] of fourth kind W , leading to $\Gamma(\xi)W_N(\vartheta) - \Gamma_0 W_{N-1}(\vartheta) = 0, \vartheta = \cos \eta_\kappa$. The eigenvectors are given by (69) for suitable κ . An alternative and direct derivation of the eigenpairs is also possible using the second method. Using an equation analogous to (63), it is easy to see that the general solution [65, 82] for normal modes is given by (26b), where λ satisfies

$$\lambda + \lambda^{-1} = \mathcal{Q}(\xi) := \frac{4(1+\gamma) - 2\cos\xi - \Gamma_0\omega^2}{\Gamma(\xi)}. \quad (72)$$

As in the case of triangular lattice waveguides, note that (21) also holds due to (72) and (68). Further, the expressions $\mathfrak{r}^2 = \mathcal{Q} + 2, \mathfrak{k}^2 = \mathcal{Q} - 2$ hold and λ is given by (19). Due to the free boundary condition, it is required that $\Gamma_0 C_1 \lambda^N + \Gamma_0 C_2 \lambda^{-N} = \Gamma(\xi)(C_1 \lambda^{N+1} + C_2 \lambda^{-N-1})$. Taking into account the symmetry, it is also required that $C_1 \lambda +$

Figure 8. Contour plot of the square lattice dispersion relation $\Omega_{\hat{\xi}}(\xi, \eta)$ and $\Omega_{\kappa}, \kappa \in \mathbb{Z}_1^N$ (first row: $\gamma = -0.25$, second row: $\gamma = 0.25$, third row: $\gamma = 1$, fourth row: $\gamma = 7.25$), for $\hat{\xi}_{\circ}$ (left) and $\hat{\xi}_{\circ}^{\bullet}$ (right) with thick curves corresponding to the curves shown in figure 3b and c, respectively. Notice the gaps (marked by dashed segments) due to the surface wave dispersion curves.

$C_2\lambda^{-1} = -C_1 - C_2$. For a non-trivial solution C_1, C_2 of both equations, it is required that $(\Gamma_0 - \lambda\Gamma(\xi))(1 + \lambda^{-1})\lambda^N - (\Gamma_0 - \lambda^{-1}\Gamma(\xi))(1 + \lambda)\lambda^{-N} = 0$. Using the Chebyshev polynomials U_n of second kind (after an application of (20)), it is easy to see that (71) follows, where ϑ is given by (21) and (72). The eigenvectors are also, consequently, described by the expression

$$a_{N+v} = -a_{(\kappa)N-v+1} = \frac{\cos(v - \frac{1}{2})\eta_{\kappa}}{\sqrt{N + (\vartheta + 1)\gamma(1 - \cos\xi)(1 + \gamma(1 + \cos\xi))^{-1}U_{N-1}^2(\vartheta)}}, v \in \mathbb{Z}_1^N, \quad (73)$$

where η_{κ}, ϑ satisfy (71). The normalization in (73) is derived in Appendix 2.2.

For $N = 2N + 1$ and odd modes, it is required that $a_v = a_{2N+2-v}$. Using the ansatz (16), the equations based on free-boundary condition and the symmetry admit a non-trivial C_1, C_2 , provided $(\Gamma_0 - \lambda\Gamma(\xi))(1 + \lambda^{-2})\lambda^{N+1} - (\Gamma_0 - \lambda^{-1}\Gamma(\xi))(1 + \lambda^2)\lambda^{-N-1} = 0$. After an application of (20) and (21), it follows that

$$\Gamma(\xi)U_N(\vartheta) - \Gamma_0U_{N-1}(\vartheta) = 0, \quad (74)$$

where ϑ is given by (21). Referring to the expressions derived in section 5.3, (74) is same as that in the case of $\hat{\xi}_{\circ}^{\bullet}$ of width N ; moreover, the same result is obtained using (10), where $\mathbf{A}_{N+1}$ has the form (9) with definitions provided in (36) and $\delta_{NE} = \delta_{SW} = 0, \delta_{NW} = -1 - 2\gamma, \delta_{SE} = 0$. The eigenvectors corresponding to the dispersion relations, based on the value of η_{κ} satisfying (74), are given by (69), or in specific form (see Appendix 2.2 for details concerning the normalization), for $\kappa \in \mathbb{Z}_1^N$,

$$a_{(\kappa)N+1+v} = C_N \sin v\eta_{\kappa}, a_{(\kappa)v+1} = -a_{(\kappa)2N+1-v}, \quad v \in \mathbb{Z}_0^N, \quad (75)$$

$$\text{where } C_N^{-2} = N + \frac{1}{2} - 2 \frac{\gamma(1 - \cos\xi)(1 + \gamma(1 + \cos\xi))}{\Gamma_0^2} U_N^2(\vartheta). \quad (76)$$

5.2b Even modes: For even modes in $\hat{\xi}_{\circ}$ with $N = 2N$, it is required that $a_v = a_{2N+1-v}$. The problem of finding propagating waves is equivalent to finding the eigenvectors of (10) where $\mathbf{A}_N$ has the form (9) and $\delta_{NE} = \delta_{SW} = 0, \delta_{NW} = -1 - 2\gamma, \delta_{SE} = -1 - 2\gamma \cos\xi$. After the application of (12) and (13), it is found that

$$\Gamma(\xi)(U_N(\vartheta) - U_{N-1}(\vartheta)) - \Gamma_0(U_{N-1}(\vartheta) - U_{N-2}(\vartheta)) = 0. \quad (77)$$

Using the Chebyshev polynomials V_n of third kind, the condition (77) can also be expressed as $\Gamma(\xi)V_N(\vartheta) - \Gamma_0V_{N-1}(\vartheta) = 0$, $\vartheta = \cos \eta_\kappa$. The eigenvectors are given by (69). An alternative derivation of the dispersion relation using the second method is included here as well. Using the definition of λ in (20) and (72), the general solution for normal modes is given by (26b). For non-trivial C_1, C_2 , in order to satisfy the free-boundary condition and the symmetry requirement, it is necessary that $(\Gamma_0 - \lambda\Gamma(\xi))(1 - \lambda^{-1})\lambda^N - (\Gamma_0 - \lambda^{-1}\Gamma(\xi))(1 - \lambda)\lambda^{-N} = 0$. using (20), it is found that (77) follows. With η_κ that satisfies (77), the eigenvectors are described by the expression

$$a_{N+v} = a_{(\kappa)N-v+1} = C_N \cos\left(v - \frac{1}{2}\right)\eta_\kappa, v \in \mathbb{Z}_1^N, \quad \text{where} \\ C_N^{-2} = N + (\vartheta - 1) \frac{1 + \gamma(1 + \cos \xi)}{\gamma(1 - \cos \xi)} U_{N-1}^2(\vartheta). \quad (78)$$

See Appendix 2.3 for details concerning the normalization.

