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A hyperpower iterative method for computing the generalized Drazin inverse of Banach algebra element

SHWETABH SRIVASTAVA[1,*], DHARMENDRA K GUPTA[2], PREDRAG STANIMIROVIĆ[3], SUKHJIT SINGH[4] and FALGUNI ROY[2]

[1]Department of Mathematics, School of Arts & Sciences, Amrita Vishwa Vidyapeetham, Amrita University, Amritapuri, India
[2]Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur, India
[3]Faculty of Sciences and Mathematics, University of Niš, Visegradska 33, 18000 Nis, Serbia
[4]Department of Mathematics, NIT Hamirpur, Hamirpur, India
e-mail: shwetabhiit@gmail.com; shwetabhs@am.amrita.edu

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Abstract. A quadratically convergent Newton-type iterative scheme is proposed for approximating the generalized Drazin inverse b^{d} of the Banach algebra element b. Further, its extension into the form of the hyperpower iterative method of arbitrary order $p \geq 2$ is presented. Convergence criteria along with the estimation of error bounds in the computation of b^{d} are discussed. Convergence results confirms the high order convergence rate of the proposed iterative scheme.

Keywords. Generalized inverse; Drazin inverse; generalized Drazin inverse; Banach algebra; iterative method; convergence analysis.

1. Introduction

The computation of the Drazin inverse of a square complex matrix is one of the most important and challenging problems that has been extensively studied. Further details can be found in [1–5]. The Drazin inverse of $A \in \mathbb{C}^{n\times n}$, denoted by A^{D}, is the unique matrix $X \in \mathbb{C}^{n\times n}$ that satisfies the matrix equations $A^tXA = A^t, XAX = X, AX = XA,$ where $t = \mathrm{ind}(A)$ denotes the index of A. In particular, a nonsingular matrix A satisfies $t = \mathrm{ind}(A) = 0$. The Drazin inverse has numerous applications in statistics, cryptography, control theory, numerical analysis, etc. [6, 7]. A number of direct and iterative methods for computation of the Drazin inverse were developed in [8–12]. Its extension to Banach algebras is known as the generalized Drazin inverse and was established in [13]. Let $\mathcal{J}$ denote the complex Banach algebra with the unit 1. The generalized Drazin inverse of an element $b \in \mathcal{J}$, denoted by b^{d}, is the unique element of $\mathcal{J}$ such that $b^{\mathrm{d}}bb^{\mathrm{d}} = b^{\mathrm{d}}$, $bb^{\mathrm{d}} = b^{\mathrm{d}}b$ and $b(1 - bb^{\mathrm{d}})$ is quasi-nilpotent. An element $b \in \mathcal{J}$ is called a quasi-nilpotent if $\lim_n \|b^n\|^{1/n} = 0$. Moreover, b^{d} exists if and only if 0 is not the accumulation point of the spectrum $\sigma(b)$. It is worth mentioning that the generalized Drazin inverse b^{d} reduces to the ordinary Drazin inverse b^{D} in the case when $b(1 - bb^{\mathrm{d}})$ is nilpotent. Numerous properties of the generalized Drazin inverse b^{d} and the generalized resolvent in Banach algebras were studied in [14]. The Laurent expansion of the generalized resolvent based on the usage of b^{d} is also discussed in [14]. Representations of the generalized Drazin inverse of an anti-triangular operator matrix, defined under some constraints, were proposed in [15]. Some representations of b^{d} of an anti-triangular block matrix b in a Banach algebra in terms of the generalized Banachiewicz–Schur form were considered in [16]. Additive properties of the generalized Drazin inverse in a Banach algebra were investigated in [17]. Conditions under which the generalized Drazin inverse $(a + b)^{\mathrm{d}}$ could be explicitly expressed in terms of a, a^{d}, b and b^{d} were also established in [16]. The author of [18] derived some results for the generalized Drazin inverse of the 2×2 block matrix of the form $x = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ in a Banach algebra, under certain conditions. An explicit representation of the Drazin inverse of $\alpha p + \beta q$, where $\alpha, \beta \in \mathbb{C} \setminus 0$ and p, q are idempotents in $\mathcal{J}$, was derived in [19]. The authors of [20] characterized pre-orders on the set of all bounded linear operators between Banach spaces involving the generalized Drazin inverse. As a consequence, some recent results on pre-orders involving the Drazin inverse of a square complex matrix were established under more general settings. In [21], the authors described representations of the generalized Drazin inverse of the block matrix $x = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathcal{J}$, where $a_1 = \gamma x \gamma$, $a_2 = \gamma x(1 - \gamma)$, $a_3 = (1 - \gamma)x\gamma$, $a_4 = (1 - \gamma)x(1 - \gamma)$ and γ is the idempotent in $\mathcal{J}$.

