A two-step iterative method and its acceleration for outer inverses

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Abstract. A two-step iterative method and its accelerated version for approximating outer inverse $A_{T,S}^{(2)}$ of an arbitrary matrix A are proposed. A convergence theorem for its existence is established. The rigorous error bounds are derived. Numerical experiments involving singular square, rectangular, random matrices and a sparse matrix obtained by discretization of the Poisson's equation are solved. Iterations count, computational time and the error bounds are used to measure the performance of our method. On comparing our results with those of other iterative methods, it is seen that significantly better performance is achieved. Thus, enhanced speed and accuracy from the computational points of view has resulted for our methodology.

Keywords. Random matrices; sparse matrices; Penrose equations; group inverse; Poisson's equation; outer inverse.

1. Introduction

Let $\mathbb{C}_r^{m \times n}$ be the set of $(m \times n)$ matrices of rank r over the complex field $\mathbb{C}$. For $A \in \mathbb{C}^{m \times n}$, let $\|A\|_2$, A^* , A^t , $R(A)$ and $N(A)$ represent the matrix 2-norm, the conjugate transpose, the transpose, the range space and the null space, respectively. Further, $\rho(X)$ and $\lambda_{\max}(X)$ denote spectral radius and the maximal eigenvalue of a square matrix $X \in \mathbb{C}^{n \times n}$. One of the most important and challenging tasks in numerical analysis is finding efficient iteration for computing outer inverse $A_{T,S}^{(2)}$ and develop its convergence analysis, where T and S stand for the subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. For any matrix $A \in \mathbb{C}^{m \times n}$, the matrix $P \in \mathbb{C}^{n \times m}$ such that $PAP = P$, $R(P) = T$ and $N(P) = S$ is known as outer inverse $A_{T,S}^{(2)}$ of A with the prescribed range T and null space S . It is unique and important as some of the most commonly used generalized inverses can be derived from it by means of appropriate choices of T and S . For example, for $T = R(A^*)$ and $S = N(A^*)$, we get the Moore–Penrose inverse $A^\dagger$ of $A \in \mathbb{C}^{m \times n}$ represented as $A_{T,S}^{(2)} = A^\dagger$. We define $A^\dagger$ as the unique matrix $X \in \mathbb{C}^{n \times m}$ such that $AXA = A$, $XAX = X$, $(AX)^* = AX$, $(XA)^* = XA$. For $T = R(A^j)$ and $S = N(A^j)$, where the index j of $A \in \mathbb{C}^{n \times n}$ is defined as $\min\{j : \text{rank}(A^{j+1}) = \text{rank}(A^j)\}$, we get the Drazin inverse

A^d such that $A_{T,S}^{(2)} = A^d$. It is defined as the unique matrix $X \in \mathbb{C}^{n \times n}$ such that $A^jXA = A^j$, $XAX = X$, $AX = XA$. For $j = 1$, X is known as the group inverse A_g of A . Many other

important generalized inverses such as, the weighted Moore–Penrose inverse $A_{M,N}^\dagger$, the Bott–Duffin inverse $A_{(L)}^{(-1)}$ and the generalized Bott–Duffin inverse $A_{(L)}^{(\dagger)}$ are also its special cases (see, [1]). A number of application areas (see [2–4]) that give rise to the computation of $A_{T,S}^{(2)}$ are statistics, optimization, multi-body system dynamics, differential algebraic equations, image processing, etc. In literature, many researchers [5–11] have proposed many direct and iterative methods for computing outer inverses. Some of the direct methods studied are usually based on the QR factorization [12], Cholesky factorization [13], and Gaussian elimination [14]. In [14], authors presented a direct method for computing outer inverse $A_{T,S}^{(2)}$ based on the choice of proper matrix G and the Gauss–Jordan elimination of the augmented matrix $[GI]$, where $G \in \mathbb{C}^{n \times m}$ be such that $R(G) = T$ and $N(G) = S$ and I be the identity matrix of suitable order. Based on the full-rank LU decomposition of an appropriate matrix M , a new direct method is developed in [15]. The most frequently used simple and numerically stable Newton's iterative method [16, 17] for approximating $A_{T,S}^{(2)}$ is given for $k = 0, 1, 2, \dots$, by,

$$X_{k+1} = X_k(2I - AX_k), \quad (1)$$

where X_0 be a suitably chosen initial matrix and I denotes the identity matrix of appropriate order. A method based on splitting of matrices is described in [18]. A quadratically convergent iterative scheme proposed in [19] based on matrix equations $XAX = X$ and $(XA)^* = XA$ is given for $k = 0, 1, 2, \dots$, by

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$$X_{k+1} = (1 + \beta)X_k - \beta X_k A X_k, \beta \in (0, 1].$$