For even modes in $\hat{\mathfrak{E}}_o^\circ$ with $N = 2N + 1$, it is required that $a_v = a_{2N+2-v}$. Using (16), the free-boundary condition and the symmetry, non-trivial C_1, C_2 exist provided $(\Gamma_0 - \lambda\Gamma(\xi))(1 - \lambda^{-2})\lambda^{N+1} - (\Gamma_0 - \lambda^{-1}\Gamma(\xi))(1 - \lambda^2)\lambda^{-N-1} = 0$. Using the Chebyshev polynomials U_n of second kind (20), it follows that

$$\Gamma(\xi)(U_{N+1}(\vartheta) - U_{N-1}(\vartheta)) - \Gamma_0(U_N(\vartheta) - U_{N-2}(\vartheta)) = 0. \quad (79)$$

In terms of the Chebyshev polynomials T_n of the first kind, (79) becomes $\Gamma(\xi)T_{N+1}(\vartheta) - \Gamma_0T_N(\vartheta) = 0$, $\vartheta = \cos \eta_\kappa$. The eigenvectors are given by (69) or, in simpler form analogous to (75), for $\kappa \in \mathbb{Z}_1^N$,

$$a_{(\kappa)N+1+v} = C_N \cos v \eta_\kappa, a_{(\kappa)v+1} = a_{(\kappa)2N+1-v}, \quad v \in \mathbb{Z}_0^N, \quad (80)$$

where the normalization constant (Appendix 2.3) is

$$C_N^{-2} = N + \frac{1}{2} + 2\gamma(\vartheta^2 - 1) \frac{1 - \cos \xi + \gamma \sin^2 \xi}{(\vartheta - 1 + 2\gamma(\vartheta \cos \xi - 1))^2} U_{N-1}^2(\vartheta). \quad (81)$$

5.2c Surface modes: A critical value of ξ , a ‘threshold’ in the surface wave dispersion relation, is given by $\vartheta = \cos \eta_\kappa = \pm 1$, assuming $\gamma \neq 0$. Recall the well-known identity [58] $U_N(\pm 1) = (N+1)(-1)^{(\frac{1}{2} \mp \frac{1}{2})N}$ (included as (102b)₃ in Appendix 1); its substitution in (67) implies that (recall that $\Gamma_0 > 0$)

$$\alpha + \alpha^{-1} \mp 2 + (\alpha - \alpha^{-1})N^{-1} = 0, \alpha = \Gamma(\xi)/\Gamma_0. \quad (82)$$

Hence, either $\cos \xi = 1$ or $\cos \xi = \frac{N-1-\gamma^{-1}}{N+1}$, and either $\cos \xi = -1 - \gamma^{-1}$ or $\cos \xi = -\frac{N-1+N\gamma^{-1}}{N+1}$. Note that the first such critical point ξ approaches 0, as $N \rightarrow \infty$; however, the second critical point ξ approaches $\cos^{-1}(-1 - \gamma^{-1})$, as $N \rightarrow \infty$, but since $\Gamma_0 > 0$, this implies that there is only one surface wave dispersion relation for the semi-infinite case for $\xi \in [-\pi, \pi] \setminus \{0\}$. Let

$$\xi_{s1} = \cos^{-1} \frac{N-1-\gamma^{-1}}{N+1} \in [0, \pi], \xi_{s2} \\ = \cos^{-1}(-1) \frac{N-1+N\gamma^{-1}}{N+1} \in [0, \pi]. \quad (83)$$

For real ξ_{s1} , it is required that $-1 < (N-1-\gamma^{-1})/(N+1) < 1$, which can be reduced to the equivalent condition $\gamma > \frac{1}{2}N^{-1} =: \gamma_*$. Similarly, for real ξ_{s2} , it is required that $-1 < (N-1+N\gamma^{-1})/(N+1) < 1$ which also reduces to an equivalent condition $\gamma > \frac{1}{2}N =: \gamma^*$. Thus, as $N \rightarrow \infty$, and the surface wave dispersion relation for the semi-infinite case exists only for $\gamma > 0$. Thus, $-\frac{1}{2} < \gamma \leq 0$ implies no such surface wave dispersion relation for the semi-infinite case as observed earlier for $\gamma = 0$.

Assuming $\gamma_* < \gamma < \gamma^*$, a critical value of ξ , a ‘threshold’ in the surface wave dispersion relation is given by $\vartheta = \cos \eta_\kappa = +1$, assuming $\gamma \neq 0$. Then one of the surface modes exists only for $\xi \in (\xi_{s1}, \pi] \cup [-\pi, -\xi_{s1})$, while the other surface mode exists for all $\xi \in [-\pi, \pi] \setminus \{0\}$. In the alternate case, assuming $\gamma > \frac{1}{2}N$, there are two critical points (thresholds) as both ξ_{s1}, ξ_{s2} are real. Thus, one of the surface modes exists for $\xi \in [-\xi_{s2}, \xi_{s2}]$ while another surface mode exists for $\xi \in [\xi_{s1}, \pi] \cup [-\pi, -\xi_{s1}]$. Analogous to the triangular lattice (section 4.2c), for the odd modes (71) in case of $\hat{\mathfrak{E}}_o^\circ$ with $N = 2N$, it is found that $W_N(+1) = (2N+1)$, $W_N(-1) = (-1)^N$; again it is easy to see that either $\cos \xi = (2N-1-\gamma^{-1})/(2N+1)$ or $\cos \xi = -1 - \gamma^{-1}$, so that one of the two surface modes is odd while the other is even. Notice that the surface mode dispersion curves in figure 7b lie entirely in the stop band [12, 54].

Remark 12 In the context of figure 7b, it is worth observing that the first choice $\gamma = -0.25 > -\frac{1}{2}$, the second choice $\gamma = 0.25 \in (\gamma_*, \gamma^*)$ but $\gamma < \frac{1}{2}$ (recall Remark 11), the third choice $\gamma = 1 \in (\gamma_*, \gamma^*)$ but $\gamma > \frac{1}{2}$, and, finally, the fourth choice $\gamma = 7.25 > \gamma^*$. Note that for $N = 6$, $\gamma_* \approx 0.08$ and $\gamma^* = 3$.

