Semi-analytical solution to plane strain loading of elastic layered coating on an elastic substrate

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MS received 28 July 2014; revised 8 January 2015; accepted 11 May 2015

Abstract. The plane strain loading of a linear elastic layered coating halfspace is solved semi-analytically through a combination of Airy stress function and Fourier transforms and highly simplified and compact expressions for displacement and stresses in layer and substrate are presented in terms of pressure distribution in the loaded region. The results are applied to study the influence of layer thickness and mismatch in elastic modulus between layer and substrate on the stresses and displacement during loading of layered coating system. Lastly the effect of degree of smoothness of the pressure profile on the plane strain loading response of layered coating system is simulated through three different pressure profiles for a fixed total load and loading zone length.

Keywords. Layered coatings; plane strain, semi-analytical.

1. Introduction

Due to the ease of manufacturing and assembly, layered coatings (LCs) with thickness in micro to nano-meter range are being actively pursued by designers as hybrid material for multifunctional devices. Few notable applications of LCs include insulators, thermal and corrosion barrier coatings (Suresh 2001). Response of LCs to loading and consequent failure via yielding, cracking and delamination modes is usually acquired through micro-nano indentation experiments which necessitates the knowledge of stresses in LCs and motivates the present work (Oliver & Pharr 1992). Calculation of stresses in LCs subjected to prescribed surface loading and indentation loading has been investigated by many researches adopting a variety of analytical and numerical techniques. Burmister (1945) calculated stresses under axisymmetric conditions in a LC system subjected to prescribed surface loading using the stress function representation of displacement and stresses. Barovich *et al* (1964) studied the stresses in LCs due to elliptically distributed load using Fourier transform and presented numerical results for varying elastic and shear modulus of coatings with respect to the substrate. Dhaliwal and co-workers (Dhaliwal 1970; Dhaliwal & Rua 1970) used the harmonic function representation of displacements and

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stresses in elasticity and expressed the axisymmetric contact problem of LCs as a Fredholm integral equation in single unknown and numerically calculated the results for load-penetration, pressure in the contact region and deformed shape outside the contact region for a variety of axisymmetric punch shapes. Chen (1971) also employed the harmonic function representation in elasticity and expressed the displacement and stresses in LCs in terms of the pressure acting in the two dimensional region of the surface and specialised the results under axisymmetric and plane strain conditions. Chen & Engel (1972) proposed an approximate solution for a contact problem of LCs whereby pressure in the contact region is represented as sum of homogeneous solution and perturbation expressed as a set of base functions chosen so as to approximate the actual displacement boundary condition in the contact region via least square minimization. Gupta & Walowit (1974) considered the plane strain parabolic indentation of a LC on a substrate using Airy stress function and Fourier integral transforms and solved for the unknown pressure in the contact region using numerical integration. Following (Gupta & Walowit 1974), Gupta *et al* (Gupta & Finkin 1973) studied the differences in resulting stresses during plane strain indentation of LCs if instead of the actual pressure distribution, elliptical pressure distribution is used. King (1987) applied the approach of Chen & Engel (1972) to the indentation of LCs by flat ended cylindrical, quadrilateral and triangular punches and proposed a method for estimation of Young's modulus of thin films from the initial unloading compliance curve in an indentation test. King & Sullivan (1987) calculated the stresses in a LC on substrate subjected to plane strain normal and sliding contact combining the approaches of Chen & Engel (1972) and Gupta and co-workers (Gupta & Finkin 1973; Gupta & Walowit 1974). By a-priori assuming pressure distribution in the contact region, Li & Chou (1997) analyzed the axisymmetric loading of an elastic LC on a substrate by employing stress function representation and Hankel transform and presented highly simplified expressions for stresses and displacement in layer and substrate. For an assumed smooth pressure distribution, Li & Chou (1997) studied the influence of layer thickness and elastic modulus mismatch between layer and substrate on the stresses and displacements in LC and substrate. Ragione *et al* (2008) used a slightly different form of stress function compared to Li & Chou (1997) and were able to study axisymmetric indentation of a layered compressible and incompressible half-space. Gao *et al* (1992) studied the contact of multilayered systems using perturbation method by assuming very little mismatch in the elastic properties of the layers and showed good agreement with finite element results for modulus ratio ranging from 0.5 to 2 amongst layers. Gao & Wu (1993) showed that as long as the cross-section shape of flat-ended punch is close to circle, the actual cross section shape is immaterial during the indentation of a layered medium. Schwarzer (2000) calculated the complete displacement and stress field during the contact of LC on a substrate possessing either isotropic or transversely isotropic elastic properties, using "method of image loads" and compared the results with experiments and finite element calculations. Fabrikant (2011) studied the contact problem of transversely isotropic elastic layers with transversely isotropic elastic half-space using generalized images method and integral transform method. Perriot (2004) utilized the Green's function obtained by Li & Chou (1997) and the auxiliary function approach to exactly solve the mixed boundary value problem of axisymmetric contact of thin-film/substrate system for a variety of indenter shapes and offered a relationship between effective elastic constant and real penetration of a coated elastic half-space. Hsueh & Miranda (2004) proposed a combined empirical analytical procedure for calculating the contact radius and indenter displacement during Hertzian contact of LC systems and verified their results with finite element analysis results. Transfer matrix method is one of the popular techniques to deal with elastostatic problems in multilayered systems and is used by many researchers to deal with the contact problem of LCs (Yue 1995; Constantinescu *et al* 2013). Constantinescu *et al* (2013) presented

a semi-analytical solution for the axisymmetric contact of multilayered elastic half-space using Papkovitch Neuber potentials, Hankel transform and transfer matrix method and in particular analysed the effect of number of layers and order of soft and stiff coatings for different types of indenter shapes. The numerical solution to contact problem of LCs has also been widely reported using finite element method under normal and or tangential loading, plane strain or axisymmetric deformation with material behaviour ranging from elastic to elastic–plastic (Van der Zwaag & Field 1982; Bhattacharya & Nix 1988; Tian & Saka 1991a, b; Djabella & Arnell 1992; Lardner *et al* 1992; Anderson & Collins 1995; Lovell 1998; Sen & Ozel 1998; Zhang *et al* 2006; Lovell & Morrow 2007).