Starting with $X_0 = \alpha G$, where α, β are real constants and $G \in \mathbb{C}^{n \times m}$ is a chosen matrix of rank s such that $0 < s \leq r$, it converges to outer inverse $A_{R(G), N(G)}^{(2)}$. A higher-order iterative scheme investigated in [20] for $A_{T,S}^{(2)}$ for arbitrary positive integer $p \geq 2$ is given for $k = 0, 1, 2, \dots$, by

$$X_k = \left[C_p^1 I - C_p^2 X_{k-1} A + \dots + (-1)^{p-1} C_p^p (X_{k-1} A)^{p-1} \right] X_{k-1}, \quad (2)$$

where $C_n^i = \frac{n!}{i!(n-i)!}$, $i = 0, 1, 2, \dots, n$ denote the binomial coefficients. It was shown that starting with $X_0 = \alpha G$, where $G \in \mathbb{C}^{n \times m}$ satisfies $R(G) = T$ and $N(G) = S$ and α be a non-zero complex scalar, method (2) converges to $A_{T,S}^{(2)}$ if and only if $\rho(\alpha G A - P_T) < 1$, where the transformation P_T denotes the orthogonal projector on T . In [21], author studied a linear order iterative scheme for computing $A_{T,S}^{(2)}$, given for $k = 0, 1, 2, \dots$ by, $X_{k+1} = X_k + \beta G(I - A X_k)$ for $G \in \mathbb{C}^{n \times m}$ satisfying $R(G) \subset T$, $N(G) \supset S$ and a non-zero real scalar β . It was shown that starting with any initial approximation $X_0 \in \mathbb{C}^{n \times m}$ satisfying $R(X_0) \subset T$, the sequence $\{X_k\}$ converges to $A_{T,S}^{(2)}$ if and only if $\rho(P_T - \beta G A) < 1$. Its p th order extension was also developed by him. An iterative algorithm to compute $A_{T,S}^{(2)}$ under the concepts of restricted inner product is presented in [22]. An algorithm to compute $A_{T,S}^{(2)}$ using successive matrix squaring (SMS) is presented in [23].

In this paper, a two-step iterative method and its accelerated version for approximating outer inverse $A_{T,S}^{(2)}$ of an arbitrary matrix A are proposed. A convergence theorem for its existence is established. The rigorous error bounds are derived. Numerical experiments involving singular square, rectangular, random matrices and a sparse matrix obtained by discretization of the Poisson's equation are solved. Iterations count, computational time and the error bounds are used to measure the performance of our methods. On comparing our methods with others, it is seen that significantly better results are obtained. Thus, our method leads to enhanced speed and accuracy from the computational points of view.

The paper is organized as follows. section 1 is the introduction. In section 2, two-step iterative method for $A_{T,S}^{(2)}$ and some basic lemmas used are described. A convergence theorem along with its error bounds is also derived here. Two-step accelerated iterative method for its computation is established in section 3. A number of numerical experiments are performed in section 4. Finally, conclusions are included in section 5.

2. Two-step iterative method and its convergence analysis

In this section, a two-step iterative method for computing $A_{T,S}^{(2)}$ is described. Its convergence analysis and error bounds are also derived. Let $\dim(T) = \dim(S^\perp) = t \leq r$, where $S^\perp$ denotes the orthogonal complement of a subspace S . Let $G \in \mathbb{C}^{n \times m}$ satisfies $R(G) \subseteq T$ and $N(G) \supseteq S$. Starting with $X_0 = \beta G$, for a non-zero real scalar β , the method is given for $k = 0, 1, 2, \dots$, by

$$Z_{k+1} = X_k + X_k R_k, \quad (3)$$

$$X_{k+1} = X_k + Z_{k+1} R_k, \quad (4)$$

where $R_k = I - A X_k$ and sequence $\{Z_{k+1}\}$ is approximated by the Newton's iteration. The following basic results are important to prove our further statements.

Lemma 2.1 [16, 24] *Let $\dim T = \dim S^\perp = t \leq r$. A has a unique outer inverse X for $R(X) = T$ and $N(X) = S$ if and only if*

$$AT \oplus S = \mathbb{C}^m, \quad (5)$$

where $\oplus$ denotes the direct sum of subspaces AT and S .

We have following remarks from Lemma 2.1.

Remark 2.1 The condition (5) is equivalent to the following conditions.

$$A^* S^\perp \oplus T^\perp = \mathbb{C}^n.$$

$$P_{S^\perp} A T = S^\perp.$$

$$P_T A^* S^\perp = T.$$

Remark 2.2 ([16, 25]) $AA_{T,S}^{(2)} = P_{AT,S}$ and $A_{T,S}^{(2)}A = P_{T,(A^*S^\perp)^\perp}$.