5.3 Fixed-free boundary $\hat{\mathfrak{E}}_o^\bullet$

Consider $\hat{\mathfrak{E}}_o^\bullet$ as shown in figure 6. The equation of motion for the lattice row at $y = 1$ is (61a), while at $y = N$ it is (62b). Naturally, there is a close connection

with the results established in section 4.3 for $\hat{\mathfrak{L}}^\bullet$. It is found that (10) holds for the normal modes, where $\mathbf{A}_N$ has the form (9) with definitions provided in (64) and $\delta_{NE} = \delta_{SW} = 0, \delta_{NW} = 0, \delta_{SE} = -1 - 2\gamma$. Applying the results of [61], i.e., using (13), in this case for $\kappa \in \mathbb{Z}_1^N$, η_κ satisfies $\Gamma(\xi) \sin(N+1)\eta_\kappa - \Gamma_0 \sin N\eta_\kappa = 0$. Using the Chebyshev polynomials U_n of second kind (20), the same condition is re-written as

$$\Gamma(\xi)U_N(\vartheta) - \Gamma_0 U_{N-1}(\vartheta) = 0. \quad (84)$$

The dispersion relations are determined with $A = \Omega_\kappa^2(\xi) = \Omega_\xi^2(\xi, \eta_\kappa)$, which are illustrated by figure 3c, along with figure 8 (right). The index κ is placed in such a way that $\Omega_\kappa(0)$ is an increasing sequence with respect to κ . Recall Remark 11 as well in the context of the appearance of a special point for ξ . As before, each eigenvector may be found in the same manner as before; in particular, it is a constant multiple of the vector with components $\sin v\eta_\kappa$. Normalizing the same (Appendix 2.2), it is found that for $\kappa \in \mathbb{Z}_1^N$,

$$a_{(\kappa)v} = C_N \sin v\eta_\kappa, v \in \mathbb{Z}_1^N, \text{ where} \\ C_N^{-2} = \frac{1}{2} \left(N + \frac{1}{2} - 2 \frac{\gamma(1 - \cos \xi)(1 + \gamma(1 + \cos \xi))}{\Gamma_0^2} U_N^2 \right). \quad (85)$$

Analogous to (70), the function $\omega^2 = \Omega_\kappa^2(\xi)$ is also implicitly given by

$$\prod_{j=1}^N \left(4(1 + \gamma) - 2 \cos \xi - \Omega_\kappa^2 - \Gamma(\xi) \cos \frac{j}{N+1} \pi \right) \\ - \Gamma_0 \prod_{j=1}^{N-1} \left(4(1 + \gamma) - 2 \cos \xi - \Omega_\kappa^2 - \Gamma(\xi) \cos \frac{j}{N} \pi \right) = 0. \quad (86)$$

Using (72), when $\Gamma(\xi) \neq 0$, η_κ can be obtained from $\cos \eta_\kappa(\xi) = (4(1 + \gamma) - 2 \cos \xi - \Gamma_0 \omega^2) / (2\Gamma(\xi))$. Indeed, η_κ is real provided it does not correspond to a surface mode (localized wave along y). For $\Gamma(\xi) = 0$, (84) can be used directly to calculate η_κ . An application of the second method, as described in section 5.2, yields the same expressions.

5.3a Surface modes: A surface wave exists below the pass band envelope (illustrated by grey-coloured shaded region in Figure 3(c)). Clearly, a critical value of ξ that signals the threshold for surface wave dispersion relation satisfies $\cos \eta_\kappa(\xi) = \pm 1$. Using (102b)₃, i.e. $U_N(\pm 1) = (N+1)(-1)^{(\frac{1}{2} \mp \frac{1}{2})N}$, it is found that $\Gamma(\xi)(N+1) = \pm \Gamma_0 N$, so that either $\cos \xi = (N - \frac{1}{2}\gamma^{-1}) / (N+1)$ or $\cos \xi = (-N - (\frac{1}{2} + N)\gamma^{-1}) / (N+1)$. Analogous to the case of $\hat{\mathfrak{E}}^\circ$, both critical values of ξ approach $0, \pi$, as $N \rightarrow \infty$. Let

$$\xi_{s1} = \cos^{-1} \frac{N - \frac{1}{2}\gamma^{-1}}{N+1} \in [0, \pi], \xi_{s2} = \cos^{-1} \frac{-N - (\frac{1}{2} + N)\gamma^{-1}}{N+1} \in [0, \pi]. \quad (87)$$

For real ξ_{s1} , it is required that $-1 < (N - \frac{1}{2}\gamma^{-1})(N+1) < 1$, which can be reduced to the equivalent condition $\gamma > \frac{1}{2}(2N+1)^{-1} =: \gamma_{**}$. Similarly, for real ξ_{s2} , it is required that $-1 < (N + (\frac{1}{2} + N)\gamma^{-1})(N+1) < 1$, which also reduces to an equivalent condition $\gamma > \frac{1}{2}(2N+1) =: \gamma^{**}$. Thus, as $N \rightarrow \infty$, the surface wave dispersion relation for the semi-infinite case exists only for $\gamma > 0$. The condition $-\frac{1}{2} < \gamma \leq 0$ implies no such surface wave dispersion relation for the semi-infinite case as observed earlier for $\gamma = 0$.

Assuming that γ satisfies $\gamma_{**} < \gamma < \gamma^{**}$, it is clear that a critical value of ξ that signals the threshold for surface wave dispersion relation satisfies $\cos \eta_\kappa(\xi) = +1$. Then only one surface mode exists for fixed-free boundary when $\xi \in [-\pi, -\xi_s) \cup (\xi_s, \pi]$. Assuming that $\gamma > \gamma^{**}$, there is only one surface wave mode and one part of its dispersion curve exists when $\xi \in [\xi_{s1}, \xi_{s2}] \cup [-\xi_{s2}, -\xi_{s1}]$ with $\cos \eta_\kappa(\xi) = +1$, while another part exists when $\xi \in [-\pi, -\xi_{s2}] \cup [\xi_{s2}, \pi]$ with $\cos \eta_\kappa(\xi) = -1$; see the region between abruptly ending dispersion curves on the contour in figure 8 (right) and corresponding dashed portions in dispersion curves in figure 7c for the latter. Notice that the surface mode dispersion curves in 7(c) lie entirely in the stop band as expected [12, 54].

Remark 13 Apart from the Remark 12 in the previous case, in the context of figure 7c, it is worth observing that the second choice $\gamma = 0.25 \in (\gamma_{**}, \gamma^{**})$, the third choice $\gamma = 1 \in (\gamma_{**}, \gamma^{**})$ and, finally, the fourth choice $\gamma = 7.25 > \gamma^{**}$; indeed, $\gamma_{**} \approx 0.39$ and $\gamma^{**} = 6.5$ for $N = 6$.

5.4 Periodic boundary $\hat{\mathfrak{E}}^\circ$

Due to the periodic boundary condition, (61d) and (62d) hold at upper and lower boundary rows, respectively. It follows that (10) holds, where $\mathbf{A}_N$ has the form (9) with (64) and $\delta_{NE} = \delta_{SW} = \gamma, \delta_{NW} = \delta_{SE} = 0$; in particular, the eigenpairs are found to be the same as in (30). An alternative derivation of this case is identical to that for the square lattice with nearest-neighbour interactions only in section 3.4 (except for replacing the expression of λ by that provided for the lattice model of this section). The expression (31) has been plotted and shown in figure 3d marking the doubly repeated dispersion relations (except for the uppermost and lowermost). Recall Remark 11 as well in the context of $\gamma > \frac{1}{2}$. As a special case, following Remark 11, note that $\gamma = 1$ is associated with a flat band; in general, there is a possibility of a flat band whenever $\cos \frac{2\kappa\pi}{N} = -\frac{1}{2\gamma}$ for $\gamma > \frac{1}{2}$, i.e., $\frac{N}{2\pi} \arccos(-\frac{1}{2\gamma})$ is an integer.

