Sādhanā Vol. 40, Part 6, September 2015, pp. 1925–1935. © Indian Academy of Sciences

Investigation of torsional vibrations in thick walled hollow poroelastic cylinder using Biot's extension theory

MALLA REDDY PERATI* and RAJITHA GURIJALA

Department of Mathematics, Kakatiya University, Warangal 506009, India
e-mail: mperati@yahoo.com; rajitha.akshu@gmail.com

MS received 30 January 2014; revised 6 March 2015; accepted 7 April 2015

Abstract. This paper deals with the study of torsional vibrations of thick walled hollow poroelastic cylinder using Biot's extension theory. Considering the boundaries to be stress free, the frequency equation is obtained in the presence of dissipation which is transcendental and complex valued in nature. In the special case of poroelastic thin shell, frequency equation is separated into two real valued equations giving propagation velocity and attenuation coefficient. From the numerical results in the case of sandy sediment, it is clear that the values of propagation velocity are, in general, less than that of attenuation coefficient.

Keywords. Torsional vibrations; poroelastic cylinder; frequency equation; propagation velocity; attenuation coefficient.

1. Introduction

The study of torsional vibrations is of importance both from theoretical and applications perspectives in several engineering fields. All the rotating systems experience torsional vibrations. In mechanical engineering, torsional vibration is due to angular twist in the shaft or drive train and causes fatigue. Excess torsional vibrations result in gear wear, gear tooth failure, and broken shafts in severe cases. The severity of vibrations depends upon the relation between speed and frequency of the vibration (Wachel & Szenasi 1993). Torsional vibration has a concern in power transmission systems which use rotating shafts or couplings where it can cause failures if not controlled. Analytical solution for the torsional vibrations of the saturated soil and pile has been derived by modelling the end bearing pile embedded in a saturated soil as elastic–poroelastic interface problem (Kuihua Wang *et al* 2008). Employing Biot's theory of wave propagation in fluid saturated porous media (Biot 1956), Tajuddin & Sarma (1980) studied torsional vibrations of poroelastic solid cylinders. The plane strain vibrations of thick-walled hollow poroelastic cylinders are investigated by Malla Reddy & Tajuddin (2000). The said paper describes

*For correspondence

transitions from the case of plate to thin shell, and to thick walled hollow cylinder, and then to solid cylinder. Frequency equation for circumferential waves of an infinite hollow poroelastic cylinder in the presence of dissipation is investigated (Tajuddin & Ahmed Shah 2006). The investigation of torsional vibrations of hollow poroelastic cylinders is made for various values of dissipations and the limiting cases namely thin shell, thick shell, and solid cylinder are drawn (Tajuddin & Ahmed Shah 2007). From the investigation, it is concluded that increase in dissipation value reduces the phase velocity, group velocity, and attenuation. There is no significance variation in attenuation as the frequency increases. Biot's theory of isotropic poroelastic solids confirms the existence of three waves, two longitudinal and one transverse. This is true when the material is saturated with a non-viscous fluid. Moreover, bulk and shear viscosities are not considered in the Biot's theory. The presence of viscosity in the Newtonian fluid results in the propagation of one more wave, which is transverse in character (Sahay 2008). The viscosity-extended Biot constitutive relations are developed by Sahay (1996). In the said paper, the volume-average equations of motion for the angular displacement fields are expressed in cylindrical coordinates. The volume-averaging approach introduces the missing fluid-strain rate term in the extended Biot's constitutive relations. Sahay (2008) developed the viscosity-extended Biot's theory for Newtonian fluids which describes the existence of second shear wave (slow S-wave). The presence of slow shear-wave plays an important role for the attenuation of seismic waves in the presence of layering. Generalized Biot's theory of wave propagation in a dissipative poroelastic medium is studied by Sharma (2013). This paper considers viscosity in the interstitial fluid that results in transverse wave as in the paper (Sahay 2008). Also, the existence and propagation of the fourth wave is proved in theoretical approach defining deviatory stress in the viscous fluid. In the framework of the volume-averaged theory of poroelasticity, complete analysis of standing torsional waves in a fully saturated porous circular cylinder is given by Solorza & Sahay (2004). Employing Biot's extension theory, an extensional wave in a poroelastic cylinder in the case of traction free open pore cylinder is studied by Solorza & Sahay (2009). This way, recently there is a much focus on extended Biot's theory. In the case of applications mentioned in the beginning of this section, if the solid is saturated with viscous fluid, classical Biot's theory is not applicable and one has to go for extended Biot's theory. The torsional vibrations of a thick walled hollow poroelastic cylinder in the frame work of extended Biot's theory are not yet investigated, therefore in this paper the same is investigated. In the present analysis, considering the boundaries to be stress free, the frequency equation of torsional vibrations is obtained in the presence of dissipation which involves fluid shear viscosity. In the limiting case of thin shell, complex valued frequency equation is reduced to two real parts that give propagation velocity and attenuation, respectively. By neglecting the fluid shear viscosity, the problem reduces to that of classical Biot's theory (1956).

The rest of the paper is organized as follows. In section 2, constitutive relations and equations of motion are given. In section 3, first boundary conditions are prescribed, next frequency equation is deduced. In section 4, numerical results in a limiting case are presented graphically. Finally, conclusion is given in section 5.