Lemma 2.2 [16] *For any $A \in \mathbb{C}^{m \times n}$, $P_{L,M}A = A$ if and only if $R(A) \subset L$ and $AP_{L,M} = A$ if and only if $N(A) \supset M$.*

Lemma 2.3 *The sequence $\{X_k\}$ for $k \geq 0$ from (3) to (4) with $X_0 = \beta G$, where $G \in \mathbb{C}^{n \times m}$ be such that $R(G) \subseteq T$ and $N(G) \supseteq S$, satisfies (i) $A_{T,S}^{(2)} A X_k = X_k$ and (ii) $AA_{T,S}^{(2)} A X_k = A X_k$.*

Proof This lemma can be proved by mathematical induction. Clearly, (i) follows for $k = 0$ from Remark 2.2 and Lemma 2.2 as

$$A_{T,S}^{(2)} A X_0 = P_{T,(A^*S^\perp)^\perp} X_0 = \beta P_{T,(A^*S^\perp)^\perp} G = \beta G = X_0.$$

Suppose $A_{T,S}^{(2)} A X_k = X_k$ holds for some positive integer k . To show that it holds for $k + 1$, we consider

$$\begin{aligned}
A_{T,S}^{(2)}AX_{k+1} &= A_{T,S}^{(2)}A[X_k + Z_{k+1}R_k], \\
&= A_{T,S}^{(2)}AX_k + A_{T,S}^{(2)}AX_kR_k + A_{T,S}^{(2)}AX_kR_k^2, \\
&= X_k + X_kR_k + X_kR_k^2, \\
&= X_k + (X_k + X_kR_k)R_k, \\
&= X_k + Z_{k+1}R_k, \\
&= X_{k+1}.
\end{aligned}$$

This proves that, (i) holds for all $k \geq 0$. Next, (ii) follows for $k = 0$ from Remark 2.2 and Lemma 2.2 as

$$AA_{T,S}^{(2)}AX_0 = P_{AT,S}AX_0 = AX_0.$$

Suppose $AA_{T,S}^{(2)}AX_k = AX_k$ holds for some positive integer k . To show that it holds for $k + 1$, we consider

$$\begin{aligned}
AA_{T,S}^{(2)}AX_{k+1} &= AA_{T,S}^{(2)}A[X_k + Z_{k+1}R_k], \\
&= AA_{T,S}^{(2)}AX_k + AA_{T,S}^{(2)}AX_kR_k + AA_{T,S}^{(2)}AX_kR_k^2, \\
&= AX_k + AX_kR_k + AX_kR_k^2, \\
&= A[X_k + Z_{k+1}R_k], \\
&= AX_{k+1}.
\end{aligned}$$

Thus, (ii) holds for all $k \geq 0$. $\square$

Theorem 2.1 Let the initial approximation $X_0 \in \mathbb{C}^{n \times m}$ be such that $R(X_0) \subset T$ and $X_0 = \beta G$ for non-zero real scalar β and $G \in \mathbb{C}^{n \times m}$ such that $R(G) \subseteq T$ and $N(G) \supseteq S$. Starting with X_0 , the iterative method given by (3)–(4) leads to a sequence $\{X_k\}$ converging to

$$X_\infty = A_{T,S}^{(2)} = X_0(I - F_0)^{-1}, \quad (6)$$

if and only if $\rho(F_0) = \rho(P_{AT,S} - AX_0) < 1$. Further, let $\|P_{AT,S} - AX_0\| = q < 1$, then X_k has the error estimation

$$\|X_{k+1} - A_{T,S}^{(2)}\| \leq q^{3k}(1 - q)^{-1}\|X_0\|. \quad (7)$$

Proof Denote the residual matrix as, $F_k = P_{AT,S} - AX_k = AA_{T,S}^{(2)} - AX_k$. Using (3)–(4), Lemma 2.2 and Remark 2.2, we get

$$\begin{aligned}
F_{k+1} &= P_{AT,S} - AX_{k+1}, \\
&= AA_{T,S}^{(2)} - A(X_k + Z_{k+1}R_k), \\
&= AA_{T,S}^{(2)} - (AX_k)^3 - 3AX_k + 3(AX_k)^2, \\
&= (AA_{T,S}^{(2)} - AX_k)^3.
\end{aligned}$$

This gives,

$$F_{k+1} = F_k^3 = F_0^{3^{k+1}} = (P_{AT,S} - AX_0)^{3^{k+1}}. \quad (8)$$

Since, $\rho(P_{AT,S} - AX_0) < 1$, this gives $X_k \rightarrow A_{T,S}^{(2)}$ as $k \rightarrow \infty$ for any $X_0 = \beta G$. Converse part of this theorem is easy to prove, hence omitted. Next, from Lemma 2.2 and (4) we get,

$$X_{k+1} = X_k + Z_{k+1}(P_{AT,S} - AX_k).$$

This implies that

$$X_{k+1} = X_0 \left[I + (P_{AT,S} - AX_0) + \dots (P_{AT,S} - AX_0)^{3^{k+1}-1} \right]. \quad (9)$$