Sneddon's classical solution for homogeneous material (Sneddon 1995) forms the foundation of the present work and utilizes Airy stress function approach, Fourier cosine transform to solve for displacements and stresses in an elastic LC system (figure 1) subject to prescribed pressure loading under plane strain condition. The methodology and analysis in the present work bears similarities with the work of Li & Chou (1997) on the axisymmetric loading study of LC system. In all the existing works on plane strain loading of LCs based on similar procedure notably references (Chen 1971; Chen & Engel 1972; Gupta & Finkin 1973; Gupta & Walowit 1974), the evaluation of Airy stress function and consequently the displacements–stresses even in Fourier domain is all numerical. In contrast, in the present work, all these quantities and in particular the Green's function relating loading region displacement and pressure profile are all presented in closed form in the Fourier domain. The compact nature of the expressions for stress and displacement in each layer greatly reduces the computation and its effectiveness is recently demonstrated through the example of LC system being subjected to uniform pressure in the loaded region and reported elsewhere (Vasu & Bhandakkar 2013).

The plan of the paper is as follows. Section 2 presents the formulation of the plane strain loading of LC system in terms of Airy stress function and solution using Fourier cosine transform. Highly simplified expressions for stresses and displacements in the coating and substrate are presented in terms of the applied pressure in the loaded region. In the next section, a smooth variation of pressure is assumed in the loaded region and the obtained expressions of stresses and

Figure 1. Normal loading $q(y)$ of an isotropic elastic layered system in the region $x = 0$, $|y| \leq y_0$, with y_0 as the semi loading zone length. E_i , ν_i denote the elastic constants of i^{th} layer with $i = 1, 0$ implies layer and substrate respectively. t is the thickness of the layer and the resultant of the applied load $q(y)$ is referred to as P .

displacements are used to study the effect of coating thickness and elastic modulus mismatch between coating and substrate on the resultant stresses and displacements in LCs (figure 1). The effect of degree of smoothness of the pressure profile on the loading response of LCs is simulated through three different pressure profiles in the loaded region for a fixed total load and loading zone length. Finally, section 4 provides summary and conclusion of the present work.

2. Formulation

Figure 1 shows the schematic of layered coating system subjected to pressure $q(y)$ acting in the region $x = 0, |y| \leq y_0$, under quasi-static plane strain condition. The deformation in layer and substrate is assumed to be infinitesimal with $2y_0$ referred to as the loading zone length and the resultant of the applied load $q(y)$ is referred as P . The thickness of the layer is denoted as t . The material behaviour of the layer and substrate is supposed to be linear elastic and isotropic with the elastic modulus, Poissons ratio and shear modulus given by $E_i, \nu_i, \mu_i = E_i/[2(1 + \nu_i)]$ where subscript “ i ” takes values of 1 and 0 for layer and substrate respectively. This subscript representation will be used hereafter to differentiate whether any material property or field variable like displacement or stress corresponds to layer or substrate. The displacements along x and y directions in the layered coating system are denoted by u_i and w_i respectively while the normal and shear stresses are represented as σ_{ai} and τ_{abi} respectively where a and b can denote x, y or z co-ordinates and $a \neq b$. The stress equilibrium equations under plane strain condition in the layer and substrate are given by (Timoshenko & Goodier 1987),

$$\frac{\partial \sigma_{xi}}{\partial x} + \frac{\partial \tau_{xyi}}{\partial y} = 0, \quad (1)$$

$$\frac{\partial \tau_{xyi}}{\partial x} + \frac{\partial \sigma_{yi}}{\partial y} = 0. \quad (2)$$

The boundary conditions along the layer surface $x = 0$ are

$$\sigma_{x1} = -q(y); \quad \tau_{xy1} = 0; \quad |y| \leq y_0, \quad (3)$$

$$\sigma_{x1} = 0; \quad \tau_{xy1} = 0; \quad |y| > y_0. \quad (4)$$

The boundary condition Eq. (3) corresponds to the assumed pressure distribution $q(y)$ in the loaded region $|y| \leq y_0$ while Eq. (4) refer to traction free boundary condition. Since the layer and substrate interface ($x = t$) is perfectly bonded, the continuity of displacements and tractions provides

$$u_1(t, y) = u_0(t, y); \quad w_1(t, y) = w_0(t, y), \quad (5)$$

$$\sigma_{x1}(t, y) = \sigma_{x0}(t, y); \quad \tau_{xy1}(t, y) = \tau_{xy0}(t, y). \quad (6)$$