2. Equations of motion and constitutive relations

Let (r, θ, z) be the cylindrical coordinates. Consider a homogenous, isotropic thick walled hollow poroelastic cylinder with inner and outer radii r_1 and r_2 respectively, whose axis is in the direction of z axis. For torsional vibrations, the volume-averaged equation of motion of poroelasticity in terms of angular displacement field, $\vec{\mathbf{u}}$ expressed in cylindrical coordinates is as follows (Solorza & Sahay 2009):

$$\left(C + N\frac{\partial}{\partial t}\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} - \frac{1}{r^2}\right]\vec{\mathbf{u}} = \Omega_i I_0 \frac{\partial \vec{\mathbf{u}}}{\partial t} + I\frac{\partial^2 \vec{\mathbf{u}}}{\partial t^2}, \tag{1}$$

here $C = \begin{pmatrix} 1 & m^f \\ d^f & d^f m^f \end{pmatrix} \beta_c^2$, $N = \begin{pmatrix} \alpha_\mu & -(\eta_0 - m^f \alpha_\mu) \\ -d^s \alpha_\mu & d^s(\eta_0 - m^f \alpha_\mu) \end{pmatrix} \beta_c^2 \frac{1}{\Omega^\beta}$, $d^f = \frac{1}{S - m^f}$, and $d^s = \frac{\phi_0 \rho^s}{\rho^i}$, ϕ_0 is unperturbed volume fraction of solid, S is tortuosity factor, $\beta_c^2 = \frac{\mu_0}{\rho^m}$ is Gassmann S-wave velocity, μ_0 is dry solid frame shear modulus, α_μ is Biot shear coefficient, Ω^β is saturated frame shear relaxation frequency, $\Omega^i = d^f \Omega^b$ is Biot relaxation frequency, $\Omega^b = \frac{\eta_0 \vartheta_f}{K}$ is the Biot critical frequency, η_0 is porosity, K is permeability, and $\vartheta_f = \frac{\mu_f}{\rho^f}$ is the shear viscosity, μ_f is the fluid shear viscosity. The matrix I_0 is 2×2 matrix whose element (2 2) is unity and rest of the elements are equal to zero, and I is the second order identity matrix. The vector $\vec{\mathbf{u}}$ is $\left(u_j^m \ \ u_j^i\right)^T = m^{-1}\left(u_j^s \ \ u_j^f\right)^T$. The notation u_j^m is the sum of mass-weighted angular displacement of the solid and fluid phases. The notation u_j^i is the difference of the angular displacements of two phases. The transformation matrix is defined by $m = \begin{pmatrix} 1 & m^f \\ 1 & -m^s \end{pmatrix}$, m^f, m^s are the solid and fluid mass fractions, respectively, and ρ^m, ρ^r are the total and reduced densities of the porous medium, respectively, given by $\rho^m = \phi_0\, \rho^s + \eta_0 \rho^f$, $\frac{1}{\rho^r} = \frac{1}{\phi_0\, \rho^s} + \frac{1}{\eta_0\, \rho^f}$, $\rho^i = \rho^r - \rho^{12}$ is the modified reduced density, and ρ^s, ρ^f are the solid and fluid densities. ρ^{12} is the induced mass coefficient which is linked to the tortuosity, as $\rho^{12} = -\ (S-1)\ \eta_0\, \rho^f$. In the frequency domain, Eq. (1) becomes

$$\left[\beta\left(\nabla^2 - \frac{1}{r^2}\right) + \omega^2 I\right]\vec{\mathbf{u}} = 0, \tag{2}$$

where $\beta = \Omega^{-1}(C - i\omega\ N) = \begin{pmatrix} \beta^{mm} & \beta^{mi} \\ \beta^{im} & \beta^{ii} \end{pmatrix}$, is a non-symmetric second order matrix associated with S motion, respectively, whose elements are dimensionally equal to velocity squared. The expressions of these parameters are given in Appendix A. The notation $I + i\ \left(\frac{\Omega^i}{\omega}\right)\ I_0 = \Omega$ is a 2×2 diagonal matrix associated with the Biot relaxation frequency Ω^i, and $\rho^{-1}\mu \equiv C_\beta$, $\rho^{-1}\vartheta \equiv N_\beta$. For torsional vibrations, the displacement components which are functions of r, z and t are introduced as follows:

$$\vec{\mathbf{u}} = R_\beta\ F(r)\ e^{ikz}\,. \tag{3}$$

Here $R_\beta = [r_{\beta\, I}, r_{\beta\, II}]$ and $L_\beta = [I_{\beta\, I}, I_{\beta\, II}]$ is the right- and left-hand eigen vector matrices of the β matrix, respectively, their explicit expressions are given in Appendix A. These eigen vectors are orthonormal to each other, therefore $\mathbf{L}_\beta^T\ \ R_\beta = I$ (Sahay 2008). Here $F(r) = [F_1(r), F_2(r)]^T$, and k is the wave number. Substituting Eq. (3) in Eq. (2), and then multiplying $\mathbf{L}_\beta^T$ on both sides, the equations for S motion are obtained as,

$$\left[\Lambda_\beta\left(\nabla^2 - \frac{1}{r^2}\right) + \omega^2 I\right]\ F(r)\, e^{ikz} = 0, \tag{4}$$

here $\Lambda_\beta = \mathbf{L}_\beta^T\, \beta\; R_\beta = \begin{pmatrix} \beta_I^2 & 0 \\ 0 & \beta_{II}^2 \end{pmatrix}$,