Taking $k \rightarrow \infty$ in (9) and using $\rho(P_{AT,S} - AX_0) < 1$, we get

$$X_\infty = A_{T,S}^{(2)} = X_0(I - (P_{AT,S} - AX_0))^{-1} = X_0(I - F_0)^{-1}. \quad (10)$$

Now, to get the error estimates, from (9) and (10), we have

$$\begin{aligned}
X_k - A_{T,S}^{(2)} &= -X_0 \left[(P_{AT,S} - AX_0)^{3^k} + (P_{AT,S} - AX_0)^{3^{k+1}} + \dots \right] \\
&= -X_0(P_{AT,S} - AX_0)^{3^k} \left[I + (P_{AT,S} - AX_0) + \dots \right]
\end{aligned}$$

Now taking norm on both the sides of above equation and using $\|P_{AT,S} - AX_0\| = q < 1$, we obtain

$$\|X_k - A_{T,S}^{(2)}\| \leq \|X_0\|q^{3^k} \left[1 + q + q^2 + \dots \right], \leq q^{3^k}(1 - q)^{-1}\|X_0\|.$$

$\square$

This completes the proof. Following lemma describes the relationship between the error matrix $E_k = A_{T,S}^{(2)} - X_k$ and the residual matrix $F_k = P_{AT,S} - AX_k$.

Lemma 2.4 Let $A \in \mathbb{C}^{m \times n}$, then we have

$$\|E_{k+1}\| \leq \|A_{T,S}^{(2)}\| \|F_k^3\|. \quad (11)$$

Proof Denote $E_k = A_{T,S}^{(2)} - X_k$ as the error matrix for computing $A_{T,S}^{(2)}$, then we have,

$$AE_{k+1} = AA_{T,S}^{(2)} - AX_{k+1} = F_{k+1} = F_k^3. \quad (12)$$

Using (12) and Remark 2.2, we get

$$\begin{aligned}
\|E_{k+1}\| &= \|X_{k+1} - A_{T,S}^{(2)}\|, \\
&= \|A_{T,S}^{(2)}AX_{k+1} - A_{T,S}^{(2)}AA_{T,S}^{(2)}\|, \\
&\leq \|A_{T,S}^{(2)}\| \|AE_{k+1}\|, \\
&\leq \|A_{T,S}^{(2)}\| \|F_k^3\|.
\end{aligned}$$

This completes the proof. $\square$

Remark 2.3 Using (8), inequality (11) can be further simplified as

$$\|E_{k+1}\| \leq \|A_{T,S}^{(2)}\| \|F_k^3\| \leq \|A_{T,S}^{(2)}\| \|F_0\|^{3^{k+1}}. \quad (13)$$

Since $\rho(F_0) < 1$, from (13) we get $\|E_{k+1}\| \rightarrow 0$ as $k \rightarrow \infty$. This leads to the conclusion that iterative method (3)–(4) converges to $A_{T,S}^{(2)}$ with the third order of convergence.

Remark 2.4 The convergence of the iteration (3)–(4) requires $\rho(F_0) = \rho(P_{AT,S} - AX_0) < 1$. However, this cannot be directly checked since it involves the unknown $A_{T,S}^{(2)}$. To verify this, we need an equivalent convergence condition given in Theorem 2.2 which does not involve $A_{T,S}^{(2)}$.

Following lemma from [23] will be needed to prove our further results.

Lemma 2.5 *If $P \in \mathbb{C}^{n \times n}$ and $S \in \mathbb{C}^{n \times n}$ satisfy $P = P^2$ and $PS = SP$ then $\rho(PS) \leq \rho(S)$.*

Proof Suppose that λ is an eigenvalue of PS and x is a corresponding vector such that $PSx = \lambda x$. Then, $\lambda Px = \lambda x$, which implies that $\lambda = 0$ or $Px = x$. If $Px = x$, then $Sx = SPx = PSx = \lambda x$, i.e. λ is an eigenvalue of S . Hence, the set of eigenvalues of PS is the subset of the set of eigenvalues of S union $\{0\}$, which implies that $\rho(PS) \leq \rho(S)$. $\square$

Theorem 2.2 *For all non-negative eigenvalues of AG , we have $\rho(P_{AT,S} - AX_0) < 1$ if*

$$\max_{1 \leq i \leq r} |1 - \beta \lambda_i(AG)| < 1. \quad (14)$$

Proof Denote $P = AA_{T,S}^{(2)}$ and $S = I - AX_0$. Then from Lemmas 2.2 and 2.3, we get

$$\begin{aligned} PS &= AA_{T,S}^{(2)}(I - AX_0) = AA_{T,S}^{(2)} - AA_{T,S}^{(2)}AX_0 = AA_{T,S}^{(2)} - AX_0, \\ &= (I - AX_0)AA_{T,S}^{(2)} = AA_{T,S}^{(2)} - AX_0AA_{T,S}^{(2)} = AA_{T,S}^{(2)} - AX_0 = SP. \end{aligned}$$