The stresses in layer and substrate are represented in terms of Airy stress function φ_i as (Timoshenko & Goodier 1987),

$$\sigma_{xi} = \frac{\partial^2 \varphi_i}{\partial y^2}; \quad \sigma_{yi} = \frac{\partial^2 \varphi_i}{\partial x^2}; \quad \tau_{xyi} = -\frac{\partial^2 \varphi_i}{\partial x \partial y}. \quad (7)$$

leading to the equilibrium Eqs. (1) and (2) in the layer and substrate being satisfied trivially. The compatibility conditions for deformation in layer and substrate lead to the governing biharmonic equation for φ_i which following Sneddon's classical work (Sneddon 1995) is reduced to

linear differential equations in 'x' through the application of Fourier cosine transform along 'y' direction. The solution to the resulting differential equations gives the Airy stress function G_i in Fourier domain,

$$G_i = (A_i + B_i x) e^{-|\omega|x} + (C_i + D_i x) e^{|\omega|x}, \quad (8)$$

where A_i, B_i, C_i and D_i are the integration constants. Employing Hooke's law relating stress-strain for linear elastic material, strain-displacement relation for infinitesimal deformation and Eq. (7) (Timoshenko & Goodier 1987), the stresses and displacements in terms of G_i are given by,

$$\sigma_{xi} = \frac{-2}{\pi} \int_0^\infty \xi^2 G_i \cos(\xi y) d\xi, \quad (9)$$

$$\sigma_{yi} = \frac{2}{\pi} \int_0^\infty \frac{d^2 G_i}{dx^2} \cos(\xi y) d\xi, \quad (10)$$

$$\tau_{xyi} = \frac{2}{\pi} \int_0^\infty \xi \frac{dG_i}{dx} \sin(\xi y) d\xi, \quad (11)$$

$$u_i = \frac{2}{\pi} \int_0^\infty \frac{1}{2\mu_i} \left[(1 - \nu_i) \frac{d^3 G_i}{dx^3} - (2 - \nu_i) \xi^2 \frac{dG_i}{dx} \right] \cos(\xi y) \frac{d\xi}{\xi^2}, \quad (12)$$

$$w_i = \frac{2}{\pi} \int_0^\infty \frac{1}{2\mu_i} \left[(1 - \nu_i) \frac{d^2 G_i}{dx^2} + \nu_i \xi^2 G_i \right] \sin(\xi y) \frac{d\xi}{\xi}, \quad (13)$$

$$\bar{Q}(\xi) = \int_0^\infty q(y) \cos(\xi y) dy. \quad (14)$$

where $\xi = |\omega|$ and, $\bar{Q}(\xi)$ is the Fourier cosine transform of the applied pressure $q(y)$ in the loaded region. Substituting the solution of Fourier domain Airy stress function from Eq. (8) in Eqs. (9–13) and applying the boundary and interface conditions Eqs. (3–6) leads to system of six equations for the eight unknowns A_i, B_i, C_i, D_i ($i = 0, 1$). The fact that the stresses in substrate should vanish far away from the loaded region ($x \rightarrow \infty$), leads to the constants C_0 and D_0 being equal to zero. Thus the system of six linear equations for the solution of unknowns A_1, B_1, C_1, D_1, A_0 and B_0 based on Eqs. (3–6) and (9–13) inputted to Maple[©] is given as,

$$\begin{bmatrix} \xi^2 & 0 & \xi^2 & 0 & 0 & 0 \\ -\xi & 1 & \xi & 1 & 0 & 0 \\ \xi & 1-2\nu_1+\xi t & -\xi e^{2\xi t} & (1-2\nu_1-\xi t)e^{2\xi t} & -\alpha\xi & \alpha(2\nu_0-1-\xi t) \\ \xi & (2\nu_1-2+\xi t) & \xi e^{2\xi t} & (2-2\nu_1+\xi t)e^{2\xi t} & -\alpha\xi & \alpha(2-2\nu_0-\xi t) \\ 1 & t & e^{2\xi t} & t e^{2\xi t} & -1 & -t \\ -\xi & 1-\xi t & \xi e^{2\xi t} & (1+\xi t)e^{2\xi t} & \xi & \xi t-1 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \\ A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} \bar{Q}(\xi) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (15)$$

The lengthy and tedious expression for the solution of A_1, B_1, C_1, D_1, A_0 and B_0 and thereby the components of displacement and stresses provided by Maple[©] from Eqs. (9–13) are carefully simplified using inbuilt commands simplify, expand, clubbing of exponential terms and identifying group of constants γ_0, γ_1, k, m , etc. defined below and involving $\nu_0, \nu_1, \xi t, \alpha$. The present formulation is derived using Poisson's ratio and shear modulus ratio. The relationship between α, ν_0, ν_1 and Dunder's parameters (Barber 2004) which are usually used to express mismatch in elastic properties is given by,

$$d_1 = \frac{\alpha(1-\nu_0) - (1-\nu_1)}{(1-\nu_1) + \alpha(1-\nu_0)}, d_2 = \frac{\alpha(1-2\nu_0) - (1-2\nu_1)}{(1-2\nu_1) + \alpha(1-2\nu_0)}. \quad (16)$$

The final simplified expressions for the six unknowns A_1 , B_1 , C_1 , D_1 , A_0 and B_0 , components of stresses and displacements are given as,

$$A_0 = 8\bar{Q}\gamma_0 \left((\alpha(\gamma_4\xi t + \gamma_5) + \gamma_0 - \xi t\gamma_2) e^{2\xi t} - (\alpha - 1)(\xi t - 1)\xi t + \gamma_0 - \alpha\gamma_5 \right) / N_1(\xi) \quad (17)$$

$$B_0 = 4\bar{Q}\xi\gamma_0 \left((\alpha + \gamma_1) e^{2\xi t} + (2\xi t - 1)(\alpha - 1) \right) / N_1(\xi) \quad (18)$$

$$A_1 = \bar{Q} \left(l e^{2\xi t} - bk - p \right) / N_1(\xi) \quad (19)$$

$$B_1 = \bar{Q}\xi \left(l e^{2\xi t} + (2\xi t - 1)k \right) / N_1(\xi) \quad (20)$$

