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Optimal time policy for deteriorating items of two-warehouse inventory system with time and stock dependent demand and partial backlogging

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Abstract. We consider the problem of a two-warehouse inventory system under the effect of stock dependent demand. There are two warehouses to store the goods in which the first is rented warehouse and the second is own warehouse that deteriorates with two different rates. The aim of this study is to determine the optimal order quantity to maximize the profit of the projected model. Finally, some numerical examples and sensitivity analysis of parameters are made to validate the results of the proposed model.

Keywords. Deterioration; stock; profit; demand; warehouse; price.

1. Introduction

At the present time, various researchers have worked on single/two warehouse inventory system, which were calculated in a statics surroundings where the demand have been considered of various kinds such as constant demand, linear demand, time proportional, ramp type demand, time varying and time dependent. Many products of inventory like domestic goods, fashionable cloth, electronic product, tasty food products and domestic items are responsible for the increase in the sales after gaining costumer's acceptances. In this sense, many authors the over time have derived different strategies for single/two warehouses inventory system when shortage is allowed and partial backlogging depends on the waiting time [1–6]. Several authors have also studied inventory system without partial backlogging [7–15]. Goyal and Giri [16] developed the deteriorating inventory model without considering replacement policies for inventory which are subject to obsolescence. Jaggi and Tiwari [17] extended two-warehouse inventory model for non-instantaneous deteriorating items with price dependent demand and time-varying holding cost. Daultani *et al* [18] have done a supply chain network equilibrium model for operational and opportunism risk mitigation (table 1).

In this paper, the most important purpose of modeling the two-warehouse inventory is to determine the optimal order quantity in order to maximize the profit of the system. We have assumed that backlogging rate is exponentially decreasing with waiting time. It has been practically noticed in super market that the demand rate is regularly influenced by the current stock level. We have considered the item in the rented warehouse (RW) which has a lower deterioration rate while the item in own warehouse (OW) is decreasing with high deterioration rate. The necessary and sufficient condition for existence and uniqueness of the optimal solution are presented with some numerical example to demonstrate the developed model.

The variation of inventory level per cycle is expressed as depicted in figure 1.

Considering the two-warehouse inventory system such that the capacity of OW inventory system is Q and the capacity of RW is unlimited. Initial demand is satisfied from an RW after that items are consumed from OW. Initially during the period $[0, t_0]$, RW inventory level depletes due to deterioration as well as demand at the same period whereas OW depletes due to deterioration only. During the period $[t_0, t_1]$ inventory level of OW decreases due to demand as well as deterioration.

The following fundamental notations and assumptions are used to derive the solution of the model.

Notations

A	ordering cost per period
a	initial demand of items
b	parameter of demand governing increasing ($b > 0$) or decreasing ($b < 0$) trend, $b < a$
h_r	the holding cost per unit item of RW
h_o	the holding cost per unit item of OW
c_d	cost of deteriorated unit
c_s	shortage cost per unit per unit time
c_l	lost sale cost per unit
c	purchase cost per unit item

*For correspondence

t_0	the time when inventory of RW becomes empty
t_1	time when inventory level reaches to zero
T	the replenishment cycle
$TPF(t_0, t_1, T)$	average profit of the system
$I_r(t)$	inventory level at of RW at any time t, $0 < t < t_0$
$I_{01}(t)$	inventory level OW at any time t, $0 \leq t \leq t_0$
$I_{02}(t)$	inventory level OW at any time t, $t_0 \leq t \leq t_1$
$I_{03}(t)$	inventory level OW at any time t, $t_1 \leq t \leq T$
S	shortage level
OQ^*	optimal Ordering quantity
α	deterioration rate of RW
β	deterioration rate of OW
p	selling price per unit of the item
$D(t)$	demand function

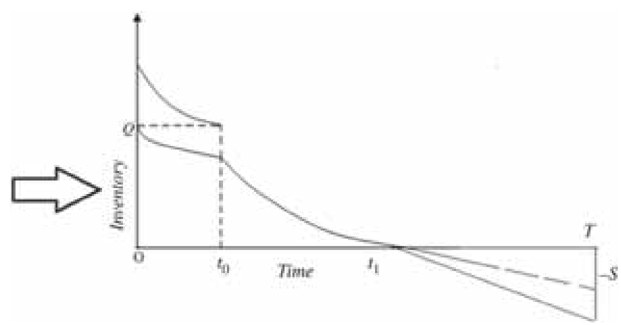

Figure 1. Geometry of the problem.