It is clear that $P^2 = P$. Now from Lemma 2.5, we have

$$\begin{aligned} \rho(P_{AT,S} - AX_0) &= \rho(AA_{T,S}^{(2)} - AX_0) \leq \rho(I - AX_0) \\ &= \max_{1 \leq i \leq r} |1 - \beta \lambda_i(AG)| < 1 \end{aligned}$$

This completes the proof. $\square$

Corollary 2.1 *Convergence condition $\max_{1 \leq i \leq r} |1 - \beta \lambda_i(AG)| < 1$ is equivalent to the following condition*

$$0 < \beta < \frac{2}{\lambda_{\max}(AG)}. \quad (15)$$

Proof The proof is obvious and hence not given here. $\square$

3. Two-step accelerated iterative method for outer inverse

The purpose of this section is to further accelerate the iterative method (3)–(4) described in section 2. Following the technique proposed in [26] for accelerating the convergence speed of Newton's method, we can define our accelerated iterative method for approximating $A_{T,S}^{(2)}$ for $k = 0, 1, 2, \dots$ by

$$Z_{k+1} = \gamma_{k+1}(X_k + X_k R_k), \quad (16)$$

$$X_{k+1} = X_k + Z_{k+1} R_k, \quad (17)$$

where, X_0 is taken as discussed in section 2, $R_k = I - AX_k$ and γ_{k+1} is the acceleration parameter. It should be noted that the acceleration parameter γ_{k+1} is selected so that it minimizes a bound on the maximum distance of any non-zero singular values of $X_{k+1}A$ from 1. Let bounds $\sigma_{\min}$ and $\sigma_{\max}$ on the singular values of A are known such that

$$\sigma_{\min} \leq \sigma_r^2 \leq \sigma_1^2 \leq \sigma_{\max}.$$

In this case, every eigenvalue of GA lies in interval $[\sigma_{\min}, \sigma_{\max}]$. Assuming that this interval has a non-zero length, otherwise, A is a scalar multiple of a unitary matrix and GA is its inverse. Taking $\rho_j^{(0)} = \frac{2\sigma_j^2}{\sigma_{\min} + \sigma_{\max}}$ and $\rho_j^{(k+1)} = \gamma_{k+1}(2 - \rho_j^{(k)})\rho_j^{(k)}$ denotes lower and upper bounds on the singular values of βGA . The values given by

$$\underline{\rho}^{(0)} = \frac{2\sigma_{\min}}{\sigma_{\min} + \sigma_{\max}}, \quad (18)$$

and

$$\overline{\rho}^{(0)} = \frac{2\sigma_{\max}}{\sigma_{\min} + \sigma_{\max}}. \quad (19)$$

are used for the acceleration parameters γ_{k+1} , $k > 0$. Now, we can define

$$\gamma_{k+1} = \overline{\rho}^{(k+1)} = \frac{2}{1 + (2 - \underline{\rho}^{(k)})\underline{\rho}^{(k)}}, \quad (20)$$

as an acceleration parameter and an upper bound of $\{\rho_j^{(k+1)}\}$, and

$$\underline{\rho}^{(k+1)} = \gamma_{k+1}(2 - \underline{\rho}^{(k)})\underline{\rho}^{(k)}, \quad (21)$$

as a lower bound on $\{\rho_j^{(k+1)}\}$. The definitions (18) and (19) gives for all $1 \leq j \leq r$, and $k \geq 1$,

$$\underline{\rho}^{(k)} \leq \rho_j^{(k)} \leq \overline{\rho}^{(k)}.$$

From (20) and (21), we get

$$\underline{\rho}^{(k)} = 2 - \overline{\rho}^{(k)}. \quad (22)$$

Likewise, it is easy to prove that $\rho_j^{(k)} \leq \overline{\rho}^{(k)}$. Finally, if $0 \leq \underline{\rho}^{(k)} \leq \rho_j^{(k)} \leq (2 - \underline{\rho}^{(k)})$ holds, then we have $(2 - \underline{\rho}^{(k)})\underline{\rho}^{(k)} \leq (2 - \rho_j^{(k)})\rho_j^{(k)} \leq 1$. Except for the last few iterations before convergence, $\underline{\rho}^{(k)} \ll 1$, which implies $\gamma_{k+1} \approx 2$ and $\rho_j^{(k+1)} \approx 4\rho_j^{(k)}$.

Following algorithm gives the implementation of accelerated iterative method (16)–(17) to compute the outer inverse $A_{T,S}^{(2)}$ for matrix A .