In the region below Biot relaxation frequency, $\omega << \Omega_i$, β_I, β_{II} are the fast and slow S wave velocities given by (Solorza & Sahay 2009)

$$\beta_I^2 \approx \beta_c^2 \left[1 - i\, \left(\frac{\omega}{\Omega^i}\right) d^f m^f\right],$$

$$\beta_{II}^2 \approx -\omega \left(\frac{\omega}{\Omega^i}\right) \left[1 + i\, \left(\frac{\omega}{\Omega^i}\right) (1 + d^f m^f)\right] d^f \vartheta_f. \tag{5}$$

The solutions for $F(r)$ are

$$\begin{aligned} F_1(r) &= C_1 J_1(pr) + C_2 Y_1(pr), \\ F_2(r) &= C_3 J_1(qr) + C_4 Y_1(qr). \end{aligned} \tag{6}$$

In the above, C_1, C_2, C_3, C_4 are constants, J_1 and Y_1 are the Bessel functions of the first and second kind of order one, respectively, and $p = \left(\frac{\omega^2}{\beta_I^2} - k^2\right)^{\frac{1}{2}}$, $q = \left(\frac{\omega^2}{\beta_{II}^2} - k^2\right)^{\frac{1}{2}}$.

The modified constitutive relations in terms of natural dynamic fields are expressed as (Sahay 2008)

$$\begin{pmatrix} \sigma_{r\theta}^m \\ \sigma_{r\theta}^i \end{pmatrix} = 2\,(\mu + \vartheta\,\partial_t)\, \begin{pmatrix} u_{r\theta}^m \\ u_{r\theta}^i \end{pmatrix}, \tag{7}$$

here $m^T \mu_b m \equiv \mu, \quad m^T \vartheta_b m \equiv \vartheta, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mu_0 \equiv \mu_b$, and $\begin{pmatrix} 0 & 0 \\ \alpha_\mu - \eta_0 & \eta_0 \end{pmatrix} \mu_f \equiv \vartheta_b$.

In the frequency domain Eq. (7) can be written as

$$\sigma_{r\theta} = 2(C_\beta + \mathbf{N}_\beta \partial_t)\vec{\mathbf{u}}_{r\theta}, \tag{8}$$

Eq. (8) can also be expressed as

$$\sigma_{r\theta} = 2\rho\, \Omega\, \beta\, \vec{\mathbf{u}}_{r\theta}, \tag{9}$$

here $\rho = \begin{pmatrix} \rho^m & 0 \\ 0 & \rho^i \end{pmatrix}$, $\quad \vec{\mathbf{u}}_{r\theta} = \frac{1}{2}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right) \vec{\mathbf{u}}$, $\quad \vec{\mathbf{u}}$ is given by Eq. (3).

3. Boundary conditions and frequency equation

The stress free boundary conditions for torsional vibrations at the inner and outer surface of the hollow poroelastic cylinder i.e. $r = r_1$ and $r = r_2$ are

$$\sigma_{r\theta} = 0. \tag{10}$$

These boundary conditions lead to the following system of homogeneous equations:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{11}$$

where
$a_{11} = b_{11} p\, J_2(pr_1)$,
$a_{12} = b_{11} p\, Y_2(pr_1)$,
$a_{23} = b_{12}\, q\, J_2(qr_1)$,
$a_{24} = b_{12} q\, Y_2(qr_1)$,
$a_{21} = b_{21} p\, J_2(pr_1)$,
$a_{22} = b_{21} p\, Y_2(pr_1)$,
$a_{23} = b_{22}\, q\, J_2(qr_1)$,
$a_{24} = b_{22} q\, Y_2(qr_1)$,
$a_{31},\ a_{32},\ a_{33},\ a_{34} =$ similar expression as $a_{11},\ a_{12},\ a_{13},\ a_{14}$ with r_1 replaced by r_2,
$a_{41},\ a_{42},\ a_{43},\ a_{44} =$ similar expression as $a_{21},\ a_{22},\ a_{23},\ a_{24}$ with r_1 replaced by r_2,
and $b_{11} = \frac{\rho^m \beta_I^2}{N_{\beta_I}}$,

$$b_{12} = \rho^m \left(\frac{\beta^{mm}}{N_{\beta_{II}}} \left(\frac{\beta^{mi}}{\beta_{II}^2 - \beta^{mm}} \right) + \frac{\beta^{mi}}{N_{\beta_{II}}} \right),$$

$$b_{21} = \rho^i \left(1 + \frac{\Omega_i}{\omega} \right) \left(\frac{\beta^{im}}{N_{\beta_I}} + \frac{\beta^{ii}}{N_{\beta_I}} \left(\frac{\beta_I^2 - \beta^{mm}}{\beta^{mi}} \right) \right),$$

$$b_{22} = \rho^i \left(1 + \frac{\Omega_i}{\omega} \right) \left(\frac{\beta^{im}}{N_{\beta_{II}}} \left(\frac{\beta^{mi}}{\beta_{II}^2 - \beta^{mm}} \right) + \frac{\beta^{ii}}{N_{\beta_{II}}} \right).$$

Eq. (11) results in a system of four homogeneous equations in four arbitrary constants C_1, C_2, C_3, C_4. For a non-trivial solution, determinant of coefficients is zero. Accordingly, we obtain the complex valued frequency equation which is transcendental in nature. Therefore, this equation cannot be solvable so easily. If $\frac{h}{r_1} << 1$, the thick-walled hollow cylinder becomes thin shell and the following asymptotic expansions for Bessel functions can be used (Milton Abramowitz & Stegun 1964).