*For correspondence

Iterative methods for computing b^{d} of $b \in \mathcal{J}$ and their convergence analysis have been studied in recent years. An iterative method for computing b^{d}, given by

$$y_{k+1} = y_k + \beta s(I - by_k), k = 0, 1, 2 \ldots, \beta \in \mathbb{C} \setminus 0, \quad (1)$$

was proposed in [22]. Starting from the initial approximation x_0 and $s \in \mathcal{J}$ satisfying $(1-\gamma)x_0 = x_0$ and $(1-\gamma)s = s(1-\gamma) = s$, the iterative procedure (1) approximates b^{d} if and only if $(1-\gamma-\beta sb)^k \to 0$ as $k \to \infty$. An iterative method faster than (1) was developed in [23]. A family of hyperpower iterative procedures was investigated in [24]. This family uses an arbitrary positive integer $p \geq 2$ and possesses the form

$$y_k = \left[{C_p}^1 I - {C_p}^2 y_{k-1} b + \cdots + (-1)^{p-1} {C_p}^p (y_{k-1}b)^{p-1}\right] y_{k-1}, \quad (2)$$

for each $k = 0, 1, 2, \ldots$, where ${C_n}^i = \frac{n!}{i!(n-i)!}$, $i = 0, 1, 2, \ldots, n$ denote the binomial coefficients. If the initial guess is chosen as $y_0 = \alpha s$, where $\alpha \in \mathbb{C} \setminus 0$ and $s \in \mathcal{J}$ satisfies $(1-\gamma)s = s(1-\gamma) = s$, the method (2) converges to b^{d} provided that $\rho(1-\gamma-\alpha sb) < 1$ holds.

This paper is motivated by the results obtained in [25]. Namely, the authors of [25] developed an iterative procedure for approximating the generalized inverse ${A^{(2)}}_{T,S}$ of $A \in \mathcal{B}(X, Y)$, where $\mathcal{B}(X, Y)$ denotes the set of bounded linear operators between Banach spaces X and Y. However, defined iteration is not explored in the approximation of the generalized Drazin inverse b^{d} of $b \in \mathcal{J}$. The aim of this work is to explore such a possibility. More precisely, a quadratically convergent iterative method is developed for approximating b^{d} of $b \in \mathcal{J}$. The proposed method is based on Newton's iterative method for computing generalized inverses. Convergence results of improved approximate solutions as well as the error estimate are derived. Further, its extension to a kind of the hyperpower iterative method is also described and the results concerning its convergence and the error estimate are established.

The rest of the paper is structured as follows. The proposed second order iterative method for approximating b^{d} of the Banach algebra element b and its convergence properties are discussed in section 2. In section 3, an adaptation of the hyperpower iterative method of an arbitrary order is defined and its convergence results are discussed. This method extends the iterative method described in section 2. Final conclusions are given in section 4.