Algorithm 3.1 (Algorithm for computing $A_{T,S}^{(2)}$ from iterative method (16)–(17))

- 1: Initialization: Input matrix A , $X_0 = \beta G$, $0 < \beta < \frac{2}{\lambda_{\max}(AG)}$ and $\rho^{(0)}$.
- 2: Perform the following steps 3,4,5 until the termination criteria are satisfied.
- 3: $\gamma_{k+1} = \frac{2}{1+(2-\rho^{(k)})\rho^{(k)}}$.
- 4: $Z_{k+1} = \gamma_{k+1}(X_k + X_k R_k)$.
- 5: Compute $X_{k+1} = X_k + Z_{k+1} R_k$.
- 6: Return the output $A_{T,S}^{(2)}$.

The proposed two-step accelerated iteration(16)–(17) and general two-step iteration (3)–(4) require three matrix–matrix multiplications per iteration. The only additional cost in (16)–(17) is the computation of $\gamma_{k+1} = \bar{\rho}^{(k+1)}$ which can be done with few operations by their formulas.

4. Numerical experiments

Numerical experiments involving singular square, rectangular, random matrices and a sparse matrix obtained by discretization of the Poisson's equation are solved. Iterations count (NN), computational time in seconds (CT) and the error bounds are used to measure the performance of our methods. As a case studies of the $A_{T,S}^{(2)}$, the $A^\dagger$ and A_g are used in the numerical examples.

Example 4.1 ([10]) (Academical test) Let A be the rectangular matrix of order (3×4) given by

$$A = \begin{pmatrix} 1 & 0 & 0 & -6 \\ 2 & 6 & 0 & -6 \\ 7 & 8 & 9 & -6 \end{pmatrix}$$

Starting with $X_0 = \beta A^*$, where $G = A^*$, the proposed iteration method converges to the outer inverse $A_{R(A^*),N(A^*)}^{(2)} = A^\dagger$ given by

$$A^\dagger = \begin{pmatrix} \frac{28}{1931} & -\frac{143}{3862} & \frac{84}{1931} \\ \frac{653}{3862} & \frac{1335}{7724} & \frac{14}{1931} \\ \frac{57}{1931} & -\frac{249}{1931} & \frac{171}{1931} \\ -\frac{1903}{11586} & -\frac{143}{23172} & \frac{14}{1931} \end{pmatrix}$$

where $\beta = \frac{1}{\|A\|^2}$, which satisfies the convergence condition $0 < \beta < \frac{2}{\rho(A^*A)}$, discussed in Corollary 2.1. The stopping criteria given by $\|X_k A X_k - X_k\|_2 \leq \epsilon = 10^{-1000}$ with matrix

2-norm is used for error estimates. To clearly demonstrate the convergence behavior of the proposed method, we used 1000 fixed point digits in our computations analysis by using MATLAB's multiprecision package MATLAB vpa. The NN and estimation of error bounds obtained by accelerated method given by (16)–(17) compared with those obtained by Newton's method (1) and the method (2), for $p = 3$ to compute $A^\dagger$ are summarized in tables 1, 2 and 3, respectively. It can be observed that proposed method converges considerably faster in comparison to other methods.

Example 4.2 Let A be a randomly generated rectangular matrix of order 1000×500 generated by MATLAB function rand(1000, 500). The condition number of generated matrices is of the order 10^2 and its entries are from the uniform distribution $U(0, 1)$.

Starting with $X_0 = \beta G$, where $G = A^*$ the proposed iteration method converges to the outer inverse $A_{R(A^*),N(A^*)}^{(2)} = A^\dagger$. The choice of β is taken such that the

Table 1. Comparison results for error norm $E_k = \|A^\dagger - X_k\|_2$ for computing $A^\dagger$.

NN	Method (1)	Method (2)	Method (16)–(17)
6	8.0750e–002	1.5761e–005	1.4406e–006
7	2.2108e–002	4.5011e–014	3.4370e–017
8	1.6572e–003	1.0484e–039	4.6674e–049
9	9.3120e–006	1.3246e–116	1.1689e–144
10	2.9401e–010	2.6717e–347	7.4999e–358

Table 2. Comparison results for residual norm $F_k = \|AA^\dagger - AX_k\|_2$ for computing $A^\dagger$.

NN	Method (1)	Method (2)	Method (16)–(17)
6	2.7379e+001	5.3440e–005	4.8845e–006
7	7.4960e–002	1.5261e–013	1.1653e–016
8	5.6190e–003	3.5545e–039	1.5825e–048
9	3.1573e–005	4.4911e–116	3.9632e–144
10	9.9685e–010	9.0584e–347	2.5429e–357

Table 3. Comparison results for norm $\|X_k A X_k - X_k\|_2$ for computing $A^\dagger$.

