MHD unsteady GO–water-squeezing nanofluid flow—heat and mass transfer between two infinite parallel moving plates: analytical investigation

MOHAMMADREZA AZIMI and ROUZBEH RIAZI*

Faculty of New Sciences and Technologies, University of Tehran, 14395-1561 Tehran, Iran
e-mail: m_r_azimi1991@yahoo.com; ro_riazi@ut.ac.ir

MS received 7 May 2015; revised 12 July 2016; accepted 27 September 2016

Abstract. Investigation for unsteady squeezing viscous flow is one of the most important research topics due to its wide range of engineering applications such as polymer processing and lubrication systems. The aim of the present paper is to study the unsteady squeezing viscous graphene oxide–water nanofluid flow with heat transfer between two infinite parallel plates. The governing equations, continuity, momentum and energy for this problem are reduced to coupled nonlinear ordinary differential equations using a similarity transformation. The transmuted model is shown to be controlled by a number of thermo-physical parameters, viz., moving parameter, graphene oxide nanoparticles solid volume fraction, Eckert and Prandtl numbers. Nusselt number and skin friction parameter are obtained for various values of GO solid volume fraction and Eckert number. Comparison between analytical results and numerical ones achieved by fourth order Runge–Kutta method revealed that our analytical method can be a simple, powerful and efficient technique for finding analytical solutions in science and engineering nonlinear differential equations.

Keywords. Heat transfer enhancement; squeezing flow; nanofluid flow; analytical solution; graphene oxide.

1. Introduction

The term nanofluid refers to fluids in which nano-scale particles are suspended in the base fluid, which was initially suggested by Choi [1]. Soleimani *et al* [2] studied natural convection heat transfer in a semi-annulus enclosure filled with nanofluid using the control-volume-based finite-element method. They found that the angle of turn has an important effect on the streamlines, isotherms and maximum or minimum values of local Nusselt number. Sheikholeslami *et al* [3] investigated the flow of nanofluid and heat transfer characteristics between two horizontal plates in a rotating system. Azimi *et al* [4] studied the effect of graphene oxide nanoparticle solid volume fraction, the moving parameter and the Eckert number on heat transfer characteristics of nanofluid flow between two moving parallel plates. In another work by Azimi *et al* [5], the MHD Jeffery Hamel flow problem with graphene oxide nanoparticles for various Hartman numbers is investigated. They concluded that the dimensionless velocity for water–GO nanofluid is an increasing function of Re , in divergent channels.

Hamza [6] considered the squeezing flow between two discs in the presence of a magnetic field. The problem of

squeezing flow between rotating discs has been studied by Hamza [7] and Bhattacharyya and Pal [8].

Most scientific problems in fluid mechanics such as MHD viscous fluid flow are inherently nonlinear by nature, and except in a limited number of cases, most of them do not have exact solutions. Accordingly, the nonlinear equations are usually solved using other methods, including numerical techniques, or by using analytical methods. Some of these methods are homotopy perturbation method [9], reconstruction of variational iteration method (RVIM) [4], Galerkin optimal homotopy asymptotic method [5], differential transformation method [10] and others [11, 12].

Squeezing flows have many applications in food industry, especially in chemical engineering. The determination of squeeze flow characteristics has attracted the attention of several investigators due to its importance in the practical problems of improving the performance of hydraulic machine elements, food industry, chemical engineering, polymer processing, compression and injection moulding [13].

The unsteady squeezing flow of a viscous fluid between parallel plates in motion normal to their own surfaces is of great interest in hydrodynamical machines. Practically it is involved in polymer processing, compression and injection modeling.

*For correspondence

In spite of importance and practical implications of the squeezing flows between parallel plates, researchers of mechanical fluid society have not had a sensible vision regarding its phenomenological properties.

The rotor–stator system of turbo-machinery can be idealized by a system of two plates, in which one plate is in rotation and the other is stationary. The movement of underground water through hydraulic pumps can be simulated by the motion of two impermeable plates moving towards or apart from each other. The high performance of a turbo-machine is attributable to the occurrence of self-sustained, vortex-induced oscillating flow in the radial direction coupled with the effect of the centrifugal and Coriolis forces. These forces induce the heat transfer processes. The two-plate problem with magnetic field has promising application in the field of hydromagnetic lubrication. These squeezing flows are also useful in polymer processing, compression and injection moulding.

However, the proposition and development of advanced computational approaches have enabled engineers to perform some practical analyses, which in turn have provided them with sufficient information regarding the physical behaviour of the phenomenon.

A seminal work on the squeezing flow under lubrication approximation was reported by Stefan [14]. Magnetohydrodynamic squeezing flow of a viscous fluid between parallel disks was analysed by Domairry and Aziz [15]. Siddiqui *et al* [16] discussed the hydromagnetic squeezing flow of a viscous fluid between parallel plates.