When $N = 2N$, the analysis and the resulting expressions for the odd modes and even modes are identical to those for the triangular lattice (sections 4.4a and 4.4b); the details are omitted. The case $N = 2N + 1$ is considered later.

5.4a ‘Odd’ modes: For odd modes, (10) holds where $\mathbf{A}_N$ has the form (9) with definitions provided in (64) and $\delta_{NE} = \delta_{SW} = 0, \delta_{NW} = -\mathcal{T}, \delta_{SE} = 0$. This results in the condition $\mathcal{T}^2 \sin(N+1)\eta_\kappa + \mathcal{T}^2 \sin N\eta_\kappa = 0$, which is, in terms of the Chebyshev polynomials of second kind,

$$U_N(\vartheta) + U_{N-1}(\vartheta) = 0, \vartheta = \cos \eta_\kappa. \quad (88)$$

The same relation, in terms of the Chebyshev polynomials of fourth kind, is $W_N(\vartheta) = 0$. Hence, $\eta_\kappa = \kappa\pi/(N + \frac{1}{2})$, κ lies in $\mathbb{Z}_1^N$ and $a_{(\kappa)N+1+v} = \sqrt{2/(2N+1)} \sin v\eta_\kappa, a_{(\kappa)v+1} = -a_{(\kappa)2N+1-v}, v \in \mathbb{Z}_0^N$.

5.4b ‘Even’ modes: For even modes in $\hat{\mathcal{E}}^\odot$ with $N = 2N + 1$, it is required that $a_v = a_{2N+2-v}$. Using (16), due to the periodic boundary condition at $v = 1$, and the symmetry at $v = N + 1$, it is required that $C_1\lambda + C_2\lambda^{-1} = C_1 + C_2$ as well as $C_1\lambda^N + C_2\lambda^{-N} = C_1\lambda^{N+2} + C_2\lambda^{-N-2}$. Thus, non-trivial C_1, C_2 exist provided $(\lambda - 1)(\lambda^{-N} - \lambda^{-N-2}) - (\lambda^{-1} - 1)(\lambda^N - \lambda^{N+2}) = 0$. Equivalently, using the Chebyshev polynomials U_n of second kind, the condition is $U_N(\vartheta) + U_{N-1}(\vartheta) - U_{N-2}(\vartheta) - U_{N+2}(\vartheta) = 0$, i.e.,

$$2(\vartheta - 1)(U_N(\vartheta) + U_{N-1}(\vartheta)) = 0, \quad (89)$$

after an application of (102a)₁. Using the Chebyshev polynomials W_n of fourth kind, the dispersion relation can be also expressed as $(\vartheta - 1)W_N(\vartheta) = 0, \vartheta = \cos \eta_\kappa$. Hence, $\eta_\kappa = \kappa\pi/(N + \frac{1}{2})$, κ lies in $\mathbb{Z}_0^N$ and the eigenvectors are given by $a_{(\kappa)N+1+v} = ((\frac{1}{\sqrt{2}} - 1)\delta_{\kappa,0} + 1)\sqrt{2/(2N+1)} \cos v\eta_\kappa, a_{(\kappa)v+1} = a_{(\kappa)2N+1-v}, v \in \mathbb{Z}_0^N$.

Remark 14 The reasoning analogous to Remark 8 holds for the periodic boundary condition on $\hat{\mathcal{E}}_N$ also. Note that Remark 4 holds for the normal modes (including the surface modes wherever they exist) of all square lattice waveguides stated in table 1.

6. Discussion and related analysis

6.1 Even–odd factors of dispersion relations for symmetric boundary conditions

In case the upper and lower boundary conditions are identical in the lattice waveguides, the forms of the appearance of the dispersion relations for even and odd modes, in each case studied earlier in the paper, are not unrelated. In fact, these results provide a factorization of

the expression of the complete dispersion relation for the lattice waveguide, in terms of Chebyshev polynomials, as a product of two expressions, one of which corresponds to the even modes and other to the odd modes (similar to the identities stated as part of the expansion of the Wiener–Hopf kernels encountered in [26]). In the following, the argument ϑ (21) (with (18), (44) and (68)) of the Chebyshev polynomials is suppressed, and η is used in place of η_κ , for brevity.

1. $\mathcal{E}^\bullet, \mathcal{T}^\bullet$ and $\hat{\mathcal{E}}^\bullet$: using (22b) and (23b) for $N = 2N$,

$$(U_N - U_{N-1})(U_N + U_{N-1}) = V_N W_N = U_{2N} = U_N \text{ (by (103a))}, \quad (90)$$

while, using (22a) and (23a) for $N = 2N + 1$,

$$(U_{N+1} - U_{N-1})U_N = 2T_{N+1}U_N = U_{2N+1} = U_N \text{ (by (103b))}, \quad (91)$$

as expected with reference to (15).

2. $\mathcal{E}^\circ$: Indeed, using (26c) and (27a) for $N = 2N$,

$$U_{N-1}(U_N - U_{N-2})U_N = 2T_N U_{N-1} = U_{2N-1} = U_{N-1} \text{ (by (103b))}, \quad (92)$$

while using (26a) and (27b) for $N = 2N + 1$,

$$(U_N + U_{N-1})(U_N - U_{N-1}) = W_N V_N = U_{2N} = U_{N-1} \text{ (by (103a))}, \quad (93)$$

as expected with reference to (25).

3. $\mathcal{T}^\circ$: Using (48) and (43) for $N = 2N$,

$$\begin{aligned} & (\cos \xi V_N - V_{N-1})(\cos \xi W_N - W_{N-1}) \\ &= \cos^2 \xi V_N W_N - \cos \xi (V_{N-1} W_N + V_N W_{N-1}) + V_{N-1} W_{N-1} \\ &= \cos^2 \xi U_N + U_{N-2} - 2 \cos \xi U_{N-1} \text{ (by (103a) and (103c))}, \end{aligned} \quad (94)$$

while using (50) and (46) for $N = 2N + 1$,

$$\begin{aligned} & 2(\cos \xi T_{N+1} - T_N)(\cos \xi U_N - U_{N-1}) \\ &= 2 \cos^2 \xi T_{N+1} U_N - 2 \cos \xi (T_N U_N + T_{N+1} U_{N-1}) + 2 T_N U_{N-1} \\ &= \cos^2 \xi U_N + U_{N-2} - 2 \cos \xi U_{N-1} \text{ (by (103b) and (103d))}, \end{aligned} \quad (95)$$

as expected with reference to (39).