$$C_1 = \bar{Q} \left(m e^{-2\xi t} - kc - p \right) / N_1(\xi) \quad (21)$$

$$D_1 = -\bar{Q}\xi \left(m e^{-2\xi t} - (2\xi t + 1)k \right) / N_1(\xi) \quad (22)$$

$$N_1(\xi) = -\xi^2 \left(l e^{2\xi t} + m e^{-2\xi t} - 2ak - 2p \right) \quad (23)$$

$$a_1 = 2\xi^2 t^2 + 1, \quad b_1 = 2\xi^2 t^2 - 2\xi t + 1, \quad c_1 = 2\xi^2 t^2 + 2\xi t + 1, \quad (24)$$

$$\sigma_{xi}(X, Y) = \frac{-2}{\pi} \int_0^\infty \frac{\bar{Q}(\eta) \bar{\sigma}_{xi}^*(X, \eta)}{N(\eta)} \cos(\eta Y) d\eta, \quad (25)$$

$$\sigma_{yi}(X, Y) = \frac{-2}{\pi} \int_0^\infty \frac{\bar{Q}(\eta) \bar{\sigma}_{yi}^*(X, \eta)}{N(\eta)} \cos(\eta Y) d\eta, \quad (26)$$

$$\tau_{xyi}(X, Y) = \frac{-2}{\pi} \int_0^\infty \frac{\bar{Q}(\eta) \bar{\tau}_{xyi}^*(X, \eta)}{N(\eta)} \sin(\eta Y) d\eta, \quad (27)$$

$$u_i(X, Y) = \frac{-2}{\pi} \int_0^\infty \frac{\bar{Q}(\eta) \bar{u}_i^*(X, \eta)}{N(\eta)} \cos(\eta Y) d\eta, \quad (28)$$

$$w_i(X, Y) = \frac{-2}{\pi} \int_0^\infty \frac{\bar{Q}(\eta) \bar{w}_i^*(X, \eta)}{N(\eta)} \sin(\eta Y) d\eta, \quad (29)$$

where

$$N(\eta) = -\eta^2 \left(-2ak - 2p + l e^{2\eta} + m e^{-2\eta} \right), \quad (30)$$

$$\alpha = \mu_1/\mu_0, \quad \eta = \xi t, \quad x = Xt, \quad y = Yt, \quad y_0 = Y_0t, \quad (31)$$

$$\gamma_0 = 1 - \nu_1, \quad \gamma_1 = 3 - 4\nu_1, \quad \gamma_2 = 1 - 2\nu_1, \quad \gamma_3 = 3 - 4\nu_0, \quad \gamma_4 = 1 - 2\nu_0, \quad \gamma_5 = 1 - \nu_0, \quad (32)$$

$$a = 2\eta^2 + 1, \quad b = 2\eta^2 - 2\eta + 1, \quad c = 2\eta^2 + 2\eta + 1, \quad (33)$$

$$e = 2\eta + 1, \quad f = \eta X - 1, \quad g = \eta X + 1, \quad (34)$$

$$M_1 = \eta X + \gamma_2, \quad M_2 = \gamma_2 - \eta X, \quad M_3 = 2\gamma_0 + \eta X, \quad M_4 = 2\gamma_0 - \eta X, \quad (35)$$

$$K = \gamma_0 - \alpha\gamma_5, \quad L = \gamma_4\eta + \gamma_5, \quad M = \gamma_0 - \eta\gamma_2, \quad (36)$$

$$K_1 = kc + p, \quad K_2 = kb + p, \quad K_3 = \alpha + \gamma_1, \quad K_4 = 2\alpha L + 2M, \quad (37)$$

$$k = \alpha^2 \gamma_3 - 2\alpha \gamma_4 - 1, \quad l = \alpha^2 \gamma_3 + 2\alpha (\gamma_1 \gamma_4 + 2\gamma_0) + \gamma_1, \quad (38)$$

$$m = \alpha^2 \gamma_3 - 2\alpha (\gamma_4 + 2\gamma_0) + \gamma_1, \quad p = 4\gamma_0 (\alpha \gamma_4 - \gamma_2), \quad (39)$$

$$\bar{\sigma}_{x1}^*(X, \eta) = \frac{\eta^2}{t} \left\{ (K_2 - k\eta X d - fl e^{2\eta}) e^{-\eta X} + (K_1 - \eta X k e + g m e^{-2\eta}) e^{\eta X} \right\}, \quad (40)$$

$$\bar{\sigma}_{x0}^*(X, \eta) = \frac{4\eta^2 \gamma_0 e^{-\eta X}}{t} \left\{ (\alpha - 1) (b - 1 - \eta X d) - 2K - (K_4 + K_3 \eta X) e^{2\eta} \right\}, \quad (41)$$

$$\bar{\sigma}_{y1}^*(X, \eta) = \frac{\eta^2}{t} \left\{ (g l e^{2\eta} - K_2 + k(g - 1)d) e^{-\eta X} - ((1 - g) p e + K_1 + f m e^{-2\eta}) e^{\eta X} \right\}, \quad (42)$$

$$\bar{\sigma}_{y0}^*(X, \eta) = \frac{4\eta^2 \gamma_0 e^{-\eta X}}{t} \left\{ 2K + (K_4 + K_3 (g - 1)) e^{2\eta} - (\alpha - 1) ((c - 3) - \eta X d) \right\}, \quad (43)$$

$$\bar{\tau}_{xy1}^*(X, \eta) = \frac{-\eta^2}{t} \left\{ (\eta X l e^{2\eta} - K_2 + k g d) e^{-\eta X} + (K_2 - k f e + \eta X m e^{-2\eta}) e^{\eta X} \right\}, \quad (44)$$