The Brownian motion of nanoparticles is too slow to directly transfer heat through nanofluid; however, it could have an indirect role in producing a convection-like microenvironment around the nanoparticles and particle clustering to increase the heat transfer [17, 18].

This paper reports the heat and mass transfer characteristics in a viscous fluid that is squeezed between parallel plates. The governing partial differential equations for unsteady two-dimensional flow with heat and mass transfer of a viscous fluid are reduced to ordinary differential equations by similarity transformations. RVIM is employed to construct the series solution of the problem. An analytical investigation is carried out for different governing parameters, namely, the squeeze number, Hartmann number, Schmidt number, Brownian motion parameter, thermophoretic parameter and Eckert number.

2. Mathematical formulation

The unsteady flow and heat transfer in a two-dimensional nanofluid between two infinite parallel plates is considered in this study. Figure 1 shows the problem schematic. The two plates are placed at $z = \pm l(1 - \alpha t)^{1/2} = \pm h(t)$ for $\alpha > 0$, and move until they touch $t = 1/\alpha$. For $\alpha < 0$ the two plates are separated. This behaviour occurs at high Eckert

number ($\ll 1$). The Eckert number expresses the relationship between a flow's kinetic energy and enthalpy. Also, it is assumed that a uniform magnetic field ($\vec{B} = B\vec{e}_y$) is applied, where $\vec{e}_y$ is a unit vector in the Cartesian coordinate system. The electric current J and the electromagnetic force F are defined by $J = \sigma(\vec{V} \times \vec{B})$ and $F = \sigma(\vec{V} \times \vec{B}) \times \vec{B}$, respectively.

The fluid is a water-based nanofluid containing graphene oxide. The nanofluid is a two-component mixture with the following assumptions: incompressible; no chemical reaction; negligible viscous dissipation; negligible radiative heat transfer; nano-solid particles; the base fluid is in thermal equilibrium and no slip condition. The governing equations are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$\rho_{nf} \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu_{nf} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \sigma B^2 u \quad (2)$$

$$\rho_{nf} \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu_{nf} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (3)$$

$$\begin{aligned} \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = & \frac{k_{nf}}{(\rho C_p)_{nf}} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \frac{\mu_{nf}}{(\rho C_p)_{nf}} \left[4 \left(\frac{\partial u}{\partial x} \right)^2 \right. \\ & + \frac{(\rho C_p)_p}{(\rho C_p)_{nf}} \left[D_B \left(\frac{\partial C}{\partial x} \right) \cdot \left(\frac{\partial T}{\partial x} \right) + D_B \left(\frac{\partial C}{\partial y} \right) \cdot \left(\frac{\partial T}{\partial y} \right) \right. \\ & \left. \left. + \left(\frac{D_T}{T_c} \right) \cdot \left\{ \left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right\} \right] \end{aligned} \quad (4)$$

$$\begin{aligned} \frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = & D_B \left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right) + \left(\frac{D_T}{T_c} \right) \\ & \cdot \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right] \end{aligned} \quad (5)$$

where u and v are the velocities in the x and y directions, respectively. T is the temperature, C is the concentration, P is the pressure, C_p is the specific heat of nanofluid and D_B is the diffusion of the diffusing species. Effective density (ρ_{nf}) , effective dynamic viscosity (μ_{nf}) , effective heat capacity $(\rho_{nf})_{nf}$ and effective thermal conductivity k_{nf} of the nanofluid are used. The relevant boundary conditions for the problem are

$$\begin{aligned} C = 0, \quad v = v_w = dh/dt, \quad T = T_H, \quad C = C_H \quad \text{at } y = h(t) \\ v = \partial u / \partial y = \partial T / \partial y = \partial C / \partial y = 0 \quad \text{at } y = 0. \end{aligned} \quad (6)$$

We can simplify the above equations by introducing following parameters:

Figure 1. Geometry of the problem.

$$\begin{aligned} \eta &= \frac{y}{[l(1-\alpha t)^{1/2}]}, \quad u = \frac{\alpha x}{[2(1-\alpha t)]} f'(\eta), \\ v &= -\frac{\alpha l}{[2(1-\alpha t)^{1/2}]} f(\eta), \\ \theta &= \frac{T}{T_H}, \quad \phi = \frac{C}{C_H}. \end{aligned} \quad (7)$$