$$J_2(x) \approx \sqrt{\frac{2}{\pi x}} \left[\cos\left(x - \frac{\pi}{4}\right) - \frac{15}{8x} \sin\left(x - \frac{\pi}{4}\right) \right],$$

$$Y_2(x) \approx \sqrt{\frac{2}{\pi x}} \left[\sin\left(x - \frac{\pi}{4}\right) + \frac{15}{8x} \cos\left(x - \frac{\pi}{4}\right) \right].$$

In this case, frequency equation is resolved as

$$|c_{l\,m}| + i\ \ |d_{l\,m}| = 0 \quad (l, m = 1,\ 2,\ 3,\ 4). \tag{12}$$

In Eq. (12), the elements $c_{l\,m}$ and $d_{l\,m}$ are real valued and are given by,

$$c_{11} = (C_{10}Q_{11} - C_{20}Q_{22})(D_1T_1 - D_2T_2) - (C_{10}Q_{22} + C_{20}Q_{11})\,(D_1T_2 + D_2T_1)\,,$$

c_{12} is similar expression as c_{11} with D_1, D_2 replaced by D_3, D_4, respectively,

$$c_{13} = (C_{30}Q_{33} - C_{40}Q_{44})(R_1T_3 - R_2T_4) - (C_{30}Q_{44} + C_{40}Q_{33})\,(R_1T_4 + R_2T_3)\,,$$

c_{14} is similar expression as c_{13} with R_1, R_2 replaced by R_3, R_4, respectively,

$$c_{21} = (C_{50}Q_{11} - C_{60}Q_{22})(D_1T_1 - D_2T_2) - (C_{50}Q_{22} + C_{60}Q_{11})\,(D_1T_2 + D_2T_1)\,,$$

c_{22} is similar expression as c_{21} with D_1, D_2 replaced by D_3, D_4, respectively,

$$c_{23} = (C_{70}Q_{33} - C_{80}Q_{44})(R_1T_3 - R_2T_4) - (C_{70}Q_{44} + C_{80}Q_{33})\,(R_1T_4 + R_2T_3)\,,$$

c_{24} is similar expression as c_{23} with R_1, R_2 replaced by R_3, R_4, respectively,

$$c_{31} = (C_{10}Q_{11} - C_{20}Q_{22})(P_1D_5 - P_2D_6) - (C_{10}Q_{22} + C_{20}Q_{11})\,(P_1D_6 + P_2D_5)\,,$$

c_{32} is similar expression as c_{31} with D_5, D_6 replaced by D_7, D_8, respectively,

$$c_{33} = (C_{30}Q_{33} - C_{40}Q_{44})(P_3R_5 - P_4R_6) - (C_{30}Q_{44} + C_{40}Q_{33})\,(P_3R_6 + P_4R_5)\,,$$

c_{34} is similar expression as c_{33} with R_5, R_6 replaced by R_7, R_8, respectively,

$$c_{41} = (C_{50}Q_{11} - C_{60}Q_{22})(P_1D_5 - P_2D_6) - (C_{50}Q_{22} + C_{60}Q_{11})\,(P_1D_6 + P_2D_5)\,,$$

c_{42} is similar expression as c_{41} with D_5, D_6 replaced by D_7, D_8, respectively,

$$c_{43} = (C_{70}Q_{33} - C_{80}Q_{44})(P_3R_5 - P_4R_6) - (C_{70}Q_{44} + C_{80}Q_{33})\,(P_3R_6 + P_4R_5)\,,$$

c_{44} is similar expression as c_{43} with R_5, R_6 replaced by R_7, R_8, respectively,

$$d_{11} = (C_{10}Q_{11} - C_{20}Q_{22})(D_1T_2 + D_2T_1) + (C_{10}Q_{22} + C_{20}Q_{11})\,(D_1T_1 - D_2T_2)\,,$$

d_{12} is similar expression as d_{11} with D_1, D_2 replaced by D_3, D_4, respectively,

$$d_{13} = (C_{30}Q_{33} - C_{40}Q_{44})(R_1T_4 + R_2T_3) + (C_{30}Q_{44} + C_{40}Q_{33})\,(R_1T_3 - R_2T_4)\,,$$

d_{14} is similar expression as d_{13} with R_1, R_2 replaced by R_3, R_4, respectively,

$$d_{21} = (C_{50}Q_{11} - C_{60}Q_{22})(D_1T_2 + D_2T_1) + (C_{50}Q_{22} + C_{60}Q_{11})\,(D_1T_1 - D_2T_2)\,,$$

d_{22} is similar expression as d_{21} with D_1, D_2 replaced by D_3, D_4, respectively,

$$d_{23} = (C_{70}Q_{33} - C_{80}Q_{44})(R_1T_4 + R_2T_3) + (C_{70}Q_{44} + C_{80}Q_{33})\,(R_1T_3 - R_2T_4)\,,$$

d_{24} is similar expression as d_{23} with R_1, R_2 replaced by R_3, R_4, respectively,

$$d_{31} = (C_{10}Q_{11} - C_{20}Q_{22})(P_1D_6 + P_2D_5) + (C_{10}Q_{22} + C_{20}Q_{11})\,(P_1D_5 - P_2D_6)\,,$$

d_{32} is similar expression as d_{31} with D_5, D_6 replaced by D_7, D_8, respectively,