2. The second-order iterative method

In this section, we propose a second-order iterative procedure for approximating the generalized Drazin inverse b^{d} of $b \in \mathcal{J}$. Let $\gamma \in \mathcal{J}$ be the idempotent element satisfying $\gamma b = b\gamma$. The initial approximations are given by $y_0 \in \mathcal{J}$ and $s_0 = \alpha s$, where $\alpha \in \mathbb{C} \backslash 0$ and $s \in \mathcal{J}$ is chosen such that $s = (1-\gamma)s = s(1-\gamma)$. The iterative rule is proposed as

$$s_k = s_{k-1} + s_{k-1}(1 - bs_{k-1}), \quad (3)$$

$$y_k = y_{k-1} + s_k(1 - by_{k-1}), k = 1, 2, \ldots. \quad (4)$$

The convergence results for the iterative method (3)–(4) are established in Theorem 2.1.

Theorem 2.1 *The iterative method defined by (3)–(4) converges to b^{d} if and only if either $\rho(1-\gamma-bs_0) < 1$ or $\rho(1-\gamma-s_0 b) < 1$. Therefore, if $(1-\gamma)y_0 = y_0$ then b^{d} exists and (3)–(4) converges to b^{d} if and only if $b\gamma$ is quasi-nilpotent element in $\mathcal{J}$ and*

$$b^{\mathrm{d}} = (\gamma + s_0 b)^{-1} s_0 = s_0(\gamma + bs_0)^{-1}.$$

Also, if

$$m = \min\{\|1-\gamma-bs_0\|, \|1-\gamma-s_0 b\|\} < 1$$

the error estimate of the sequence y_k is given by

$$\|b^{\mathrm{d}} - y_k\| \leq |\alpha| m^{2^{k+1}-3}(1-m^{2k+1})^{-1}\|s\|\|g_0\|, \quad (5)$$

where $g_0 = (1-\gamma)(1-by_0)$.

Proof Since $(1-\gamma)s = s(1-\gamma) = s$, it is not difficult to prove that $(1-\gamma)s_k = s_k(1-\gamma) = s_k$. Denote $g_k = (1-\gamma)(1-by_k)$; then $(1-\gamma)g_k = g_k$ and

$$\begin{aligned} g_k &= (1-\gamma)[(1-by_{k-1}) - bs_k(1-by_{k-1})], \\ &= (1-\gamma-bs_0)^{2^{k+1}-2} g_0. \end{aligned}$$

Starting from (3), it is possible to verify that

$$\begin{aligned} s_k &= s_{k-1} + s_{k-1}(1-bs_{k-1}) \\ &= s_{k-1} + s_{k-1}(1-\gamma-bs_{k-1}). \end{aligned} \quad (6)$$

Further, the following follows from (4):

$$\begin{aligned} y_k &= y_{k-1} + s_k(1-by_{k-1}) \\ &= y_{k-1} + s_k g_{k-1}. \end{aligned} \quad (7)$$

Now, using (6) and (7), we obtain

$$\begin{aligned} y_k - y_0 &= s_0 \sum_{i=0}^{2^{k+1}-3} (1-(1-\gamma-bs_0))^i \, g_0 \\ &= \sum_{i=0}^{2^{k+1}-3} (1-(1-\gamma-s_0 b))^i \, s_0 g_0. \end{aligned}$$

Further, the last identity can be written as

$$y_k = y_0 + s_0 e_k g_0, \quad (8)$$

$$= y_0 + f_k s_0 g_0, \quad (9)$$

where

$$e_k = \sum_{i=0}^{2^{k+1}-3} (1-(1-\gamma-bs_0))^i$$

and

$$f_k = \sum_{i=0}^{2^{k+1}-3} (1-(1-\gamma-s_0b))^i.$$

Also, we get

$$\begin{aligned}(1-(1-\gamma-bs_0))e_k &= (\gamma+bs_0)m_k \\ &= 1-(1-\gamma-bs_0)^{2^{k+1}-2}\end{aligned} \tag{10}$$

and

$$\begin{aligned}(1-(1-\gamma-s_0b))f_k &= (\gamma+s_0b)f_k \\ &= 1-(1-\gamma-s_0b)^{2^{k+1}-2}.\end{aligned} \tag{11}$$