Table 1. Summary of related literature for inventory models with partial backlogging and without partial backlogging.

References	Demand rate	Deterioration rate	Single/two warehouse	Under inflation	With partial backlogging	Allow for shortage
Chakrabarty *et al* [7]	Linear demand	Three-parameter Weibull distribution	Single warehouse	No	No	Yes
Covert and Philip [15]	Constant	Two-parameter Weibull distribution	Single warehouse	No	No	No
Dave and Patel [8]	Time proportional	Constant	Single warehouse	No	No	No
Ghare and Shrader [9]	Constant	Constant	Single warehouse	No	No	No
Giri *et al* [10]	Ramp-type demand	Three-Parameter Weibull distribution	Single warehouse	No	No	Yes
Hariga [11]	Time varying (log concave)	Constant	Single warehouse	No	No	Yes
Lo *et al* [4]	Constant	Two-parameter Weibull distribution	Single warehouse	Yes	Yes	Yes
Misra [12]	Constant	Two-parameter Weibull distribution	Single warehouse	No	No	No
Philip [13]	Constant	Three-parameter Weibull distribution	Single warehouse	No	No	No
Skouri and Konstantaras [5]	Time dependent	Two-parameter Weibull distribution	Single warehouse	No	Yes	Yes
Skouri *et al* [1]	Ramp-type	Two-parameter Weibull distribution	Single warehouse	No	Yes	Yes
Teng *et al* [14]	Time varying	Constant	Single warehouse	No	No	Yes
Wee *et al* [2]	Constant	Two-parameter Weibull distribution	Two warehouse	Yes	Yes	Yes
Yang [3]	Constant	Constant	Two warehouse	Yes	Yes	Yes
Yang [6]	Constant	Three-parameter Weibull distribution	Two warehouse	Yes	Yes	Yes
Jaggi and Tiwari [17]	Selling price dependent	Constant	Two warehouse	No	No	Yes
Present paper (2015)	Stock dependent	Constant	Two warehouse	No	Yes	Yes

$$HC_o = h_0\left[\int_0^{t_0} I_{01}(t)\mathrm{d}t + \int_{t_0}^{t_1} I_{02}(t)\mathrm{d}t\right]$$

$$\Rightarrow HC_0 = h_o\left[\frac{Q}{\alpha}(1-e^{-\alpha t_0}) + \frac{a}{(\alpha+b)} \times \left(t_0 - t_1 + \frac{1}{(\alpha+b)}\left(e^{(\alpha+b)(t_1-t_0)} - 1\right)\right)\right].$$

(iii) The deterioration cost per cycle is represented by

$$DC = c_d\left[\beta\int_0^{t_0} I_r(t)\mathrm{d}t + \alpha\int_0^{t_0} I_{01}(t)\mathrm{d}t + \alpha\int_0^{t_0} I_{02}(t)\mathrm{d}t\right]$$

$$\Rightarrow DC = c_d\left[-\frac{(\beta a)}{(\beta+b)}\left\{t_0 + \frac{1}{(\beta+b)}\left(1-e^{(\beta+b)t_0}\right)\right\} + Q(1-e^{-\alpha t_0}) + \frac{\alpha a}{(\alpha+b)}\{t_0 - t_1 + \frac{1}{(\alpha+b)}\left(e^{(\alpha+b)(T-t_0)} - 1\right)\}\right].$$

(iv) Purchase cost PC for the order quantity is expressed as

$$PC = c(OQ)$$

$$\Rightarrow PC = c\left[\left\{Q + \frac{a}{(\beta+b)}\left[e^{(\beta+b)t_0} - 1\right]\right\} + S\right],$$