$$d_{33} = (C_{30}Q_{33} - C_{40}Q_{44})(P_3R_6 + P_4R_5) + (C_{30}Q_{44} + C_{40}Q_{33})\,(P_3R_5 - P_4R_6)\,,$$

d_{34} is similar expression as d_{33} with R_5, R_6 replaced by R_7, R_8, respectively,

$$d_{41} = (C_{50}Q_{11} - C_{60}Q_{22})(P_1D_6 + P_2D_6) + (C_{50}Q_{22} + C_{60}Q_{11})\,(P_1D_5 - P_2D_6)\,,$$

d_{42} is similar expression as d_{41} with D_5, D_6 replaced by D_7, D_8, respectively,

$$d_{43} = (C_{70}Q_{33} - C_{80}Q_{44})(P_3R_6 + P_4R_5) + (C_{70}Q_{44} + C_{80}Q_{33})\,(P_3R_5 - P_4R_6)\,,$$

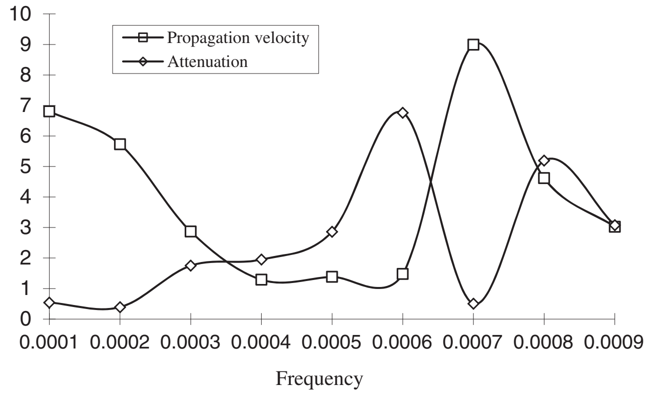

Figure 1. Variation of propagation velocity and attenuation with frequency.

$$d_{44} \text{ is similar expression as } d_{43} \text{ with } R_5, R_6 \text{ replaced by } R_7, R_8, \text{ respectively,} \quad (13)$$

The notations of Eq. (13) are given in Appendix (B).

4. Numerical results

The complex frequency Eq. (12) gives propagation velocity and attenuation coefficient as a function of frequency. The real part of Eq. (12) gives propagation velocity of the wave, whereas imaginary part and real part of Eq. (12) give attenuation coefficient. For the illustration purpose, the parameter values for water-saturated sandy sediment are used (Byoung Nam Kim *et al* 2004), which are given in Appendix C. Employing these values in Eq. (12), propagation velocity and attenuation coefficient are computed as a function of frequency. The values are computed using bisection method implemented in MATLAB, and are depicted in figure 1. From the figure, it is clear that the values of propagation velocity are, in general, less than that of attenuation coefficient values. The values are available at very low frequencies. By neglecting the shear viscosity, the problem reduces to that of classical Biot's theory (Tajuddin & Ahmed Shah 2007).

5. Conclusion

Torsional vibrations of thick walled hollow poroelastic cylinder are investigated using viscosity-extended Biot's theory. Because of the mathematical complexity, some approximations are made in the case of thick walled hollow cylinder to obtain the case of thin shell, and the complex valued frequency equation is reduced to two real valued equations which give propagation velocity and attenuation coefficient. In the case of water saturated sandy sediments, it is clear that propagation velocity is greater than attenuation coefficient. The values are available at low frequencies which may not be the case with the other poroelastic materials. This kind of study can be made for any poroelastic solid if the pertinent parameters are known.

Appendix A

In the region below Biot relaxation frequency, $\omega << \Omega_i$

In the region below Biot relaxation frequency the following approximations are valid (Solorza & Sahay 2009):

$$\beta^{mm} \approx \beta_c^2\left(1 - i\,\frac{\omega}{\Omega^\beta}\alpha_\mu\right),$$
$$\beta^{mi} \approx \beta_c^2\left[1 + i\,\frac{\omega}{\Omega^\beta}\left(\frac{\eta_0}{m^f} - \alpha_\mu\right)\right] m^f,$$
$$\beta^{im} \approx \beta_c^2\left[\left(\frac{\omega}{\Omega^i}\right)^2 - i\,\left(\frac{\omega}{\Omega^i}\right)\right]\left(d^f + i\,\frac{\omega}{\Omega^\beta} d^s \alpha_\mu\right),$$
$$\beta^{ii} \approx \beta_c^2\left[\left(\frac{\omega}{\Omega^i}\right)^2 - i\,\left(\frac{\omega}{\Omega^i}\right)\right]\left[d^f - i\,\frac{\omega}{\Omega^\beta} d^s\left(\frac{\eta_0}{m^f} - \alpha_\mu\right)\right] m^f,$$
$$r_{\beta_I} = \frac{1}{N_{\beta_I}}\begin{pmatrix} 1 \\ \frac{\beta_I^2 - \beta^{mm}}{\beta^{\mathrm{mi}}} \end{pmatrix}, \qquad r_{\beta_{II}} = \frac{1}{N_{\beta_{II}}}\begin{pmatrix} \frac{\beta^{\mathrm{mi}}}{\beta_{II}^2 - \beta^{mm}} \\ 1 \end{pmatrix},$$
$$I_{\beta_I} = \frac{1}{N_{\beta_I}}\begin{pmatrix} 1 \\ \frac{\beta^{\mathrm{mi}}}{\beta_I^2 - \beta^{ii}} \end{pmatrix}, \qquad I_{\beta_{II}} = \frac{1}{N_{\beta_{II}}}\begin{pmatrix} \frac{\beta_{II}^2 - \beta^{ii}}{\beta^{\mathrm{mi}}} \\ 1 \end{pmatrix},$$
$$N_{\beta_I} = \sqrt{\frac{\beta_I^2 - \beta_{II}^2}{\beta_I^2 - \beta^{ii}}}, \qquad N_{\beta_{II}} = \sqrt{\frac{\beta_I^2 - \beta_{II}^2}{\beta^{mm} - \beta_{II}^2}}.$$