4. $\hat{\mathcal{E}}^\circ$: Using (77) and (71) for $N = 2N$,

$$\begin{aligned} & (\Gamma(\xi) V_N - \Gamma_0 V_{N-1})(\Gamma(\xi) W_N - \Gamma_0 W_{N-1}) \\ &= \Gamma(\xi)^2 V_N W_N - \Gamma(\xi) \Gamma_0 (V_{N-1} W_N + V_N W_{N-1}) + \Gamma_0^2 V_{N-1} W_{N-1} \\ &= \Gamma(\xi)^2 U_N + \Gamma_0^2 U_{N-2} - 2 \Gamma_0 \Gamma(\xi) U_{N-1} \text{ (by (103a) and (103c))}, \end{aligned} \quad (96)$$

while using (79) and (74) for $N = 2N + 1$,

$$\begin{aligned}
& 2(\Gamma(\xi)T_{N+1} - \Gamma_0 T_N)(\Gamma(\xi)U_N - \Gamma_0 U_{N-1}) \\
& = 2\Gamma(\xi)^2 T_{N+1} U_N - 2\Gamma(\xi)\Gamma_0(T_N U_N + T_{N+1} U_{N-1}) + 2\Gamma_0^2 T_N U_{N-1} \\
& = \Gamma(\xi)^2 U_N + \Gamma_0^2 U_{N-2} - 2\Gamma_0 \Gamma(\xi) U_{N-1} \text{ (by (103b) and (103d))},
\end{aligned} \tag{97}$$

as expected with reference to (67).

5. $\mathfrak{S}_\odot$, $\mathfrak{T}_\odot$ and $\hat{\mathfrak{S}}_\odot$: Using (57) and (56) for $N = 2N$,

$$\begin{aligned}
4(\vartheta^2 - 1)U_{N-1}^2 &= U_{2N} - U_{2N-2} - 2 \\
&= U_N - U_{N-2} - 2 \text{ (by (103e) and (102b)}_1),
\end{aligned} \tag{98}$$

while using (89) and (88) for $N = 2N + 1$,

$$\begin{aligned}
2(\vartheta - 1)(U_N + U_{N-1})^2 &= U_{2N+1} - U_{2N-1} - 2 \\
&= U_N - U_{N-2} - 2 \text{ (by (103f) and (102b)}_1),
\end{aligned} \tag{99}$$

as expected with reference to (31).

6.2 Features of $\hat{\mathfrak{S}}_N$ and its relation with the first two models

As discussed in the context of the square lattice model with next-nearest-neighbour interactions in section 5, there exist certain special values of the parameter γ . Summarizing these observations, the set of critical values of γ are

1. $\gamma = -\frac{1}{2}$: $\gamma \leq -\frac{1}{2}$ leads to an unphysical situation when the shear wave speed is imaginary or zero;
2. $\gamma = \frac{1}{2}$: $\gamma \geq \frac{1}{2}$ is associated with the appearance of a feature similar to that for the triangular lattice encountered before; in this context ξ_* defined in Remark 11 can be inspected in figure 8;
3. $\gamma = \gamma_{**} = \frac{1}{2}(2N+1)^{-1}$: $\gamma > \gamma_{**}$ is associated with the surface waves in the waveguide with fixed-free boundary conditions with dispersion curve of type shown in figure 7(c) for $\gamma = 1$;
4. $\gamma = \gamma_* = \frac{1}{2}N^{-1}$: $\gamma > \gamma_*$ is associated with the surface waves in the waveguide with free boundary conditions with dispersion curve of type shown in figure 7b for $\gamma = 1$;
5. $\gamma = \gamma^* = \frac{1}{2}N$: $\gamma > \gamma^*$ is associated with the surface waves in the waveguide with free boundary conditions with dispersion curve of type shown in figure 7b for $\gamma = 7.25$;
6. $\gamma = \gamma^{**} = \frac{1}{2}(2N+1)$: $\gamma > \gamma^{**}$ is associated with the surface waves in the waveguide with fixed-free boundary conditions with dispersion curve of type shown in figure 7(c) for $\gamma = 7.25$.

As $\gamma \rightarrow 0$, the earlier results for the square lattice model (with nearest-neighbour interactions only) can be easily recovered. Moreover, and quite elegantly, for $\gamma \gg 1$, it is also easy to recognize the relevance of the results for

the triangular lattice model (with nearest-neighbour interactions only). Thus, the square lattice model with nearest-neighbour and next-nearest-neighbour interactions can be interpreted as an intermediate case. A relook at the schematic in figures 2 and 6 for the square and triangular lattice models reveals the same immediately.

6.3 Expressions for the dispersion relations

Following the manipulations leading to the type of expressions (42), (70), etc, in order to obtain alternate expressions for the dispersion relations stated so far, consider the definition, for given integers j and n :

$$\begin{aligned}
& F_{j,n}(\xi) \\
& = \begin{cases} 4 - 2\cos\xi - \omega^2 - 2\cos\frac{j}{n+1}\pi & \text{for } \mathfrak{S}_\mathfrak{R} \text{ in section 2,} \\ 6 - 2\cos 2\xi - \frac{3}{2}\omega^2 - 4\cos\xi\cos\frac{j}{n+1}\pi & \text{for } \mathfrak{T}_\mathfrak{R} \text{ in section 3,} \\ 4(1+\gamma) - 2\cos\xi - \Gamma_0\omega^2 - 2\Gamma(\xi)\cos\frac{j}{n+1}\pi & \text{for } \hat{\mathfrak{S}}_N \text{ in section 4,} \end{cases}
\end{aligned} \tag{100}$$

where $\kappa = 1, 2, \dots, N$. Using the expansion of Chebyshev polynomials [56, 58], the dispersion relations for all the four cases (excluding the even-odd cases) can be simplified further; a summary of the resulting expressions is provided in table 3.

7. Concluding remarks

The paper presents an analysis of the propagating modes of linear waves in several infinite lattice waveguides of square and triangular structure. The dispersion relations, the normal modes and detailed specifications of the surface waves are included. Some of the expressions stated in the paper are explicit, especially for the square lattice waveguides and most of the triangular lattice waveguides, while for some triangular lattice waveguides, these expressions are provided in terms of the roots of various kinds of linear combinations of the Chebyshev polynomials. The influence of next-nearest-neighbour interactions is studied in the context of square lattice waveguides, where the corresponding mathematical formulation and analysis is found to be a natural extension of that for the square lattice and the triangular lattice with nearest-neighbour interactions. The techniques used in the paper can be applied and extended to other situations, such as bimaterial strips, etc, as well as for more realistic lattice models. It is possible to apply the results stated in this paper on a varied set of problems arising in diverse physical phenomena, including the current scientific and technological interests that involve the motion of crystalline solids at small scales.

Table 3. Summary for $\mathfrak{S}_N$, $\mathfrak{T}_N$ and $\hat{\mathfrak{S}}_N$ (see also tables 1 and 2) using (100).