(v) The cost due to lost sales is given by

$$LC = c_l\int_{t_1}^{T} D(p)\left(1-e^{-\delta(T-t)}\right)\mathrm{d}t$$

$$\Rightarrow LC = ac_l\left[(T-t_1) + \frac{1}{\delta}\left(e^{-\delta(T-t_1}\right)\right]$$

(vi) The shortage cost in the entire cycle is described by

$$SC = -c_s\int_{t_1}^{T} I_{03}(t)\mathrm{d}t$$

$$= -\frac{ac_s}{\delta}\int_{t_1}^{T}\left[e^{-\delta(T-t_1)} - e^{\delta(T-t)}\right]\mathrm{d}t$$

$$\Rightarrow SC = -\frac{ac_s}{\delta}\left[\frac{1}{\delta}\left(1-e^{\delta(T-t_1)}\right) + Te^{-\delta(T-t_1)} - t_1 e^{-\delta(T-t_1)}\right]$$

(vii) Total sales revenue SR is represented by

$$SR = p\left[\int_0^{t_1} D(t)\mathrm{d}t - I_{03}(T)\right]$$

$$\Rightarrow SR = p\left[-\frac{ab}{(\beta+b)}\left\{t_0 + \frac{1}{(\beta+b)}\left(1-e^{(\beta+b)t_0}\right)\right\} + at_1 + b\left\{t_0 - t_1 - \frac{1}{(\alpha+b)}\left(1-e^{(\alpha+b)(T-t_0)}\right)\right\} - \frac{a}{\delta}\left\{e^{-\delta(T-t_1)} - 1\right\}\right]$$

Total profit of the system per unit time $TPF(t_0^*, t_1^*, T^*)$ is given by

$$TPF(t_0, t_1, T) = \frac{1}{T}[SR - (A + HC_r + HC_O + DC + PC + SC + LC)] \quad (11)$$

Case 1: The profit of the system per unit time per cycle $TPF(t_0, t_1, T)$ becomes a function of only t_0.

Now in this case the profit of the system per unit time per cycle is expressed as

$$TPF(t_0) = \frac{1}{T}[SR - (A + HC_r + HC_O + DC + PC + SC + LC)] \quad (12)$$

The optimal value of t_0 that is t_0^* obtained by satisfying the following necessary condition $\frac{\partial TPF(t_0)}{\partial t_0} = 0$, along with the following sufficient condition $\left[\frac{\partial^2 TPF(t_0)}{\partial t_0^2}\right]_{at\left(t=t_0^*\right)} < 0$

Case 2: The profit of the system per unit time per cycle $TPF\left(t_0, t_{1,}T\right)$ is a function of only t_1.

Now in this case the profit of the system per unit time per cycle is expressed as

$$TPF(t_1) = \frac{1}{T}[SR - (A + HC_r + HC_O + DC + PC + SC + LC)] \quad (13)$$

The optimal value of t_1 that is t_1^* obtained by satisfying the following necessary condition $\frac{\partial TPF(t_1)}{\partial t_1} = 0$, along with the following sufficient condition $\left[\frac{\partial^2 TPF(t_1)}{\partial t_1^2}\right]_{at\left(t=t_1^*\right)} < 0$

Case 3: The profit of the system per unit time per cycle $TPF(t_0, t_1, T)$ is a function of only T.

Now in this case the profit of the system per unit time per cycle is expressed as

$$TPF(T) = \frac{1}{T}[SR - (A + HC_r + HC_O + DC + PC + SC + LC)] \quad (14)$$

The optimal value of T that is T^* obtained by satisfying the following necessary condition $\frac{\partial TPF(T)}{\partial T} = 0$, along with the following sufficient condition $\left[\frac{\partial^2 TPF(T)}{\partial T^2}\right]_{at(t=T^*)} < 0$

2. Numerical result, graphical representation and sensitivity analysis

The required optimal values for a two-warehouse deteriorating inventory system with the parameter a, b, δ, α, and β, are calculated using the Newton Rapshon method. We have computed numerical values in example 2.1, 2.2, and 2.3 as well as their graphical represent which are shown in figure 2, figure 3, and figure 4 using Mathematica software to study the effects of various parameters on profit used in the model that can be seen from table 2, table 3, and table 4 given below. Table 2, table 3, and table 4 show the sensitivity of the various parameters on optimal average profit TPF^*, optimal ordering quantity $OQ^*$and optimal time of on hand inventory.