Using (10), it is clear that $\rho(1-\gamma-bs_0)<1$ holds if and only if $(1-\gamma-bs_0)^{2^{k+1}-2} \to 0$, or analogously e_k converges. This immediately proves that y_k converges. Similarly, from (11), it follows that $\rho(1-\gamma-z_0a)<1$ is satisfied if and only if $(1-\gamma-s_0b)^{2^{k+1}-2} \to 0$. This means that f_k converges, which further implies the convergence of y_k. Moreover from $\rho(1-\gamma-bs_0)<1$ and $\rho(1-\gamma-s_0b)<1$ it follows that $(\gamma+bs_0)$ and $(\gamma+s_0b)$ are invertible. Consequently

$$s_0(\gamma+bs_0)^{-1} = (\gamma+s_0b)^{-1}s_0. \tag{12}$$

With respect to the identities (9), (11) and (12), by taking the limit as $k\to\infty$ and using

$$(1-\gamma)y_0 = y_0,\ s_0(1-\gamma) = (1-\gamma)s_0 = s_0,$$

it can be concluded that

$$\begin{aligned} y_\infty &= y_0 + (\gamma+s_0b)^{-1}s_0g_0 \\ &= y_0 + (\gamma+s_0b)^{-1}s_0(1-\gamma)(1-by_0) \\ &= (\gamma+s_0b)^{-1}s_0 \\ &= s_0(\gamma+bs_0)^{-1}. \end{aligned} \tag{13}$$

Since, $b\gamma = \gamma b$ and $(1-\gamma)s = s$, it follows that

$$\gamma(\gamma+bs_0) = \gamma = (\gamma+s_0b)\gamma. \tag{14}$$

In accordance with (12), (13) and (14), the three identities $y_\infty b = by_\infty$, $y_\infty by_\infty = y_\infty$ and $b(1-ay_\infty) = b\gamma$ follow immediately. This proves that $y_\infty = b^{\mathrm{d}}$ if and only if $b\gamma$ is quasi-nilpotent. To derive the error estimation, let us assume that $\|1-\gamma-bs_0\| = m<1$. Then (8) and (10) clearly imply

$$b^{\mathrm{d}} = y_0 + s_0(1-(1-\gamma-bs_0))^{-1}g_o. \tag{15}$$

From (8) and (15), we get

$$b^{\mathrm{d}} - y_k = s_0\Big[(1-\gamma-bs_0)^{2^{k+1}-3} + (1-\gamma-bs_0)^{2^{k+2}-3} + \cdots\Big]g_o. \tag{16}$$

By taking a norm on (16) and using $\|1-\gamma-bs_0\| = m<1$, the following inequalities hold:

$$\begin{aligned}&\|b^{\mathrm{d}}-y_k\| \le \|s_0\|\Big[m^{2^{k+1}} + m^{2^{k+2}} + \cdots\Big]m^{-3}\|g_0\| \\ &\le \|s_0\|\,m^{2^{k+1}-3}(1-m^{2^{k+1}})^{-1}\,\|m_0\| \le |\alpha|\,m^{2^{k+1}-3}(1-m^{2^{k+1}})^{-1} \\ &\quad \|s\|\,\|g_0\|.\end{aligned}$$

This proves the result.

Dually, let $b\in\mathcal{J}$ and $\gamma\in\mathcal{J}$ be an idempotent element satisfying $\gamma b = b\gamma$. The iterative method is given for $k = 1,2,\ldots$ by

$$s_k = s_{k-1} + (1-s_{k-1}b)s_{k-1}, \tag{17}$$

$$y_k = y_{k-1} + (1-y_{k-1}b)s_k, \tag{18}$$

where the initial guess $y_0\in\mathcal{J}$ and $s_0 = \alpha s$ satisfy $(1-\gamma)s = s(1-\gamma) = s$, where α is a non-zero complex scalar.

The following result is valid for the iterative method (17)–(18).