$$\bar{\tau}_{xy0}^*(X, \eta) = \frac{-4\eta^2 \gamma_0 e^{-\eta X}}{t} \left\{ 2K + (K_4 + K_3 g) e^{2\eta} - (\alpha - 1) (2\eta^2 - 1 - \eta X d) \right\}, \quad (45)$$

$$\bar{u}_1^*(X, \eta) = \frac{\eta}{2\mu_1} \left\{ (dk M_1 + M_3 l e^{2\eta} - K_2) e^{-\eta X} + (ek M_2 + K_1 - M_4 m e^{-2\eta}) e^{\eta X} \right\}, \quad (46)$$

$$\bar{u}_0^*(X, \eta) = \frac{2\eta \gamma_0 e^{-\eta X}}{\mu_0} \left\{ 2K + (K_4 + K_3 (\eta X + \gamma_4)) e^{2\eta} - (\alpha - 1) (b + 2\eta - d (\eta X - 2v_0)) \right\}, \quad (47)$$

$$\bar{w}_1^*(X, \eta) = \frac{\eta}{2\mu_1} \left\{ (-M_2 l e^{2\eta} - K_2 - dk M_4) e^{-\eta X} + (ek M_3 - K_1 - M_1 m e^{-2\eta}) e^{\eta X} \right\}, \quad (48)$$

$$\bar{w}_0^*(X, \eta) = \frac{2\eta \gamma_0 e^{-\eta X}}{\mu_0} \left\{ 2K + (K_4 + K_3 (\eta X - 2\gamma_5)) e^{2\eta} - (\alpha - 1) (c - 3 - d(2v_0 + \eta X)) \right\}. \quad (49)$$

α in Eq. (15) is the shear modulus ratio and along with the Poissons ratio ν_i signifies the mismatch in elastic properties of layer and substrate. Value of $\alpha > 1$ corresponds to stiff coating while $\alpha < 1$ represents soft coating with respect to the substrate. For $\alpha = 1$ and $v_0 = \nu_1$, the LC system in figure 1 reduces to homogeneous elastic isotropic half-space. As a sanity check, for $\alpha = 1$ and $v_0 = \nu_1$, Eqs. (9–45) indeed reduced to the expressions of Sneddon (1995). The expressions for stresses and displacements in Eqs. (25–29) also compared favorably with the results of Chen (1971) specialized to the case of a single layer coated substrate system under plane strain loading condition. It should be mentioned that the work of Chen (1971) had a broader scope and dealt with the case of normal loading acting on a two dimensional patch of the surface of a linear elastic half space composed of one or two layers perfectly bonded to another homogeneous medium. Compared to the work of Chen (1971), expressions in the present study are much more simpler and well-suited for computer implementation. The integrals required for calculating stresses and displacements in Eqs. (25–29) are numerically evaluated using *quad* and *integral* function in Matlab[®]. For the results discussed in next section, the Poisson ratio of layer (ν_1) and substrate (ν_0) is taken same and equal to 0.33.

3. Results and discussion

3.1 Effect of layer thickness and elastic modulus mismatch between layer and substrate

The effect of layer thickness and mismatch in elastic modulus between layer and substrate on the stresses and displacement in layer and substrate is investigated by assuming the following pressure distribution $q(y)$ in the loading region ($|y| \leq y_0$),

$$q(y) = \frac{2P(y_0^2 - y^2)^{1/2}}{\pi y_0^2}, \quad (50)$$

The above pressure distribution hereafter referred to as Case-1 results if the homogeneous half space ($\alpha = 1$ and $\nu_0 = \nu_1$) is indented by a parabolic indenter (Sneddon 1995). The Fourier cosine transform of the Case-1 pressure profile Eq. (50) using Eq. (14) is Maple[©],

$$\bar{Q}(\eta) = \frac{PJ_1(\eta Y_0)}{\eta Y_0}, \quad (51)$$

Figure 2. Effect of (a) shear modulus ratio α for a given layer thickness $t/y_0 = 1$, (b) layer thickness t of the stiff coating with shear modulus ratio $\alpha = 10$ and (c) layer thickness t of the soft coating with shear modulus ratio $\alpha = 0.1$, on the relative surface displacement $u^* = u(y/y_0, 0) - u(0, 0)$ of a layered coating substrate system (figure 1) subjected to Case-1 pressure profile in Eq. (46) for a given total load P and semi loading zone size y_0 . Curves corresponding to Layer and Substrate in (b) and (c) refer to the situation when the entire coating-substrate system is a homogeneous half space with elastic properties of layer and substrate respectively. Poissons ratio for layer and substrate $\nu_0 = \nu_1 = 0.33$. Due to symmetry of the problem about x -axis, results are shown for positive y values only.