S. no.	Strip	Dispersion relation	Modes
1	$\mathfrak{S}^\bullet$	$\prod_{j=1}^N F_{j,N}(\xi) = 0$	(14b)
2	$\mathfrak{S}^\circ$	$(2 - 2 \cos \xi - \omega^2) \prod_{j=1}^N F_{j,N}(\xi) = 0$	(24)
3	$\mathfrak{S}^\circ$	$\prod_{j=1}^N F_{j,N}(\xi) - \prod_{j=1}^{N-1} F_{j,N-1}(\xi) = 0$	(29)
4	$\mathfrak{S}^\ominus$	$\prod_{j=1}^N F_{j,N}(\xi) - 4 \prod_{j=1}^{N-2} F_{j,N-2}(\xi) - 2^{N+1} = 0$	(30)
1	$\mathfrak{T}^\bullet$	$\prod_{j=1}^N F_{j,N}(\xi) = 0$	(38)
2	$\mathfrak{T}^\circ$	$\prod_{j=1}^N F_{j,N}(\xi) + 4 \prod_{j=1}^{N-2} F_{j,N-2}(\xi) - 4 \prod_{j=1}^{N-1} F_{j,N-1}(\xi) = 0$	(41)
3	$\mathfrak{T}^\circ$	$\prod_{j=1}^N F_{j,N}(\xi) - 2 \prod_{j=1}^{N-1} F_{j,N-1}(\xi) = 0$	(54)
4	$\mathfrak{T}^\ominus$	$\prod_{j=1}^N F_{j,N}(\xi) - (2 \cos \xi)^2 \prod_{j=1}^{N-2} F_{j,N-2}(\xi) - 2(2 \cos \xi)^N = 0$	(30)
1	$\hat{\mathfrak{S}}^\bullet$	$\prod_{j=1}^N F_{j,N}(\xi) = 0$	(66c)
2	$\hat{\mathfrak{S}}^\circ$	$\prod_{j=1}^N F_{j,N}(\xi) + \Gamma_0^2 \prod_{j=1}^{N-2} F_{j,N-2}(\xi) - 2\Gamma_0 \prod_{j=1}^{N-1} F_{j,N-1}(\xi) = 0$	(69)
3	$\hat{\mathfrak{S}}^\circ$	$\prod_{j=1}^N F_{j,N}(\xi) - \Gamma_0 \prod_{j=1}^{N-1} F_{j,N-1}(\xi) = 0$	(85)
4	$\hat{\mathfrak{S}}^\ominus$	$\prod_{j=1}^N F_{j,N}(\xi) - \Gamma(\xi)^2 \prod_{j=1}^{N-2} F_{j,N-2}(\xi) - 2\Gamma(\xi)^N = 0$	(30)

Several applications of the expressions catalogued in this paper appear in a series of forthcoming papers by the author on dislocation kinetics in lattice strips, mode III brittle fracture of lattice strips (for example, [24]), wave propagation in bifurcated lattice waveguides [25, 26] and scattering by an infinite array of parallel semi-infinite cracks and rigid constraints. The results analogous to those presented in this paper for the honeycomb lattice structure appear in [85].

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Appendix 1: Chebyshev polynomials

Let n be a positive integer. Recall that [56–58]

$$\begin{aligned}
 T_n(\vartheta) &= \frac{1}{2}((\vartheta + \sqrt{\vartheta^2 - 1})^n + (\vartheta - \sqrt{\vartheta^2 - 1})^n), U_n(\vartheta) \\
 &= \frac{1}{2}((\vartheta + \sqrt{\vartheta^2 - 1})^n - (\vartheta - \sqrt{\vartheta^2 - 1})^n)/\sqrt{\vartheta^2 - 1}, \\
 V_n(\vartheta) &= (w^{n+1} + w^{-n})/(w + 1), W_n(\vartheta) \\
 &= (w^{n+1} - w^{-n})/(w - 1), \vartheta = \frac{1}{2}(w + w^{-1}).
 \end{aligned}
 \tag{101}$$

The following identities are useful also:

$$\begin{aligned}
 U_n &= 2\vartheta U_{n-1} - U_{n-2}, 2T_n = V_n + V_{n-1}, V_n = U_n - U_{n-1}, \\
 W_n &= U_n + U_{n-1},
 \end{aligned}
 \tag{102a}$$

$$2T_n = U_n - U_{n-2}, U_n^2 = 1 + U_{n-1}U_{n+1}, U_n(\pm 1) = (n+1)(-1)^{(\frac{n}{2}+\frac{1}{2})n}.
 \tag{102b}$$

A few more identities can be proved [58] as follows:

$$V_n W_n = \frac{\cos(n + \frac{1}{2})\eta \sin(n + \frac{1}{2})\eta}{\cos \frac{1}{2}\eta \sin \frac{1}{2}\eta} = \frac{\sin(2n+1)\eta}{\sin \eta} = U_{2n},
 \tag{103a}$$

$$2T_{n+1}U_n = 2\cos(n+1)\eta \frac{\sin(n+1)\eta}{\sin \eta} = \frac{\sin(2n+2)\eta}{\sin \eta} = U_{2n+1}, \quad (103b)$$

$$\begin{aligned} V_{n-1}W_n + V_nW_{n-1} &= \frac{\cos(n-\frac{1}{2})\eta}{\cos \frac{1}{2}\eta} \frac{\sin(n+\frac{1}{2})\eta}{\sin \frac{1}{2}\eta} \\ &+ \frac{\sin(n-\frac{1}{2})\eta}{\sin \frac{1}{2}\eta} \frac{\cos(n+\frac{1}{2})\eta}{\cos \frac{1}{2}\eta} = U_{2n-1}, \end{aligned} \quad (103c)$$

$$\begin{aligned} T_nU_n + T_{n+1}U_{n-1} &= \cos n\eta \frac{\sin(n+1)\eta}{\sin \eta} \\ &+ \cos(n+1)\eta \frac{\sin n\eta}{\sin \eta} = U_{2n}, \end{aligned} \quad (103d)$$

$$U_{n-1}^2 = \frac{\sin^2 n\eta}{\sin^2 \eta} = \frac{1}{2} \frac{1 - \cos 2n\eta}{1 - \cos^2 \eta} = -\frac{1}{2} \frac{1 - T_{2n}}{(\vartheta^2 - 1)}, \quad (103e)$$

$$W_n^2 = \frac{\sin^2(n+\frac{1}{2})\eta}{\sin^2 \frac{1}{2}\eta} = \frac{1 - \cos(2n+1)\eta}{1 - \cos \eta} = -\frac{1 - T_{2n+1}}{(\vartheta - 1)}. \quad (103f)$$