Theorem 2.2 *The iterative method defined by (17)–(18) converges to b^{d} if and only if either $\rho(1-\gamma-bs_0)<1$ or $\rho(1-\gamma-s_0b)<1$. This implies that if $y_0(1-\gamma) = y_0$ then b^{d} exists and the iterative sequence converges to b^{d} if and only if γb is quasi-nilpotent in $\mathcal{J}$ and*

$$b^{\mathrm{d}} = (\gamma+s_0b)^{-1}s_0 = s_0(\gamma+bs_0)^{-1}.$$

Also, if

$$m_1 = \min\{\|1-\gamma-bs_0\|, \|1-\gamma-s_0b\|\}<1$$

then the error estimation of y_k is given by

$$\|b^{\mathrm{d}}-y_k\| \le |\alpha|\,m_1^{2^{k+1}-3}(1-m_1^{2^{k+1}})^{-1}\,\|s\|\,\|g_0\|,$$

where $g_0 = (1-y_0b)(1-\gamma)$.

Proof Proof of this theorem is similar to the proof of Theorem 2.1.

3. A hyperpower iterative method

In this section, the second-order iterative method (3)–(4) is extended to a generalization of the hyperpower iterative method that is applicable in approximating the generalized Drazin inverses of the Banach algebra element $b\in\mathcal{J}$. Using arbitrary $y_0\in\mathcal{J}$ and the initial approximation $s_0 =$

αs satisfying $\alpha \in \mathbb{C}\backslash 0$, where $s \in \mathcal{J}$ is chosen such that $s = (1-\gamma)s = s(1-\gamma)$, the hyperpower iterative method of the order $p \geq 2$ is given for $k = 1, 2, \ldots$ by

$$s_k = s_{k-1} \sum_{i=0}^{p-1} (1 - bs_{k-1})^i, \tag{19}$$

$$y_k = y_{k-1} + s_k(1 - by_{k-1}). \tag{20}$$

The next theorem describes the convergence properties of the hyperpower iterative method defined by (19)–(20).

Theorem 3.1 *The iterative method defined by* (19)–(20) *generates a sequence* $\{y_k\}$ *in* $\mathcal{J}$ *converging to* b^{d} *if and only if either* $\rho(1-\gamma-bs_0) < 1$ *or* $\rho(1-\gamma-s_0 b) < 1$. *This means that if* $(1-\gamma)y_0 = y_0$ *then* b^{d} *exists and* (19)–(20) *converges to* b^{d} *if and only if* $b\gamma$ *is quasi-nilpotent in* $\mathcal{J}$ *and*

$$b^{\mathrm{d}} = (\gamma + s_0 b)^{-1} s_0 = s_0(\gamma + bs_0)^{-1}.$$

Also, if

$$m = \min\{\|1-\gamma-bs_0\|, \|1-\gamma-s_0 b\|\} < 1$$

then x_k *has the error estimation given by*

$$\|b^{\mathrm{d}} - y_k\| \leq |\alpha|\, m^{\frac{p^{k+1}-(2p-1)}{p-1}} (1 - m^{\frac{p^{k+1}}{p-1}})^{-1} \|s\| \|g_0\|, \tag{21}$$

where $g_0 = (1-\gamma)(1-by_0)$.

Proof The proof is a generalization of the proof of Theorem 2.1. It is not difficult to show that $(1-\gamma)s_k = s_k(1-\gamma) = s_k$ for any positive integer k and $(1-\gamma)g_k = g_k$, where $g_k = (1-\gamma)(1-by_k)$. Further,

$$\begin{aligned} g_k &= (1-\gamma)[(1-by_{k-1}) - bs_k(1-by_{k-1})] \\ &= (1-\gamma-bs_0)^{\frac{p^{k+1}-p}{p-1}}. \end{aligned}$$

The identity (19) afterwards implies

$$\begin{aligned} s_k &= s_{k-1} + s_{k-1}(1-bs_{k-1}) \\ &= s_{k-1} + s_{k-1}(1-\gamma-bs_{k-1}). \end{aligned} \tag{22}$$

Subsequently, from (20) one can obtain

$$\begin{aligned} y_k &= y_{k-1} + s_k(1-by_{k-1}) \\ &= y_{k-1} + s_k g_{k-1}. \end{aligned} \tag{23}$$