where $J_1(\eta Y_0)$ is the Bessel function of order one (Abramowitz & Stegun 1972; Watson 1958). Figure 2(a) shows the effect of shear modulus ratio on the relative surface displacement $u^* = u(y/y_0, 0) - u(0, 0)$ of the layered coating substrate system for normalized layer thickness $t/y_0 = 1$. As expected, the soft coating case ($\alpha = 0.1$) exhibits highest compliance and maximum deformation compared to the homogeneous ($\alpha = 1$) and stiff coating case ($\alpha = 10$). Figures 2(b) and 2(c) show the effect of layer thickness on the relative surface displacement for stiff ($\alpha = 10$) and soft ($\alpha = 0.1$) coating substrate system respectively. Displacement curves corresponding to “Layer” and “Substrate” in these figures refer to the situation when the entire layered coating system is a homogenous half space with elastic properties of layer ($t \rightarrow \infty$) and substrate ($t \rightarrow 0$) respectively. Hence the curves “Layer” and “Substrate” act as bound and enclose the displacement curves for entire spectrum of layer thickness of the layered coating system. Based on figures 2(b) and 2(c), it is clear that for a layer with ratio of $t/y_0 \geq 10$, the effect of substrate is negligible while a layer with ratio of $t/y_0 \leq 0.1$ is as good as no layer being present. In the case of stiff coating system ($\alpha = 10$) in figure 2(b), the compliance of the system varies inversely with thickness of the layer and the response of the layered coating system veers from that of the substrate alone case to that of the layer alone case as the thickness of the layer is increased. Exactly opposite conclusions can be drawn for the case of soft coating system ($\alpha = 0.1$) as seen in figure 2(c). Figures 3(a) and 3(b) show the variation of normal stress components σ_x and σ_y along the axis of the loading ($y = 0$) for shear modulus ratio of $\alpha = 0.1, 1$, and 10 and thickness ratio $t/y_0 = 1$ respectively. In spite of the amount of mismatch in the elastic properties of layer and substrate, as expected, the stresses decay away from the loading region. From figure 3(a), the normal stress σ_x is compressive all through the x -axis, attains its maximum at the layer surface equal to the applied pressure (Eq. (3)) and its rate of decay with respect to the distance from the layer surface is higher if the layer is stiffer compared to the substrate. From figure 3(b), component σ_y is maximum compressive at the layer surface but its variation in the interior of layer depends strongly on the shear modulus ratio α . For homogenous ($\alpha = 1$) and soft coating case ($\alpha = 0.1$), σ_y is compressive all along the x -axis while for stiff coating case ($\alpha = 10$), σ_y transitions from maximum compressive at the layer surface to maximum tensile at the layer–substrate interface. Thus in case of stiff coatings, cracks can initiate near the layer–substrate interface if σ_y near the layer–substrate interface equals or exceeds the fracture strength of the layer. Figures 4(a)–4(c) show the stresses along the interface ($x/t = 1$) approaching from the layer side for $\alpha = 0.1, 1$, and 10 with $t/y_0 = 1$. The normal stress component σ_x along the interface in figure 4(a) is compressive below the loaded region with its magnitude decreasing with increased stiffness of coating and decays quickly away from the loaded region to be consistent with the traction free boundary condition outside the loaded region (Eq. (4)). The tensile nature of the normal stress component σ_y along the interface in loading region of figure 4(b) for $\alpha = 10$, again reemphasizes the possibility of fracture near the layer–substrate interface while using stiff coatings. For soft coating and homogeneous cases, the location of maximum shear stress along the interface as shown in figure 4(c) lies within the loaded region while it lies outside the loaded region for stiff coating. Figures 5 and 6 show the effect of layer thickness t on the stresses of a soft ($\alpha = 0.1$) and stiff ($\alpha = 10$) coating respectively at two different locations along the layer–substrate interface ($x/t = 1$) viz. loading axis $y = 0$ and edge of the loaded region $y = y_0$. As shown in figures 5(a) and 6(a), shear stress τ_{xy} is zero along the intersection of layer–substrate interface and the x -axis $x/t = 1, y/y_0 = 0$ owing to symmetry. For the case of soft coating ($\alpha = 0.1$) as shown in figure 5, the stress component σ_x dominates while for stiff coating ($\alpha = 10$) as shown in figure 6, the stress component σ_y is dominant in terms of magnitude. Based on the shear modulus mismatch (α), there exists a particular layer thickness t for which the stresses at a distance of y_0 away from the loading axis and along the layer–substrate

Figure 3. Effect of shear modulus ratio α on the normal components of stresses (a) σ_x , and (b) σ_y along the axis of the loading ($y = 0$) of a layered coating substrate system (figure 1) subjected to Case-1 pressure profile in Eq. (46) for a given normalized layer thickness $t/y_0 = 1$, with y_0 as the semi loading zone length. Poissons ratio for layer and substrate $\nu_0 = \nu_1 = 0.33$. The stress discontinuity at the interface ($x/t = 1$) is as expected in case of σ_y component of stress.

interface ($x/t = 1$, $y/y_0 = 0$) attain extremum in figures 5(b) and 6(b). Similar to the previous cases in figures 3(b) and 4(b), the normal component σ_y becomes tensile for stiff coating ($\alpha = 10$) as shown in figure 6(b).

3.2 Effect of smoothness of pressure profile

The effect of degree of smoothness of the pressure profile on the response of LCs is simulated through three different pressure profiles $q(y)$ in the loaded region ($|y| \leq y_0$) for a fixed total load P and loading zone length $2y_0$. Eq. (50) referred to as Case-1 represents the first pressure

Figure 4. Effect of shear modulus ratio α on the components of stresses (a) σ_x , (b) σ_y , and (c) τ_{xy} along the interface ($x/t = 1$) of a layered coating substrate system (figure 1) subjected to Case-1 pressure profile in Eq. (46) for a given normalized layer thickness $t/y_0 = 1$, with y_0 as the semi loading zone length. Poisson's ratio for layer and substrate $\nu_0 = \nu_1 = 0.33$. Due to symmetry of the problem about x -axis, results are shown for positive y values only.