2.1 *Example*

$c_d = \$1/unit,\ c = \$1/per\,unit, c_1 = \$1/per\,unit, h_r = 1,$
$h_0 = \$0.25/per\,unit/per\,unit\,time,$
$c_s = \$1/per\,unit/per\,unit\,time, p = \$10/unit,$
$a = 100\,unit/month, b = 0.30, \delta = 0.05,\ \alpha = 0.40,$
$\beta = 0.2, A = \$2/per\,order,\ Q = 100,$
$t_1 = 2$ month, $T = 6$ month,

Optimal time of on hand inventory in RW $t_0^* = 3.27034$ month
Optimal average profit of the system $TPF^* = \$\,228.0013$
Optimal order quantity $OQ^* = 1288.60$

Graphical representation
Graphical representation of the effect of time t_0 on profit is done in figure 2 as follows:

Sensitivity analysis
Sensitivity analysis is carried out by changing the specified parameter a, b, δ, α and β by -50%, -25%, $+25\%$, and $+50\%$ keeping the remaining parameter at their standard value.

The study manifested table 2 following facts.

1. When the change in the value of its parameters a, b and the backlogging parameter δ then optimal average profit TPF^* is highly sensitive to demand. It is slightly sensitive to the change in β and moderately sensitive to changes in α.
2. The study reflects that optimal order quantities OQ^* is highly sensitive to the change in demand parameter a and b whereas moderately sensitive of OQ^* is observed

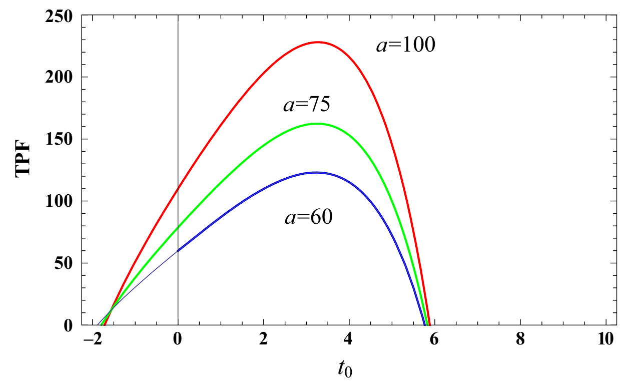

Figure 2. Variation of profit with respect to time t_0.

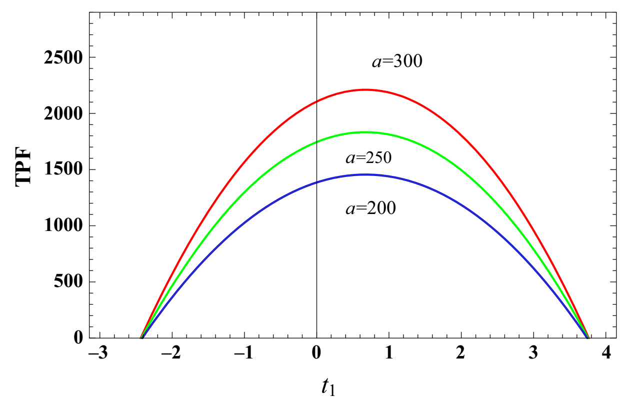

Figure 3. Variation of profit with respect to time t_1.

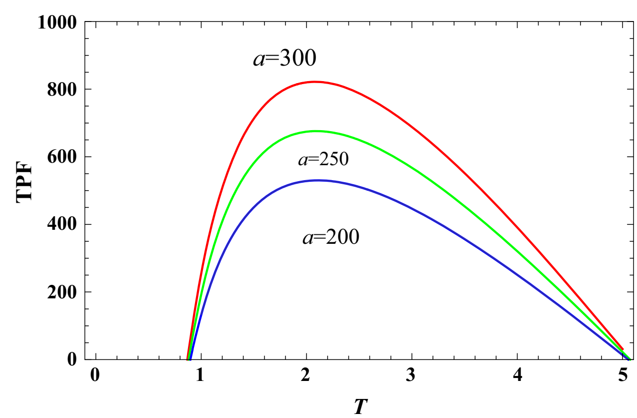

Figure 4. Variation of profit with respect to time T.