Now, using (22) and (23), it is possible to verify that

$$\begin{aligned} y_k - y_0 &= s_0 \sum_{i=0}^{\frac{p^{k+1}-(2p-1)}{p-1}} (1-(1-\gamma-bs_0))^i \, g_0 \\ &= \sum_{i=0}^{\frac{p^{k+1}-(2p-1)}{p-1}} (1-(1-\gamma-s_0 b))^i \, s_0 g_0. \end{aligned}$$

The last identities can be written in condensed form as

$$y_k = y_0 + s_0 e_k g_0 \tag{24}$$

$$= y_0 + f_k s_0 g_0, \tag{25}$$

where

$$e_k = \sum_{i=0}^{\frac{p^{k+1}-(2p-1)}{p-1}} (1-(1-\gamma-bs_0))^i$$

and

$$f_k = \sum_{i=0}^{\frac{p^{k+1}-(2p-1)}{p-1}} (1-(1-\gamma-s_0 b))^i.$$

Also, it follows that

$$\begin{aligned} (1-(1-\gamma-bs_0))e_k &= (\gamma+bs_0)e_k \\ &= 1-(1-\gamma-bs_0)^{\frac{p^{k+1}-p}{p-1}} \end{aligned} \tag{26}$$

and

$$\begin{aligned} (1-(1-\gamma-s_0 b))f_k &= (\gamma+s_0 b)f_k \\ &= 1-(1-\gamma-s_0 b)^{\frac{p^{k+1}-p}{p-1}}. \end{aligned} \tag{27}$$

From (26), it is clear that $\rho(1-\gamma-bs_0) < 1$ holds if and only if $(1-\gamma-bs_0)^{\frac{p^{k+1}-p}{p-1}} \to 0$, or analogously e_k is convergent. This immediately proves that y_k is convergent. Similarly, according to (27), the inequality $\rho(1-\gamma-s_0 b) < 1$ holds if and only if $(1-\gamma-s_0 b)^{\frac{p^{k+1}-p}{p-1}} \to 0$ or f_k converges, which again implies the convergence of y_k. Moreover, from $\rho(1-\gamma-bs_0) < 1$ and $\rho(1-\gamma-s_0 b) < 1$ it follows that $(\gamma+bs_0)$ and $(\gamma+s_0 b)$ are invertible. Consequently, we obtain

$$s_0(\gamma+bs_0)^{-1} = (\gamma+s_0 b)^{-1} s_0. \tag{28}$$

From (25), (27) and (28), taking the limit as $k \to \infty$ and using that $(1-\gamma)y_0 = y_0$, $s_0(1-\gamma) = (1-\gamma)s_0 = s_0$, we conclude that

$$\begin{aligned} y_\infty &= y_0 + (\gamma+s_0 b)^{-1} s_0 g_0, \\ &= y_0 + (\gamma+s_0 b)^{-1} s_0 (1-\gamma)(1-by_0), \\ &= (\gamma+s_0 b)^{-1} s_0, \\ &= s_0(\gamma+bs_0)^{-1}. \end{aligned} \tag{29}$$

Also, since $b\gamma = \gamma b$ and $(1-\gamma)s = s$, it follows that

$$\gamma(\gamma+bs_0) = \gamma = (\gamma+s_0 b)\gamma. \tag{30}$$

From (28), (29) and (30) we get $y_\infty b = by_\infty$, $y_\infty b y_\infty = y_\infty$ and $b(1-by_\infty) = b\gamma$. This implies that $y_\infty = b^{\mathrm{d}}$ if and only if $b\gamma$ is quasi-nilpotent. To compute the error estimate, let us assume $\|1-\gamma-bs_0\| = m < 1$. Then, using (24) and (26) we obtain

$$b^{\mathrm{d}} = y_0 + s_0(1-(1-\gamma-bs_0))^{-1}g_0. \tag{31}$$

The identities (24) and (31) give

$$\begin{aligned} b^{\mathrm{d}} - y_k = s_0\Big[&(1-\gamma-bs_0)^{2^{k+1}-3} \\ &+(1-\gamma-bs_0)^{2^{k+2}-3} + \cdots\Big]g_0. \end{aligned} \tag{32}$$