Appendix 2: Auxiliary identities and some derivations

Using the elementary properties of trigonometric functions and geometric series,

$$\sum_{j=1}^N \sin j\eta = -i \frac{1}{2} \left(\sum_{j=1}^N e^{ij\eta} - \sum_{j=1}^N e^{-ij\eta} \right) = \frac{\sin(N+1)\frac{1}{2}\eta \sin N\frac{1}{2}\eta}{\sin \frac{1}{2}\eta}, \quad (104a)$$

$$\begin{aligned} \text{and } \sum_{j=1}^N \cos j\eta &= \frac{1}{2} \left(\sum_{j=1}^N e^{ij\eta} + \sum_{j=1}^N e^{-ij\eta} \right) \\ &= \frac{\cos(N+1)\frac{1}{2}\eta \sin N\frac{1}{2}\eta}{\sin \frac{1}{2}\eta}, \end{aligned} \quad (104b)$$

so that

$$\sum_{j=1}^N \sin^2 j\eta = \frac{1}{2} \sum_{j=1}^N (1 - \cos 2j\eta) = \frac{1}{2}N - \frac{1}{2} \frac{\cos(N+1)\eta \sin N\eta}{\sin \eta}, \quad (105a)$$

$$\sum_{j=1}^N \cos^2 j\eta = \frac{1}{2} \sum_{j=1}^N (1 + \cos 2j\eta) = \frac{1}{2}N + \frac{1}{2} \frac{\cos(N+1)\eta \sin N\eta}{\sin \eta}, \quad (105b)$$

$$\begin{aligned} \sum_{j=1}^N \cos^2 \left(j - \frac{1}{2} \right) \eta &= \frac{1}{2} \sum_{j=1}^N (1 + \cos(2j-1)\eta) \\ &= \frac{1}{2}N + \frac{1}{2} \frac{\cos N\eta \sin N\eta}{\sin \eta}. \end{aligned} \quad (105c)$$

In terms of Chebyshev polynomials [56–58], with $\cos \eta = \vartheta$ as the argument (suppressed), using (102b)₁,

$$\sum_{j=1}^N \sin^2 j\eta = \frac{1}{2}N - \frac{1}{2}T_{N+1}U_{N-1} = \frac{1}{2}N - \frac{1}{4}(U_{N+1} - U_{N-1})U_{N-1}, \quad (106a)$$

$$\sum_{j=1}^N \cos^2 j\eta = \frac{1}{2}N + \frac{1}{2}T_{N+1}U_{N-1} = \frac{1}{2}N + \frac{1}{4}(U_{N+1} + U_{N-1})U_{N-1}, \quad (106b)$$

$$\sum_{j=1}^N \cos^2 \left(j - \frac{1}{2} \right) \eta = \frac{1}{2}N + \frac{1}{4}(U_N + U_{N-2})U_{N-1}. \quad (106c)$$

Also,

$$\begin{aligned} \sum_{v=1}^N \sin j\eta \sin(j-1)\eta &= \cos \eta \sum_{v=1}^N \sin^2 j\eta - \frac{1}{2} \sin \eta \sum_{v=1}^N \sin 2j\eta \\ &= \frac{1}{2}N \cos \eta - \frac{1}{4} \cos \eta (U_{N+1} - U_{N-1})U_{N-1} \\ &- \frac{1}{2} \sin^2 \eta U_N U_{N-1} \text{ (by (106a), (104a))} \\ &= \frac{1}{2} \left(N \cos \eta - \frac{1}{2} U_{N-1} (U_N + U_{N-2}) \right) \text{ (by (102a)₁).} \end{aligned} \quad (106d)$$

In the following also, the argument ϑ (21) (with (18), (44), and (68)) of the Chebyshev polynomials is suppressed, and η is used in place of η_k , for brevity.

Appendix 2.1: $\mathfrak{S}_N$

By (106a), the normalization constant in (14b) can be obtained as

$$\sum_{v \in \mathbb{Z}_1^N} \sin^2 v\eta = \frac{1}{2}N + \frac{1}{2}U_{N-1}^2 \text{ (by (102a)₁ and (15))}, \quad (107)$$

followed by (102b)₂ and (15). By (106c), the normalization constant in (24) follows as

$$\begin{aligned} \sum_{v \in \mathbb{Z}_1^N} \cos^2 \left(v - \frac{1}{2} \right) \eta &= \frac{1}{2}N \text{ (by (25) for } \vartheta \neq 1, \\ &\text{with (102b)₃ for } \vartheta = 1). \end{aligned} \quad (108)$$

By (106c), the normalization constant in (29) follows as

$$\begin{aligned} \sum_{v \in \mathbb{Z}_1^N} \cos^2 \left(v - \frac{1}{2} \right) \eta &= \frac{1}{2}N + \frac{1}{4}(U_N^2 + 1 - U_{N-1}^2) \\ &\text{(by (28) and (102b)₃),} \end{aligned} \quad (109)$$

leading to (29) after applying (28).

Appendix 2.2: $\mathfrak{T}_N$

The normalization constant in (38) follows from (107). Using (106a), (106b) and (106d), the normalization constant in (41) is given by

$$\begin{aligned} C_N^{-2} &= \cos^2 \xi \sum_{v=1}^N \sin^2 v\eta + \sum_{v=1}^N \sin^2(v-1)\eta \\ &\quad - 2 \cos \xi \sum_{v=1}^N \sin v\eta \sin(v-1)\eta \\ &= \frac{1}{2}N(1 + \cos^2 \xi - 2\vartheta \cos \xi) - \frac{1}{2} + S, \text{ where} \\ S &= -\frac{1}{4} \cos^2 \xi (U_{N+1} - U_{N-1})U_{N-1} - \frac{1}{4}(U_N - U_{N-2})U_{N-2} \\ &\quad + \frac{1}{2} \cos \xi U_{N-1}(U_N - U_{N-2}). \end{aligned} \quad (110a)$$

The expression of S can be simplified as

$$\begin{aligned} S &= -\frac{1}{4} \cos^2 \xi ((U_{N+1} - U_{N-1})U_{N-1} - U_N(U_N - U_{N-2})) \text{ (by (39))} \\ &= -\frac{1}{4} \cos^2 \xi ((U_{N+1} - U_{N-1})U_{N-1} \\ &\quad - (1 + U_{N-1}U_{N+1} - U_N U_{N-2})) \text{ (by (102b)}_3) \\ &= \frac{1}{2} \cos^2 \xi \text{ (again by (102b)}_3). \end{aligned} \quad (110b)$$

The normalization constant in (45) is derived as follows. By (43) and (102a)₁,

$$(1 + \cos \xi)U_N + (-\vartheta - 1 + \cos \xi)U_{N-1} = 0. \quad (111a)$$

Employing (106c),

$$\begin{aligned} C_N^{-2} &= 2 \sum_{v=1}^N \cos^2 \left(v - \frac{1}{2}\right) \eta = N + \frac{1}{2}(1 - \cos \xi) \\ &\quad (U_N + U_{N-1})U_{N-1} \text{ (by (43))} \\ &= N + \frac{1}{2} \frac{1 - \cos \xi}{1 + \cos \xi} (\vartheta + 2)U_{N-1}^2 \text{ (by (111a))}. \end{aligned} \quad (111b)$$