profile intended for the study in the present sub-section. The remaining two pressure profiles are given as,

$$q(y) = \frac{3P(y_0^2 - y^2)}{4y_0^3}, \quad (52)$$

$$q(y) = \frac{P}{2\pi y_0} \ln \left(\frac{y_0 + \sqrt{(y_0^2 - y^2)}}{y_0 - \sqrt{(y_0^2 - y^2)}} \right), \quad (53)$$

and hereafter referred to as Case-2 and Case-3 respectively. Similar to Case-1 profile of Eq. (50), Case-2 pressure distribution of Eq. (52) is a representative of smooth pressure profile and is the two-dimensional analogue of the pressure profile adopted by Li & Chou (1997). The Case-3 pressure distribution is obtained if the homogeneous half space ($\alpha = 1$ and $\nu_0 = \nu_1$) is indented by a sharp wedge indenter (Sneddon 1995). Compared to Case-1 and Case-2 profiles, where the pressure is finite in the loaded region ($|y| \leq y_0$), Case-3 profile exhibits unbounded pressure at the origin ($x = y = 0$). The Fourier cosine transform of the Case-2 and Case-3 pressure profiles Eqs. (52) and (53) using Eq. (14) is Maple[©],

$$\bar{Q}(\eta) = \frac{3P(\sin(\eta Y_0) - \eta Y_0 \cos(\eta Y_0))}{2(\eta Y_0)^3}, \quad (54)$$

Figure 5. Effect of layer thickness t on the components of stress σ_x , σ_y , and τ_{xy} at (a) intersection of the axis of loading and coating-substrate interface $x/t = 1$, $y/y_0 = 0$, (b) along the coating-substrate interface at a distance of semi loading edge length y_0 from the axis of loading $x/t = 1$, $y/y_0 = 1$, of the layered coating substrate system (figure 1) subjected to Case-1 pressure profile in Eq. (46) for a shear modulus ratio $\alpha = 0.1$. Poissons ratio for layer and substrate $\nu_0 = \nu_1 = 0.33$.

$$\bar{Q}(\eta) = P \left[\frac{-1}{2} J_0(Y_0 \eta) + \frac{1}{4} \pi J_0(Y_0 \eta) H_1(Y_0 \eta) - \frac{1}{4} \pi J_1(Y_0 \eta) H_0(Y_0 \eta) \right], \quad (55)$$

where $J_0(\eta Y_0)$ and $J_1(\eta Y_0)$ are the Bessel function of order 0 and 1, and $H_0(Y_0 \eta)$ and $H_1(Y_0 \eta)$ are the Struve function of order 0 and order 1 respectively (Abramowitz & Stegun 1972; Watson 1958). Appendix A provides details for evaluation of Struve function required in Eq. (55) as Matlab[®] which is used to carry out all the numerical computations in the present work does not provide inbuilt option for Struve function. Figure 7 shows the comparison of relative surface displacement $u^* = u(y/y_0, 0) - u(0, 0)$ of layered coating system of figure 1 subjected to Case-1, Case-2 and Case-3 pressure profiles of Eqs. (50), (52) and (53) for a given total load P , loading zone size $2y_0$ and normalized thickness ratio $t/y_0 = 1$ and Poissons

Figure 6. Effect of layer thickness t on the components of stress σ_x , σ_y , and τ_{xy} at (a) intersection of the axis of loading and coating-substrate interface $x/t = 1, y/y_0 = 0$, (b) along the coating-substrate interface at a distance of semi loading edge length y_0 from the axis of loading $x/t = 1, y/y_0 = 1$, of the layered coating substrate system (figure 1) subjected to Case-1 pressure profile in Eq. (46) for a shear modulus ratio $\alpha = 10$. Poissons ratio for layer and substrate $\nu_0 = \nu_1 = 0.33$.

ratio $\nu_0 = \nu_1 = 0.33$. Figure 7(a) shows the result for homogeneous case ($\alpha = 1$) where exact match was achieved for relative surface displacement $u^* = u(y/y_0, 0) - u(0, 0)$ corresponding to Case-1 and Case-3 pressure profiles with the parabolic and wedge indenter profiles respectively (Sneddon 1995), thus validating numerical methodology of the present work for a given total load (P) and loading zone length ($2y_0$), the amount of relative surface displacement (u^*) is maximum in case of the Case-3 pressure profile a representative of sharper pressure profile. Figures 7(b) and 7(c) show the effect of elastic modulus mismatch ($\alpha \neq 1$) on the relative surface displacement $u^* = u(y/y_0, 0) - u(0, 0)$ subjected to pressure profiles of Eqs. (50), (52) and (53) for soft and stiff coating case respectively. Comparison of figures 7(a)–7(c) reveals that qualitatively the relative displacement response for three different pressure

Figure 7. Comparison of the relative surface displacement profile $u^* = u(y/y_0, 0) - u(0, 0)$ for (a) homogeneous case ($\alpha = 1$), (b) soft coating ($\alpha = 0.1$), (c) stiff coating ($\alpha = 10$) of a layered coating substrate system (figure 1) subjected to Case-1, Case-2 and Case-3 pressure profile in Eqs. (46), (48), (49) respectively for a given total load P and semi loading zone size y_0 , normalized layer thickness $t/y_0 = 1$ and Poissons ratio for layer and substrate $\nu_0 = \nu_1 = 0.33$. Due to symmetry of the problem about x -axis, results are shown for positive y values only.

profiles is same irrespective of elastic modulus mismatch (α) between layer and substrate. The presence of stiffer (softer) coating relative to the substrate leads to lower (higher) compliance and thus a lower (higher) relative surface displacement is reaffirmed in figures 7(a)–7(c). It is noteworthy that for the case of stiff coating ($\alpha = 10$), the relative surface displacement (u^*) for the three pressure profiles is very close to each other thus negating the effect of pressure profile.