An application of a norm in (32) and further imposition of the condition $\|1-\gamma-bs_0\| = m < 1$ imply

$$\begin{aligned} \|b^{\mathrm{d}}-y_k\| &\le \|s_0\|\left[m^{\frac{p^{k+1}-(2p-1)}{p-1}} + m^{\frac{p^{k+2}-(2p-1)}{p-1}} + \cdots\right]\|g_0\|, \\ &\le \|s_0\|\left[m^{\frac{p^{k+1}}{p-1}} + \left(m^{\frac{p^{k+1}}{p-1}}\right)^2 + \left(m^{\frac{p^{k+1}}{p-1}}\right)^3 + \cdots\right] m^{\frac{-(2p-1)}{p-1}}\|g_0\|, \\ &\le |\alpha|\, m^{\frac{p^{k+1}-(2p-1)}{p-1}}\left(1-m^{\frac{p^{k+1}}{p-1}}\right)^{-1}\|s\|\,\|g_0\|. \end{aligned} \tag{33}$$

This proves the result. □

Dually, let $b \in \mathcal{J}$ and let $\gamma \in \mathcal{J}$ be an idempotent element satisfying $\gamma b = b\gamma$. Starting from arbitrary initial approximation $y_0 \in \mathcal{J}$ in conjunction with the initializer $s_0 = \alpha s$ that satisfies $(1-\gamma)s = s(1-\gamma) = s$, $\alpha \in \mathbb{C}\setminus 0$, the proposed hyperpower iterative process can be defined for $k = 1, 2, \ldots$ by

$$s_k = \sum_{i=0}^{p-1}(1-s_{k-1}b)s_{k-1}, \tag{34}$$

$$y_k = y_{k-1} + (1-y_{k-1}b)s_k. \tag{35}$$

The next theorem determines the convergence behaviour of the iterative procedure (34)–(35).

Theorem 3.2 *The iterative method defined by (34)–(35) converges to b^{d} if and only if either $\rho(1-\gamma-bs_0) < 1$ or $\rho(1-\gamma-s_0b) < 1$. This implies that if $y_0(1-\gamma) = y_0$, then b^{d} exists and the method converges to b^{d} if and only if γb is quasi-nilpotent in $\mathcal{J}$ and $b^{\mathrm{d}} = (\gamma+s_0b)^{-1}s_0 = s_0(\gamma+bs_0)^{-1}$. Also, under the assumption*

$$m_1 = \min\{\|1-\gamma-bs_0\|, \|1-\gamma-s_0b\|\} < 1,$$

the error bound of the sequence y_k is estimated by

$$\|b^{\mathrm{d}}-y_k\| \le |\alpha|\, m_1^{\frac{p^{k+1}-(2p-1)}{p-1}}\left(1-m_1^{\frac{p^{k+1}}{p-1}}\right)^{-1}\|s\|\,\|g_0\|,$$

where $g_0 = (1-y_0b)(1-\gamma)$.

Proof This statement follows immediately from Theorem 2.1 and hence it is omitted here.

4. Conclusions

In this work, we presented a quadratically convergent iterative procedure for approximating the generalized Drazin inverse b^{d} of a Banach algebra element $b \in \mathcal{J}$. The iterative process is based on the well-known Newton method for approximating generalized inverses of b. The convergence results are described. The iterative method is further extended into a hyperpower iterative method for approximating the generalized Drain inverse b^{d} of the Banach algebra element $b \in \mathcal{J}$. Necessary and sufficient conditions that ensure the convergence of the extended hyperpower iterative process are established. Strict error bounds for the method are also given.

Further extension of this work may be in the direction of approximating the generalized Drazin inverses of block matrices in Banach algebra. One can also work on developing different criteria in order to further accelerate the computational speed of the proposed method.

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