The above parameters are substituted into Eqs. (2) and (3). Then the pressure gradient is eliminated from the resulting equations:

$$f^{(IV)} - S(\eta f''' + 3f'' + f'f'' - ff''') - Ha^2 f'' = 0 \quad (8)$$

Using Eq. (7), Eqs. (4) and (5) simplify to the following equations:

$$\theta'' + \text{Pr} S(f\theta' - \eta\theta') + \text{Pr} Ec(f''^2) + Nb\phi'\theta' + Nt\theta'^2 = 0 \quad (9)$$

$$\phi'' + S.Sc(f\phi' - \eta\phi') + \frac{Nt}{Nb}\theta'' = 0 \quad (10)$$

with the following boundary conditions:

$$\begin{aligned} f(0) &= 0, \quad f''(0) = 0, \quad f(1) = 1, \quad f'(1) = 0, \\ \theta'(0) &= \phi'(0) = 0, \quad \theta(1) = \phi(1) = 1 \end{aligned} \quad (11)$$

where S is the squeeze number, Pr is the Prandtl number, Ec is the Eckert number, Sc is the Schmidt number, Ha is Hartman number of the nanofluid, Nb is the Brownian motion parameter and Nt is the thermophoretic parameter, which are defined as follows:

$$\begin{aligned} S &= \frac{\alpha l^2}{2v_f}, \quad \text{Pr} = \frac{\mu_f(\rho C_p)_f}{\rho_f k_f}, \quad Ec = \frac{\rho_f}{(\rho C_p)_f} \left(\frac{\alpha x}{2(1-\alpha t)} \right)^2, \\ Sc &= \frac{\mu_f}{\rho_f D} \\ Ha &= lB \sqrt{\frac{\sigma}{\mu_f} (1-\alpha t)}, \quad Nb = (\rho C_p)_f D_B (C_H) / ((\rho C)_f \alpha), \\ Nt &= (\rho C_p)_f D_T (T_H) / ((\rho C)_f \alpha T_c). \end{aligned} \quad (12)$$

3. Solution procedure

In the following section, an alternative method for finding the optimal value of the Lagrange multiplier by the use of the Laplace transform will be investigated. To facilitate our discussion of RVIM, introducing a new linear or nonlinear function $h(u(x, t)) = f(x, t) - N(u(x, t))$, considering the new equation $h(u(x, t)) = f(x, t) - N(u(x, t))$ rewritten as

$$L(u(t, x)) = h(t, x, u). \quad (13)$$

Now, for implementation of the correctional function of variational iteration method (VIM) based on new idea of Laplace transform, apply the Laplace transform to both sides of the above equation; we introduce artificial initial conditions of zero for the main problem; then the left hand side of equation after transformation is

$$L[L\{u(x, t)\}] = U(x, s)P(s) \quad (14)$$

where $P(s)$ is a polynomial with the degree of the highest order derivative of the linear operator:

$$L[L\{u(x, t)\}] = U(x, s)P(s) = L[h\{(x, t, u)\}] \quad (15)$$

$$U(x, s) = \frac{L[h\{(x, t, u)\}]}{P(s)}. \quad (16)$$

Suppose that $D(s) = 1/P(s)$. Taking the inverse Laplace transform on both side of Eq. (13) using the convolution theorem gives

$$u(x, t) = \int_0^t d(t-\varepsilon) h(x, \varepsilon, u) d\varepsilon \quad (17)$$

$$u_0(x, t) + \int_0^t d(t-\varepsilon) h(x, \varepsilon, u) d\varepsilon. \quad (18)$$

$u_0(x, t)$ is the initial solution with or without unknown parameters. In the absence of unknown parameters, $u_0(x, t)$ should satisfy initial boundary conditions.

4. Results and discussion

In this section we will discuss about the obtained results of squeezing GO–water nanofluid flow between infinite parallel plates. The effects of active parameter on heat and mass characteristics will also be examined.

Physical quantities of interest are the skin friction coefficient and Nusselt number, which are defined as follows:

$$Cf = \frac{\mu_f \left(\frac{\partial u}{\partial y} \right)_{y=h(t)}}{\rho_{nf} v_w^2}, \quad Nu = \frac{-lk_{nf} \left(\frac{\partial T}{\partial y} \right)_{y=h(t)}}{kT_H}. \quad (19)$$

In order to characterize the influence of embedding physical parameters on the velocity, temperature and concentration profiles, figures 2–8 are displayed. Figure 2 shows the combined effects of positive and negative squeeze number on the velocity profiles. For the case of squeezing flow, the velocity increases due to an increase in the absolute value of squeeze number when $\eta < 0.5$, while it decreases for $\eta > 0.5$. Also it can be seen that the absolute value of squeeze number has a reverse relationship with the absolute values of skin friction coefficient for

Figure 2. Influence of S on (a) f and (b) f' when $H = 2$.

Figure 3. Influence of S on θ when $H = 2, N_b = N_t = 0.1, Ec = 1, Pr = 7, Sc = 0.2$.

squeezing flow case. Also figure 3 shows that the opposite trend is observed for the case in which the plates are moving apart. Besides, it can be found that velocity components of nanofluid increase as a result of an increase in the energy transport in the fluid as the volume fraction increases.