Appendix B

$$C_{10} = \frac{\rho^m(B_{10}N_1 + B_{20}N_2)}{N_1^2 + N_2^2},\ C_{20} = \frac{\rho^m(B_{20}N_1 - B_{10}N_2)}{N_1^2 + N_2^2},$$
$$B_{10} = \beta_c^2,\ \ B_{20} = \frac{-\beta_c\omega\, d^f m^f}{\Omega^i},$$
$$N_1 = \left(a_3^2 + b_3^2\right)^{1/2}\cos\left[\tan^{-1}\left(\frac{b_3}{a_3}\right)\right],$$
$$N_2 = \left(a_3^2 + b_3^2\right)^{1/2}\sin\left[\tan^{-1}\left(\frac{b_3}{a_3}\right)\right],$$
$$a_3 = \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2},\ b_3 = \frac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2},$$
$$a_1 = \beta_c^2 + \frac{\omega^2 d^f \vartheta_f}{\Omega^i},$$
$$b_1 = \frac{(1 + d^f m^f)d^f\vartheta_f\omega^3}{\Omega^{i2}} - \beta_c^2 d^f m^f\left(\frac{\omega}{\Omega^i}\right),$$
$$a_2 = \beta_c^2 - \beta_c^2 m^f\left(\left(\frac{\omega}{\Omega^i}\right)^2 d^f - \left(\frac{\omega^2 d^s}{\Omega^i\Omega^\beta}\right)\left(\frac{\eta_0}{m^f} - \alpha_\mu\right)\right),$$
$$b_2 = -\beta_c^2 d^f m^f\left(\frac{\omega}{\Omega^i}\right) + \beta_c^2 m^f\left(\left(\frac{\omega}{\Omega^i}\right)^2\left(\frac{\omega\, d^s}{\Omega^\beta}\right)\left(\frac{\eta_0}{m_f} - \alpha_\mu\right) + \left(\frac{\omega}{\Omega^i}\right) d^f\right),$$
$$C_{30} = \frac{\rho^m(C_{11}N_{11} + C_{22}N_{22})}{N_{11}^2 + N_{22}^2},\ C_{40} = \frac{\rho^m(C_{22}N_{11} - C_{11}N_{22})}{N_{11}^2 + N_{22}^2},$$

$$C_{11} = B_{51}(a_1 B_{10} + b_4 B_{60}) + B_{61}(b_4 B_{10} + a_1 B_{60}),$$
$$C_{22} = B_{61}(a_1 B_{10} + b_4 B_{60}) - B_{51}(b_4 B_{10} + a_1 B_{60}),$$
$$N_{11} = N_3 a_1 - N_4 b_4, \qquad N_{22} = N_3 b_4 + N_4 a_1,$$
$$b_4 = -\beta_c^2 \alpha_\mu \left(\frac{\omega}{\Omega^\beta}\right) + (1 + d^f m^f)\omega \left(\frac{\omega}{\Omega^i}\right)^2 d^f \vartheta_f,$$
$$B_{60} = \frac{\beta_c^2 \omega \alpha_\mu}{\Omega^\beta}, \quad B_{51} = \beta_c^2 m^f, \quad B_{61} = \beta_c^2 m^f \frac{\omega}{\Omega^\beta} \left(\frac{\eta_0}{m^f} - \alpha_\mu\right),$$
$$N_3 = \left(A_3^2 + B_3^2\right)^{1/2} \cos\left[\tan^{-1}\left(\frac{B_3}{A_3}\right)\right],$$
$$N_4 = \left(A_3^2 + B_3^2\right)^{1/2} \sin\left[\tan^{-1}\left(\frac{B_3}{A_3}\right)\right],$$
$$A_3 = \frac{a_1^2 + b_1 b_4}{a_1^2 + b_4^2}, \; B_3 = \frac{a_1 b_1 - a_1 b_4}{a_1^2 + b_4^2},$$
$$C_{50} = \rho^i \left(1 + \frac{\Omega^i}{\omega}\right) \left(\frac{e_1 e_3 + e_2 e_4}{e_3^2 + e_4^2}\right), \; C_{60} = \rho^i \left(1 + \frac{\Omega^i}{\omega}\right) \left(\frac{e_2 e_3 - e_1 e_4}{e_3^2 + e_4^2}\right),$$
$$e_1 = B_{51} B_{52} - B_{61} B_{62} + B_{70}(B_{10} - B_{50}) - B_{80}(B_{20} - B_{60}),$$
$$e_2 = B_{52} B_{61} + B_{51} B_{62} + B_{70}(B_{20} - B_{60}) + B_{80}(B_{10} - B_{50}),$$
$$e_3 = N_1 B_{51} - N_2 B_{61}, \qquad e_4 = N_2 B_{51} + N_1 B_{61},$$
$$B_{52} = \beta_c^2 \left(\frac{\omega}{\Omega^i}\right) \left(\left(\frac{\omega}{\Omega^i}\right) d^f + \left(\frac{\omega\, d^s}{\Omega^\beta}\right) \alpha_\mu\right),$$
$$B_{62} = \beta_c^2 \left(\frac{\omega}{\Omega^i}\right) \left(\left(\frac{\omega}{\Omega^i}\right) \left(\frac{\omega\, d^s \alpha_\mu}{\Omega^\beta}\right) - d^f\right),$$
$$B_{70} = \beta_c^2 m^f \left(\left(\frac{\omega}{\Omega^i}\right)^2 d^f - \left(\frac{\omega}{\Omega^i}\right) \left(\frac{\omega\, d^s}{\Omega^\beta}\right) \left(\frac{\eta_0}{m^f} - \alpha_\mu\right)\right),$$
$$B_{80} = -\beta_c^2 m^f \left(\left(\frac{\omega}{\Omega^i}\right)^2 \left(\frac{\omega\, d^s}{\Omega^\beta}\right) \left(\frac{\eta_0}{m^f} - \alpha_\mu\right) + \left(\frac{\omega}{\Omega^i}\right) d^f\right),$$
$$C_{50} = \rho^i \left(1 + \frac{\Omega^i}{\omega}\right) \left(\frac{f_1 N_{11} + f_2 N_{22}}{N_{11}^2 + N_{22}^2}\right), \; C_{60} = \rho^i \left(1 + \frac{\Omega^i}{\omega}\right) \left(\frac{f_2 N_{11} - f_1 N_{22}}{N_{11}^2 + N_{22}^2}\right),$$
$$f_1 = (B_{70} a_1 - B_{80} b_4) - (B_{52} B_{51} - B_{61} B_{62}),$$
$$f_2 = (B_{80} a_1 + B_{70} b_4) - (B_{52} B_{61} + B_{62} B_{51}),$$
$$N_{11} = N_3 a_1 - N_4 b_4, \qquad N_{22} = N_3 b_4 + N_4 a_1,$$
$$Q_{11} = \frac{\omega\,(B_1 W_1 + B_2 W_2)}{V(B_1^2 + B_2^2)}, \qquad Q_{22} = \frac{\omega\,(B_1 W_2 - B_2 W_1)}{V(B_1^2 + B_2^2)},$$