Table 2. Sensitivity analysis by changing the specified parameter a, b, δ, α and β in percentages.

Parameter	Change %	t_0^*	OQ^*	PF^*
a	−50	−2.02	−47.41	−57.57
	−25	−0.64	−23.69	−28.57
	+25	+0.37	+23.68	+28.80
	+50	+0.62	+47.35	+57.60
b	−50	+83.69	+95.226	+169.84
	−25	+33.98	+35.18	+56.02
	+25	−25.35	−22.67	−29.41
	+50	−46.44	−38.77	−45.24
δ	−50	0.00	+1.40	+10.04
	−25	0.00	+0.69	+4.97
	+25	0.00	−0.66	−4.87
	+50	0.00	−1.31	−9.65
α	−50	+1.61	+2.13	+1.74
	−25	+0.72	+0.95	+0.75
	+25	−0.62	−0.80	−0.59
	+50	−1.17	−1.51	−1.06
β	−50	−9.24	−19.90	−37.07
	−25	+0.37	−50.80	−15.43
	+25	−3.17	+2.29	+10.58
	+50	−7.22	+2.81	+17.69

Table 3. Sensitivity analysis by changing the specified parameter a, b, δ, α and β in percentages.

Parameter	Change %	t_1^*	OQ^*	PF^*
a	−50	0.00	−41.89	−51.51
	−25	0.00	−20.94	−25.62
	+25	0.00	+20.94	+25.62
	+50	0.00	+41.89	+51.24
b	−50	0.00	−0.07	+50.61
	−25	0.00	−0.07	+20.24
	+25	0.00	+0.03	−14.45
	+50	0.00	+0.7	−25.28
δ	−50	−0.93	+0.47	+0.09
	−25	−0.49	+0.23	+0.04
	+25	+0.45	−0.23	−0.04
	+50	+0.93	−0.47	−0.09
α	−50	+0.34	−0.11	+0.04
	−25	+0.17	−0.05	+0.01
	+25	−0.16	+0.05	−0.01
	+50	−0.32	+0.13	−0.03
β	−50	0.00	−0.03	+13.76
	−25	0.00	−0.01	6.25
	+25	0.00	+0.01	−5.38
	+50	0.00	+0.03	−9.83

to the β while it is slightly sensitive to change in α and backlogging parameter δ.

3. t_0^* is slightly sensitive to the change in the parameter β also it is observed that it is highly sensitive to the change in the parameter a and b whereas it is moderately sensitive to δ and α.

Table 4. Sensitivity analysis by changing the specified parameter a, b, δ, α and β in percentages.

Parameter	Change %	T^*	OQ^*	PF^*
a	−50	0.00	−40.03	−52.67
	−25	0.00	−20.01	−26.33
	+25	0.00	+20.01	+26.33
	+50	0.00	+40.03	+52.67
b	−50	0.00	−0.24	+113.11
	−25	0.00	−0.12	+37.92
	+25	0.00	+0.37	−49.00
	+50	0.00	+0.24	−38.06
δ	−50	+1.81	+2.01	−0.22
	−25	+1.60	+0.97	−0.11
	+25	+0.64	−0.95	+0.11
	+50	+0.27	−1.87	+0.22
α	−50	0.00	0.00	+0.56
	−25	0.00	0.00	+0.28
	+25	0.00	0.00	−0.28
	+50	0.00	0.00	−0.56
β	−50	0.00	−0.002	+0.28
	−25	0.00	−0.001	+0.14
	+25	0.00	+0.001	−0.14
	+50	0.00	+0.002	−0.28