4. Summary and conclusion

The plane-strain loading problem of a linear elastic isotropic layered coating system (figure 1) is solved semi-analytically using Airy stress function and Fourier cosine transform for a given pressure distribution ($q(y)$) in the loaded region ($|y| \leq y_0$). Highly simplified and readily programmable expressions for stress and displacement field in layer and substrate are derived and presented. In the first part of the work, pressure profile (Case-1) corresponding to the indentation of parabolic indenter on homogeneous half space is used to assess the effect of elastic modulus mismatch (α), layer thickness (t) on the resulting stresses and displacement in layered coating system (figure 1). For a given normalized thickness ratio (t/y_0), the relative surface displacement (u^*) is maximum for a layer which is softer than the substrate ($\alpha < 1$). Regardless of the mismatch in elastic properties between layer and substrate, the loading response is independent of substrate if the thickness ratio (t/y_0) is equal to or greater than 10 and independent of layer if the thickness ratio (t/y_0) is less than or equal to 0.1. In case the layer is much stiffer than the

substrate, the compliance of the system varies inversely with thickness of the layer and response of layered coating system ranges from that of the substrate alone case to that of the layer alone case as the thickness of the layer is increased. In case the layer is much softer than the substrate, the compliance of the system is proportional to layer thickness and response of layered coating system spans from that of the layer alone case to that of the substrate alone case as the thickness of the layer is increased. Normal stress component σ_x along the loading axis ($y = 0$) and layer–substrate interface ($x = t$) is compressive and its magnitude decreases with increasing stiffness of the layer. For stiff layer, the normal stress component σ_y within the loading zone and near the layer–substrate interface is tensile and can lead to crack initiation. Variation of stresses in the layer–substrate system with layer thickness is not monotonous but depends on the location where stresses are monitored. For instance, at the intersection of loading axis and layer–substrate interface, stresses decrease monotonically with increased layer thickness while at the intersection of loading axis and loading edge, there exists a particular thickness which maximizes the magnitude of stress components. In the second part of this work, the effect of degree of smoothness of the pressure profile on the response of layered coating system is analysed by varying the pressure distribution $q(y)$ in the loading region ($|y| \leq y_0$) while keeping the total load P and loaded zone size $2y_0$ constant. Amongst the three pressure profile chosen for the study, two of them (Case-1, Case-2) are bounded and smooth throughout the loaded region while the third profile (Case-3) is unbounded at the centre of the loaded region. It is seen that despite the elastic modulus mismatch (α) between layer and substrate, for a given layer thickness (t), amongst the three pressure profile cases, the relative surface displacement (u^*) is maximum for Case-3 i.e. a representative of the sharp pressure profile. Although if a layer is relatively stiffer than the substrate, the influence of pressure profile on the relative surface displacement (u^*) becomes rather less significant. It is worthwhile to mention that all the results and observations in the present work are qualitatively consistent with Li & Chou (1997) work on axisymmetric loading study of layered coating system.

It has to be mentioned the linear-elastic-isotropic material behaviour and small deformation kinematics used in the present work may not always be the most accurate assumptions for layered systems. In such instances, it is usually difficult to obtain analytically tractable solution for loading as done here and recourse has to be taken to numerical methods (Tian & Saka 1991a,b; Sun & Bell 1995; Aksel & Karlsruhe 1986; Giannakopoulos 2002; Anand & Su 2005; Choi *et al* 2008a,b; Wu *et al* 2012). Possible future work includes extension of the current work to multi-layer (> 1) similar to work of Constantinescu *et al* (2013) for axisymmetric loading. Compact expressions in the present study can also be potentially applied to carry out numerically efficient studies for crack and dislocation nucleation in the layered coating systems subjected to indentation (Oliveira & Bower 1996; Hurtado & Kim 1999; Hills *et al* 1996; He *et al* 1994; Evans *et al* 1999). Using Eq. (28), the Green function $G(\eta)$ relating the surface displacement $u_1(0, Y)$ to $\bar{Q}(\eta)$, the cosine Fourier transform of the applied pressure $q(y)$ is given by

$$u_1(0, Y) = \frac{-2}{\pi} \int_0^\infty \bar{Q}(\eta) G(\eta) \cos(\eta Y) d\eta. \quad (56)$$

where

$$G(\eta) = \frac{\bar{u}_1^*(0, \eta)}{N(\eta)} = \frac{\eta}{2\mu_1 N(\eta)} \left\{ dk\gamma_2 + 2\gamma_0 l e^{2\eta} - K_2 + ek\gamma_2 + K_1 - 2\gamma_0 m e^{-2\eta} \right\}. \quad (57)$$

The Green's function $G(\eta)$ in Eq. (57) finds use in the solution of contact problem of LC system. The Green's function can also be used to develop an auxiliary function approach for the

solution of plane strain contact of LC system analogous to the work of Perriot and Barthel on axisymmetric indentation (Perriot 2004).

Acknowledgements

We would like to gratefully acknowledge the financial support from Industrial Research & Consultancy Centre, Indian Institute of Technology Bombay through the seed grant 11IRCCSG022.

Appendix A

Evaluation of Struve function

The Struve function of order 0 and 1 in Eq. (55) is evaluated using the power series expansion given by (Abramowitz & Stegun 1972; Watson 1958),

$$H_{\rho}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+3/2)\Gamma(k+\rho+3/2)} \left(\frac{x}{2}\right)^{2k+\rho+1} \quad (58)$$

where “ ρ ” is the order of Struve function and $\Gamma(k)$ is the Gamma (Factorial) function. The infinite series in Eq. (58) corresponding to Struve function of order 0 and 1 is calculated conveniently using piecewise continuous function representation based on the algorithm of Brun *et al* (1995). The piecewise functions with the argument x in Eq. (58) treated as symbolic variable are coded as subroutine and passed to the inbuilt quad/integral function in Matlab[®] as part of the integrand in Eqs. (25–29) for numerical integration.

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