Q_{33}, Q_{44} are similar expressions as Q_{11}, Q_{22} with B_1, B_2, W_1, W_2 replaced by B_3, B_4, W_3, W_4,

$$B_1 = \beta_c, \; B_2 = \frac{-\beta_c \omega\, d^f m^f}{2\Omega^i}, \; B_3 = \frac{(1 + d^f m^f)\omega^2}{2\Omega^i} \sqrt{\frac{d^f \vartheta_f}{\Omega^i}}, \; B_4 = -\omega \sqrt{\frac{d^f \vartheta_f}{\Omega^i}},$$

here ϑ_f is shear viscosity, V is complex velocity,

$$W_1 = \left(V_1^2 + V_2^2\right)^{1/2} \cos\left[\frac{1}{2}\tan^{-1}\left(\frac{V_2}{V_1}\right)\right],$$

$$W_2 = \left(V_1^2 + V_2^2\right)^{1/2} \sin\left[\frac{1}{2}\tan^{-1}\left(\frac{V_2}{V_1}\right)\right],$$

$W_3,\ W_4$ are similar expressions as $W_1,\ W_2$ with $V_1,\ V_2$ replaced by $V_{11},\ V_{22}$,

$$V_1 = (V^2 - \beta_c^2),\ \ V_2 = \frac{\beta_c^2 \omega\, d^f m^f}{\Omega^i},\ \ V_{11} = V^2 + \frac{\omega^2 d^f \vartheta_f}{\Omega^i},\ \ V_{22} = \left(\frac{\omega}{\Omega^i}\right)^2 (1 + d^f m^f) d^f \vartheta_f,$$

$$D_1 = 8Q_1 \cosh Q_2(\cos Q_1 + \sin Q_1) + (15\cosh Q_2 - 8Q_2 \sinh Q_2)\ (\cos Q_1 - \sin Q_1),$$
$$D_2 = 8Q_1 \sinh Q_2(\cos Q_1 - \sin Q_1) + (8Q_2 \cosh Q_2 - 15\sinh Q_2)\ (\cos Q_1 + \sin Q_1),$$
$$D_3 = 8Q_1 \cosh Q_2(\sin Q_1 - \cos Q_1) + (15\cosh Q_2 - 8Q_2 \sinh Q_2)\ (\cos Q_1 + \sin Q_1),$$
$$D_4 = 8Q_1 \sinh Q_2(\cos Q_1 + \sin Q_1) + (8Q_2 \cosh Q_2 - 15\sinh Q_2)\ (\sin Q_1 - \cos Q_1),$$

$D_5,\ D_6$ are similar expressions as $D_1,\ D_2$ with $Q_1,\ Q_2$ replaced by $Q_3,\ Q_4$,
$D_7,\ D_8$ are similar expressions as $D_3,\ D_4$ with $Q_1,\ Q_2$ replaced by $Q_3,\ Q_4$,

$$Q_1 = \frac{\omega r_1(B_1 W_1 + B_2 W_2)}{V(B_1^2 + B_2^2)},\ \ Q_2 = \frac{\omega r_1(B_1 W_2 - B_2 W_1)}{V(B_1^2 + B_2^2)},$$