NN	Method (1)	Method (2)	Method (16)–(17)
6	5.8642e–002	1.5761e–005	1.4406e–006
7	2.0451e–002	4.5011e–014	3.4370e–017
8	1.6479e–003	1.0484e–039	4.6674e–049
9	9.3117e–006	1.3246e–116	1.1689e–144
10	2.9401e–010	2.6717e–347	7.4999e–358

Figure 1. Histogram to visualize a distribution of numbers in matrix A for Example 4.2.

Figure 2. Variation of parameter γ_{k+1} for Example 4.2.

convergence condition $0 < \beta < \frac{2}{\rho(A^*A)}$ is satisfied. The termination criteria are given as $\max\{\|AX_kA - A\|_2, \|X_kAX_k - X_k\|_2, \|(AX_k)^* - AX_k\|_2, \|(X_kA)^* - X_kA\|_2\} \leq \epsilon = 10^{-7}$. Figure 1 gives the visualization of the distribution of numbers from the uniform distribution $U(0, 1)$ considered

in the matrix A . Variation of the acceleration parameter γ_{k+1} with respect to number of iterations is shown in figure 2. It can be observed that $\gamma_{k+1} \in [1, 2]$, which also confirms the theoretical observation discussed in section 3. Table 4 compares the error bounds and NN by applying the proposed accelerated method (16)–(17), and iterative methods (1) and (2). Further, figure 3 shows the comparison of computational time. Note that x-axis in figure 3 represents the number of times random matrices are selected which is 30, whereas y-axis represents the computational time in seconds. Again, it can be observed that the proposed method (16)–(17) converges faster than the methods (1) and (2) and gives improved results.

Example 4.3 Let A be a randomly generated singular matrix of the order 600×600 . The rank and index j of generated matrix are 300 and 1, respectively.

Starting with $X_0 = \beta G$, where $G = A^j = A$, the proposed iteration method converges to the outer inverse $A_{R(A), N(A)}^{(2)} = A_g$. The choice of β is taken such that the convergence condition $0 < \beta < \frac{2}{\rho(A^*A)}$ is satisfied. The termination criteria are given by $\max\{\|AX_kA - A\|_2, \|X_kAX_k - X_k\|_2, \|AX_k - X_kA\|_2\} \leq \epsilon = 10^{-7}$. Figure 4 gives the visualization of the distribution of numbers considered in the matrix A . It should be noted that entries of matrix A does not follow any particular distribution. Variation of the acceleration parameter γ_{k+1} with respect to number of iterations is shown in figure 5. It can be observed that $\gamma_{k+1} \in [1, 2]$, which again confirms the theoretical observation discussed in section 3. Table 5 gives comparison results in terms of error estimates obtained by applying the proposed accelerated method (16)–(17), iterative methods (1), (2) and method from [27]. Additionally, comparison results of computational time and number of iterations obtained by applying these methods are shown in figures 6 and 7, respectively. x-axis in figures 6 and 7 represents the number of times random matrices are selected which is 30, whereas y-axis represents the computational time in seconds and number of iterations, respectively. From table 5, it can be observed that error estimates obtained by (16)–(17) is almost same as those obtained by methods (1) and (2). However, method from [27] was not able to give correct results due to large error for $\|AX_kA - A\|_2$. On the other hand, accelerated method outperforms methods (1) and (2)

Table 4. Comparison results for computing $A^\dagger$.

Performance measures	Method (1)	Method (2)	Method (16)–(17)
NN	19	12	9
$\ AX_kA - A\ _2$	5.1492e–13	4.2628e–13	4.1733e–13
$\ X_kAX_k - X_k\ _2$	2.3421e–14	1.1294e–15	4.8129e–16
$\ (AX_k)^* - AX_k\ _2$	5.4890e–14	4.8866e–14	4.3010e–14
$\ (X_kA)^* - X_kA\ _2$	4.6644e–14	3.7960e–14	3.4129e–14

Figure 3. Comparison of CPU time for Example 4.2.

Figure 4. Histogram to visualize a distribution of numbers in matrix A for Example 4.3.

Figure 5. Variation of parameter γ_{k+1} for Example 4.3.

in terms of the computational speed, shown by figures 6 and 7, respectively. Note that computational time and number of iterations taken by method from [27] for each random run was approximately 35 s and 100, respectively, so we have omitted that from the figures. Thus one can conclude that significantly better performance is achieved by accelerated method (16)–(17).