Figure 3 shows the effect of squeeze number on temperature. Considerable reduction in the temperature field is observed for large values of S . It is quite obvious that the temperature is relatively high when the plates move towards each other. An increment in the squeeze number can be relevant to the reduction in the kinematic viscosity, an increment in the distance of plates or an increment in the speed at which the plates move.

Influence of Hartman number on velocity profiles has been presented in figure 4. It is valuable to note that the effect of magnetic field is reducing the value of the velocity of the enclosure because the presence of magnetic field defines a force that is called the Lorentz force and acts against the flow if the magnetic field is applied in the normal direction. This is a resisting force that slows down the fluid velocity.

Effect of squeeze number on concentration profile is shown in figure 5. As can be seen in figure 5, with an increment in squeeze number the concentration profile increases. In figure 6 the outcome of large values of Schmidt number on the concentration field is seen. It is noticed that the gradual increase in Sc corresponds to weak molecular diffusivity and low boundary layer thickness. The opposite trend is seen for thermophoretic parameter according to figure 7. Figure 8 shows the effect of Brownian motion on non-dimensional concentration profiles. As

Figure 4. Influence of H on f when $S = 1$.

Figure 5. Influence of S on ϕ when $H = 2, N_b = N_t = 0.5, Ec = 0.05, Pr = 7, Sc = 0.5$.

illustrated in figure 8, an increase in Brownian motion parameter increases the nanoparticles concentration. This increase is due to the effective movement of nanoparticles from the upper disk to the fluid.

Table 1 presents a comparison between the present work and Mustafa *et al* [19]. Good agreement between our study and a previous one assures us about validity and high accuracy of our results.

Figure 6. Influence of Sc on ϕ when $H = 2, N_b = N_t = 0.5, Ec = 0.05, Pr = 7, S = 0.1$.

Figure 7. Influence of Nt on ϕ when $H = 2, N_b = 0.5, Ec = 0.05, Pr = 7, S = 0.1, Sc = 3$.

5. Conclusions

In this paper, the heat and mass transfer in the unsteady nanofluid flow between two moving parallel plates was investigated using RVIM. The effect of squeeze number on heat transfer enhancement for different cases has been shown. An analytical investigation is carried out for various governing parameters such as the squeeze number, Hartman

Figure 8. Influence of Nb on ϕ when $H = 2, N_t = 0.5, Ec = 0.05, Pr = 7, S = 0.1, Sc = 3$.

Table 1. Comparison between the present work and Mustafa *et al* [19] when $H = 0$.

S	$-f''(1)$ present	$-f''(1)$ Mustafa <i>et al</i>
-1.0	2.17009125	2.179009
-0.5	2.61412563	2.614038
0.01	3.00713256	3.007134
0.5	3.33645265	3.336449

number, thermophoretic parameter, Brownian motion parameter and Eckert number. The results show that an increase in Brownian motion parameter increases the nanoparticles concentration. This increase is due to the effective movement of nanoparticles from the upper disk to the fluid. Comparison of obtained results with analytical solutions achieved by previous research assures us about the validity and accuracy of the current study.

Appendix: Application of VIM

Using the proposed procedure for variational iteration method (VIM) to solve Eq. (8) with the associated boundary conditions (11), we have the following iterative schemes:

$$f_{n+1} = f_n - \int_0^\eta \frac{(s-\eta)^3}{3!} \left[f_n^{(IV)}(s) - S \left(\begin{aligned} &sf_n'''(s) + 3f_n''(s) \\ &+ f_n'(s)f_n''(s) - f_n'''(s)f_n(s) \\ &- Ha^2 f_n''(s) \end{aligned} \right) \right] ds \quad (20)$$

The first two approximants of the solutions are given by

$$\begin{aligned} f_0(\eta) &= A + \frac{1}{2}A_1\eta^2 + \frac{1}{6}A_2\eta^3, \\ f_1(\eta) &= A + \frac{1}{2}A_1\eta^2 + \frac{1}{6}A_2\eta + \left(-\frac{1}{12}SA_2 + \frac{1}{8}SA_1 + \frac{1}{24}M^2A_1 \right) \eta^4 \\ &\quad + \left(\frac{1}{30}SA_2 + \frac{1}{120}M^2A_2 \right) \eta^5 - \frac{1}{360}SA_1A_2\eta^6 - \frac{1}{2520}SA_2^2\eta^7. \end{aligned} \quad (21)$$

According to boundary conditions we know, $f(0) = 0$. Therefore $A = 0$. We can find further approximants of the solutions in a similar fashion. It is important to note that the same trend can be used for non-dimensional energy and concentration equations.

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