$Q_3,\ Q_4$ are similar expressions as $Q_1,\ Q_2$ with B_1, B_2, W_1, W_2 replaced by B_3, B_4, W_3, W_4,

$$T_1 = \frac{T_{01}}{T_{03}},\ \ T_2 = \frac{T_{02}}{T_{03}},\ \ T_3 = \frac{T_{001}}{T_{003}},\ \ T_4 = \frac{T_{002}}{T_{003}},$$

$$T_{01} = Q_1 \cos(X) - Q_2 \sin(X),\quad T_{02} = -Q_1 \sin(X) - Q_2 \cos(X),$$

$$T_{03} = 8\sqrt{\pi}\ \left(Q_1^2 + Q_2^2\right)^{1/2}\left[(Q_1\cos(X) - Q_2\sin(X))^2 + (Q_1\sin(X) + Q_2\cos(X))^2\right],$$

$$X = \frac{1}{2}\tan^{-1}\left(\frac{Q_2}{Q_1}\right),$$

$T_{001},\ T_{002}, T_{003}$ are similar expressions as $T_{01},\ T_{02}, T_{03}$ with Q_1, Q_2 replaced by Q_3, Q_4,

$$M_1 = \frac{\omega r_2(B_1 W_1 + B_2 W_2)}{V(B_1^2 + B_2^2)},\ \ M_2 = \frac{\omega r_2(B_1 W_2 - B_2 W_1)}{V(B_1^2 + B_2^2)},$$

$M_3,\ M_4$ are similar expressions as $M_1,\ M_2$ with B_1, B_2, W_1, W_2 replaced by B_3, B_4, W_3, W_4,

$$p_1 = \frac{p_{01}}{p_{03}},\ \ p_2 = \frac{p_{02}}{p_{03}}, p_3 = \frac{p_{001}}{p_{003}},\ \ p_4 = \frac{p_{002}}{p_{003}},$$

$$p_{01} = M_1 \cos(Y) - M_2 \sin(Y),\quad p_{02} = -M_1 \sin(Y) - M_2 \cos(Y),$$

$$p_{03} = 8\sqrt{\pi}\ \left(M_1^2 + M_2^2\right)^{1/2}\left[(M_1\cos(Y) - M_2\sin(Y))^2 + (M_1\sin(Y) + M_2\cos(Y))^2\right],$$

$$Y = \frac{1}{2}\tan^{-1}\left(\frac{M_2}{M_1}\right),$$

$p_{001},\ p_{002}, p_{003}$ are similar expressions as $p_{01},\ p_{02}, p_{03}$ with M_1, M_2 replaced by M_3, M_4,

Appendix C

Porosity	$\eta_0 = 0.435$,
Tortuosity	$S = 1.655$,
Viscosity of water	$\mu_f = 0.001$ (kg/m),
Permeability	$K = 1.75 \times 10^{-11}$ (m^2),
Density of water	$\rho^f = 998$ (kg/m^3),
Density of solid	$\rho^s = 2616$ (kg/m^3),
Solid frame shear modulus	$\mu_0 = 2.178 \times 10^7$ (kg/ms^2).

References

Biot M A 1956 The theory of propagation of elastic wave in fluid-saturated porous solid. *J. Acoust. Soc. Am.* 28: 168–178

Byoung Nam Kim, Kang II Lee and Suk Wang Yoon 2004 Phase velocity and attenuation of acoustic waves in water saturated sandy sediment: Application of Biot's theory. *J. Korean Phys. Soc.* 44: 1442–1448

Kuihua Wang, Zhiqing, Chin Jian Leo and KangheXie 2008 Dynamic torsional response of an end bearing pile in saturated poroelastic medium. *Comput. Geotech.*450–458

Malla Reddy P and Tajuddin M 2000 Exact analysis of the plane strain vibrations of thick walled hollow poroelastic cylinders. *Int. J. Solids Struct.* 37: 3439–3456

Milton Abramowitz and Irene A Stegun 1964 *Handbook of mathematical functions*. National Bureau of Standard Applied Mathematics Series, 55

Sahay P N 1996 Elastodynamics of deformable porous media. *Proc. Royal Soc. Lond.* A 452: 1517–1529

Sahay P N 2008 On the Biot slow S-wave. *Geophys. J. Int.* 73: N19—N33

Solorza S and Sahay P N 2004 Standing torsional waves in a fully saturated porous circular cylinder. *Geophys. J. Int.* 157: 455–473

Solorza S and Sahay P N 2009 On extensional waves in a poroelastic cylinder within the framework of viscosity-extended Biot theory: The case of traction-free open-pore cylindrical surface. *Geophys. J. Int.* 179: 1679–1702

Sharma M D 2013 Wave propagation in a dissipative poroelastic medium. *J. Appl. Math.* 78: 59–79

Tajuddin M and Sarma 1980 Torsional vibrations of poroelastic cylinders. *J. Appl. Mech. (ASME)* 47: 214–216

Tajuddin M and Ahmed Shah 2006 Circumferential waves of infinite hollow poroelastic cylinders. *J. Appl. Mech. (ASME)* 73(4): 705–708

Tajuddin M and Ahmed Shah 2007 On torsional vibrations of infinite hollow poroelastic cylinders. *J. Mech. Mater. Struct.* 2(1): 189–200

Wachel J C and Szenasi F R 1993 Analysis of torsional vibrations in rotating machinery. In: *Proceedings of 22nd turbo machinery symposium*, Texas A & M University, 127–152