Using (106a) and (46), the normalization constant in (47) can be expressed as

$$\begin{aligned} C_N^{-2} &= 2 \sum_{v=1}^N \sin^2 v\eta = N - \frac{1}{2}(U_{N+1} - U_{N-1}) \cos \xi \\ U_N &= N - \frac{1}{2} \cos \xi (\vartheta - 2 \cos \xi)U_N^2 \text{ (by (102a)}_1) \\ &= N - \frac{1}{2} (\vartheta \cos \xi - \cos^2 \xi - 1 + \sin^2 \xi)U_N^2. \end{aligned} \quad (112)$$

By (102a)₁, using (46), $U_{N+1} = (\vartheta - \cos \xi)U_N$; also by (102b)₃, using (46), $U_N^2 = 1 + (\vartheta - \cos \xi) \cos \xi U_N^2$, so that $(1 - \vartheta \cos \xi + \cos^2 \xi)U_N^2 = 1$. Hence, the normalization in

(47) follows. The normalization constant in (49) is considered in the same manner. By (48) and (102a)₁,

$$(\cos \xi - 1)U_N + (\vartheta - 1 - \cos \xi)U_{N-1} = 0, \quad (113a)$$

so that (106c) leads to

$$\begin{aligned} C_N^{-2} &= N + \frac{1}{2}(U_N - U_{N-1} + U_{N-1} - U_{N-2})U_{N-1} \\ &= N + \frac{1}{2} \frac{1 + \cos \xi}{1 - \cos \xi} (\vartheta - 2)U_{N-1}^2 \text{ (by (48) and (113a))}. \end{aligned} \quad (113b)$$

The normalization constant in (51) is derived below. Using (102a)₁ in (50),

$$(\vartheta \cos \xi - 2)U_N + (\vartheta - 2 \cos \xi)U_{N-1} = 0, \quad (114a)$$

while using (106b) and (50),

$$\begin{aligned} C_N^{-2} &= 1 + 2 \sum_{v=1}^N \cos^2 v\eta = 1 + N + \frac{1}{2}(U_{N+1} - U_{N-1})U_{N-1} \\ &= 1 + N + \frac{1}{2}(U_N^2 - 1 - U_{N-1}^2) \text{ (by (102b)}_3) \\ &= N + \frac{1}{2} + \frac{1}{2} \frac{(1 - \cos \xi)(1 + \cos \xi)}{(\vartheta \cos \xi - 2)^2} (\vartheta + 2) \\ &\quad (\vartheta - 2)U_{N-1}^2 \text{ (by (114a))}, \end{aligned} \quad (114b)$$

resulting in (51). The derivation of the normalization constant in (54) is almost identical to that presented for (47) (see (112)).

Appendix 2.3: $\hat{\mathfrak{S}}_N$

The normalization constant in (66c) follows from (107) as in the case of the corresponding triangular lattice waveguide. Following (110a), using (106a), (106b) and (106d), the normalization constant in (69) is given by

$$\begin{aligned} C_N^{-2} &= \frac{1}{2}N(\Gamma(\xi)^2 + \Gamma_0^2 - 2\vartheta \Gamma_0 \Gamma(\xi)) - \frac{1}{2}\Gamma_0 + S, \text{ where} \\ S &= -\frac{1}{4}\Gamma(\xi)^2(U_{N+1} - U_{N-1})U_{N-1} \\ &\quad - \frac{1}{4}\Gamma_0^2(U_N - U_{N-2})U_{N-2} + \frac{1}{2}\Gamma_0 \Gamma(\xi)U_{N-1}(U_N - U_{N-2}). \end{aligned} \quad (115)$$

The expression of S can be simplified as in the same way as (110b) to yield $S = \frac{1}{2}\Gamma(\xi)^2$. The normalization constant in (73) is derived as follows. By (71) and (102a)₁,

$$(2 + 2\gamma(1 + \cos \xi))U_N + (\Gamma(\xi) - \Gamma_0(\vartheta + 1))U_{N-1} = 0. \quad (116a)$$

Following (111b), employing (106c),

$$C_N^{-2} = N + \frac{1}{2} \Gamma_0^{-1} (\Gamma_0 - \Gamma(\xi)) (U_N + U_{N-1}) U_{N-1} \text{ (by (71))}, \quad (116b)$$

which leads to (73) using (116a). Following (112), using (106a) and (74), the normalization constant in (75) can be expressed as

$$C_N^{-2} = N - \frac{1}{2} \frac{\Gamma(\xi)}{\Gamma_0} (\varrho - 2 \frac{\Gamma(\xi)}{\Gamma_0}) U_N^2 \text{ (by (102a))}_1. \quad (117)$$

By (102a)₁, using (74), $U_{N+1} = (\varrho - \alpha) U_N$, $\alpha = \Gamma(\xi)/\Gamma_0$; also by (102b)₃, using (74), $U_N^2 = 1 + U_{N+1} U_{N-1} = 1 + (\varrho - \alpha) \alpha U_N^2$, so that $(1 - \varrho \alpha + \alpha^2) U_N^2 = 1$. Hence, (76) follows.

Following (113b), by (77) and (102a)₁,

$$2\gamma(1 - \cos \xi) U_N + (\Gamma(\xi) - \Gamma_0(\varrho - 1)) U_{N-1} = 0, \quad (118a)$$

so that (106c) leads to

$$\begin{aligned} C_N^{-2} &= N + \frac{1}{2} \Gamma_0^{-1} (\Gamma_0 + \Gamma(\xi)) (U_N - U_{N-1}) U_{N-1} \text{ (by (71))}, \\ &= N + \frac{1}{2} \frac{2 + 2\gamma(1 + \cos \xi)}{2\gamma(1 - \cos \xi)} (\varrho - 2) U_{N-1}^2 \text{ (by (118a))}, \end{aligned} \quad (118b)$$

yielding (78). The normalization constant in (80) is derived here. Using (102a)₁ in (79),

$$(\varrho \Gamma(\xi) - 2\Gamma_0) U_N + (\varrho \Gamma_0 - 2\Gamma(\xi)) U_{N-1} = 0, \quad (119a)$$

while following (114b), using (106b) and (79),

$$\begin{aligned} C_N^{-2} &= 1 + N + \frac{1}{2} (U_N^2 - 1 - U_{N-1}^2) \text{ (by (102b))}_3 \\ &= N + \frac{1}{2} + 2\gamma \frac{(1 - \cos \xi + \gamma \sin^2 \xi)}{(\varrho - 2 + 2\gamma(\varrho \cos \xi - 2))^2} \\ &\quad (\varrho + 2)(\varrho - 2) U_{N-1}^2 \text{ (by (119a))}, \end{aligned} \quad (119b)$$

resulting in (81). Similar to the triangular lattice, the normalization constant in (85) can be derived in almost an identical manner as that presented for (75) (see (117)).

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