Example 4.4 [28] Consider a sparse singular symmetric matrix A of index $j = 1$, where the matrix A is an $(n + 1)^2 \times (n + 1)^2$ arises from the 5-point central difference discretization of the following Poisson's equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = c(p, q), (p, q) \in \Omega = (0, 1) \times (0, 1)$$

and all Neumann boundary conditions given by

$$\frac{\partial w}{\partial N} = g,$$

where c and g satisfy the compatibility condition $\int_{\Omega} c dx dy = \int_{\partial \Omega} g ds$. Taking mesh size $h = \frac{1}{n}$, for $n = 35$, the matrix A of order 1296×1296 is given by,

$$A = \begin{pmatrix} \frac{1}{2}V_{n+1} - W_{n+1} & & & & & \\ -W_{n+1} & V_{n+1} & -W_{n+1} & & & \\ & \ddots & \ddots & \ddots & & \\ & & & -W_{n+1} & V_{n+1} & -W_{n+1} \\ & & & -W_{n+1} & \frac{1}{2}V_{n+1} & \end{pmatrix},$$

where I_n is an $(n + 1) \times (n + 1)$ identity matrix and V_{n+1} and W_{n+1} are $(n + 1) \times (n + 1)$ matrices given as follows

$$V_{n+1} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & -1 & 4 & -1 \\ & & & -1 & 2 & \end{pmatrix},$$

$$W_{n+1} = \begin{pmatrix} \frac{1}{2} & & & & \\ & 1 & & & \\ & \ddots & \ddots & \ddots & \\ & & & 1 & \\ & & & & \frac{1}{2} \end{pmatrix}.$$

Starting with $X_0 = \beta G$, where $G = A^j = A$, the proposed iteration method converges to $A_{R(A), N(A)}^{(2)} = A_g$, the group

Table 5. Comparison results for error bounds for computing A_g .

Error bounds	Method from [27]	Method (1)	Method (2)	Method (16)–(17)
$\ AX_kA - A\ _2$	4.4819e+04	9.7926e-09	5.47061e-09	2.3926e-09
$\ X_kAX_k - X_k\ _2$	2.2326e-12	5.4083e-11	4.5776e-11	1.1567e-11
$\ AX_k - X_kA\ _2$	3.2466e-22	6.0302e-11	5.9075e-11	3.5234e-12

Figure 6. Comparison of CPU time for Example 4.3. **Figure 7.** Comparison of number of iterations for Example 4.3.

inverse of A . The choice of β is taken such that the convergence condition $0 < \beta < \frac{2}{\rho(A^t A)}$ is satisfied. The termination criteria are same as taken in Example 4.3. The sparsity structure of matrix A is illustrated in figure 8, where ‘nz’ stands for the number of non-zero entries of the matrix, which are 6300. Variation of the acceleration parameter γ_{k+1} with respect to number of iterations are presented in

Figure 8. Sparsity structure of matrix for Example 4.4. **Figure 9.** Variation of parameter γ_{k+1} for Example 4.4.

figure 9. It can be observed that $\gamma_{k+1} \in [1, 2]$, which confirms the theoretical observation discussed in section 3. The performance measures CT, NN, and error bounds are compared with the methods (1), (2) as well as with the

Table 6. Comparison results for CT, NN and error bounds for computing A_g .

Performance measures	Method from [27]	Method (1)	Method (2)	Method (16)–(17)
CT	183.5531	78.0728	52.1202	40.4580
NN	100	35	22	17
$\ AX_kA - A\ _2$	1.95756e+00	3.5475e-12	7.5044e-12	3.7460e-12
$\ X_kAX_k - X_k\ _2$	8.0522e-02	5.2762e-09	1.6572e-09	4.2130e-09
$\ AX_k - X_kA\ _2$	2.8613e-15	1.0772e-11	1.2288e-11	9.9853e-12

method studied in [27]. The results are tabulated in table 6. Again it can be observed that accelerated method (16)–(17) takes very less computational time and number of iterations to achieve almost the same level of accuracy than the methods (1) and (2) and gives improved results.

5. Conclusions

A two-step iteration and its accelerated version for computing the outer inverse $A_{T,S}^{(2)}$ of an arbitrary matrix A is described. It is based on the well-known Newton's method for approximating outer inverse of arbitrary matrix A . The convergence analysis along with the estimation of error bounds is derived. Numerical experiments involving singular square, rectangular, random matrices and a sparse matrix obtained by discretization of the Poisson's equation are solved. Iterations count (NN), computational time (CT) and the error bounds are used to measure the performance of our methods. As a case studies of the $A_{T,S}^{(2)}$, the Moore–Penrose inverse $A^\dagger$ and group inverse A_g are used in the numerical examples. The results obtained from the accelerated iterative method are compared with those obtained by some of the existing methods. It is apparent that the accelerated iterative method converges considerably faster in comparison with other existing methods. Thus, both computational speed and accuracy are significantly improved by it.

The work of this paper can be extended in various directions. Some of the future works to be carried out can be listed as follows.

- As an application, one can modify the proposed iterative methods to approximate generalized inverses of structured matrices such as Toeplitz and Hankel matrices.
- One can also extend the proposed iterative to approximate the outer inverse $A_{T,S}^{(2)}$ of bounded linear operators A between Banach spaces.

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