2.2 *Example*

$c_d = \$1/unit,\ c = \$1/per\ unit, c_1 = \$1/per\ unit,$
$h_r = \$1/per\ unit/per\ unit\ time,$
$h_o = \$\,0.25/per\ unit/per\ unit\ time,$
$c_s = \$1/per\ unit/per\ unit\ time, p = \$\,5$ /unit,
$a = 300\ unit/month, b = 0.04, \delta = 0.008, \alpha = 0.04,$
$\beta = 0.02, A = \$1/per\ order,\ Q = 100,$
$t_0 = 0.40\ month, T = 2\ month,$

Optimal time of on hand inventory $t_1^* = 0.6749\ month$
Optimal average profit of the system $TPF^* = \$\,2209.92$
Optimal order quantity $OQ^* = 616$.

Graphical representation
Graphical representation of the effect of time t_1^* on profit is done in figure 3 as follows:

Sensitivity analysis
Sensitivity analysis is carried out by changing the specified parameter a, b, δ, α and β by −50%, −25%, +25%, and +50% keeping the remaining parameter at their standard value.

The study manifested table 3 following facts.

1. t_1^* is no sensitive to the change in the parameter a, b, and β but it is also obtained that it is moderately sensitive to the vary in the parameter α and slightly sensitive in δ.
2. The study of observation that optimal order quantity OQ^* is highly sensitive to the change in demand

parameter a whereas moderately sensitive of OQ^* is observed to the b, α, β while it is slightly sensitive to change in backlogging parameter δ.

3. Optimal average profit TPF^* is highly sensitive due to the change in the value of its parameters a, b and it is slightly sensitive to the change in β and there is moderately sensitive to changes in α and the backlogging parameter δ.

2.3 *Example*

$$c_d = \$1/unit,\ c = \$1/per\,unit, c_1 = \$1/per\,unit,$$
$$h_r = \$1/per\,unit/per\,unit\,time,$$
$$h_o = \$0.25/per\,unit/per\,unit\,time,$$
$$c_s = \$1/per\,unit/per\,unit\,time, p = \$4 \text{ per unit},$$
$$a = 300\,units/month, b = 0.10, \delta = 0.05, \alpha = 0.04,$$
$$\beta = 0.001, A = \$2/per\,order, Q = 100,$$
$$t_0 = 0.40\,month,\ t_1 = 2\,month,$$

Optimal time of on hand inventory $T^* = 2.9525\,month$
Optimal average profit of the system $TPF^* = \$\,699.942$
Optimal order quantity $OQ^* = 501.513$.

Graphical representation

Graphical representation of the effect of time Ton profit is done in figure 4 as follows:

Sensitivity analysis

Sensitivity analysis is carried out by changing the specified parameter a, b, δ, α and β by -50%, -25%, $+25\%$, and $+50\%$ keeping the remaining parameter at their standard value.

The study manifested table 4 following facts.

1. T^* is no sensitive to the change in the parameter a, b, α, β but it is also obtained that it is moderately sensitive to the vary in the parameter δ.
2. The study of observation that optimal order quantity OQ^* is highly sensitive to the change in demand parameter a whereas moderately sensitive of OQ^* is observed to the b, β while it is slightly sensitive to change in backlogging parameter δ and no sensitive to the change in the parameterα.
3. Optimal average profit TPF^* is highly sensitive due to the change in the value of its parameters a, b and it is slightly sensitive to the change in β and the backlogging parameter δ and there is moderately sensitive to changes in α.

3. Conclusions

This paper presents the two-warehouse deteriorating inventory system in which it is assumed that demand rate is stock dependent. At first demand are satisfied from an RW to reduce the loss due to higher rental of RW, but RW offers a better preserving facility that result a low deteriorating rate in comparison with OW. Total optimal profit, the optimal cycle time and optimal order quantity per cycle have been calculated for developed model. The effects of different parameters on the total optimal profit, the optimal cycle time and optimal order quantity per cycle has been observed by sensitivity analysis. Hence the developed model is applicable for a stowing business in order to take economic decision which contributes to enhance the revenue of a company. Besides considering demand as a stock dependent we can take demand as price decreasing and stochastic demand too that will remain future scope of this work.

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