A computerized loop based approach for identification of isomorphism and type of mobility in planar kinematic chains

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MS received 1 September 2012; revised 18 September 2014; accepted 12 December 2014

Abstract. Some new invariants like chain string, link identification string, loop participation of joint frequency string and loop size frequency string of simple jointed kinematic chains are presented. The first step of the proposed method is to identify all possible loops of a kinematic chain and then these new invariants are developed. A modified loop-joint approach is proposed in this work as compared to the previous work in literature where loop-link matrix is formulated to search all the loops present in a kinematic chain. A computer program has been developed for formulating loop-joint matrix of kinematic chain and its all invariants. This method takes into consideration all the loops present, associated joints and links and is able to test isomorphism among kinematic chains and their inversions and also to detect type of mobility of multi degree of freedom kinematic chains. The proposed method is successfully tested for all kinematic chains upto five independent loops having any number of degree of freedom and no counter example is found. The detailed results of 9-link, $M = 2$ (mobility) and 10 link, $M = 3$ kinematic chains are provided with five types of mobility and results are in accordance with the results published in literature. The method is also explained with the help of some complex examples and presented in this paper.

Keywords. Kinematic chain; distinct mechanisms; mobility; independent loops; loop-joint matrix.

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1. Introduction

Structural synthesis (Rajesh & Linda 2006; Eric & Chris 2005) of kinematic chains has been a subject of research from last several decades. It includes the enumeration of kinematic chains and mechanisms from a set of given link and degree of freedom without any duplication. The identification of duplication or isomorphism among kinematic chains has been found tedious and troublesome job by the researchers in the past and various methods have been proposed for it. The characteristic polynomial methods (Uicker & Raicu 1975; Yan & Hall 1981, 1982; Mruthyunjaya 1984a, b, c) proposed by the researchers are efficient but some counter examples (Mruthyunjaya & Balasubramanian 1987) have been reported. Various code based (Ambekar & Agrawal 1986, 1987; Tang and Liu 1993) and distance and loop based (Rao & Varada 1991; Rao & Rao 1992a, b; Huafeng & Zhen 2007; Ding *et al* 2008; Rao & Raju 2000) approaches are also proposed by the researchers in which distance among the links or joints is the basis of characterization of kinematic chains. The hamming number (Rao & Varada 1991; Rao & Rao 1992a, b) approach is computationally complex as in so many cases computation of higher order hamming strings is required to reach the final result. Some artificial intelligence based methods such as neural networks (Kong *et al* 1999), genetic algorithm (Rao 2000a) and fuzzy logic (Rao 2000b) are also proposed but these methods are generally very complex and time consuming as they require a lot of iterations to reach the final result. The loop based methods (Huafeng & Zhen 2007; Ding *et al* 2008; Rao & Raju 2000) are also reported by the researchers in which all the loops of a kinematic chain are the basis of characterization of kinematic chains and mechanisms. The loops of kinematic chain identified by Rao & Raju (2000) are found to be a combination of loops and meshes, not loops alone and it also fails to identify distinct mechanism derived from 10 link 3-DOF kinematic chains (in chain number 61, 62, 63, 64 and 65 of Mruthyunjaya (2003) link 6 and 7 are not equivalent links but this method fails to identify it.). Ding *et al* (2008) proposed the loop based canonical degree sequence for isomorphic identification of kinematic chains. In this method, there are various steps for isomorphism identification namely; identification of all the loops of kinematic chain, formation of canonical degree sequence, relabelling of kinematic chain, forming adjacency matrix and finally matching it. The whole process looks lengthy and requires a lot of computational effort and finally a kinematic chain may have more than one maximum perimeter degree sequence. Mruthyunjaya (2003) in his review paper has summarized the efficiency and limitations of various methods dealing with this problem. Irrespective of so many available methods, the isomorphism identification among kinematic chains remains a hot area for research in last few decades.

The multi degree of freedom kinematic chains have potential applications in the area of robot design. For this, there is always a need to classify the kinematic chains on the basis of type of mobility or degree of freedom. Many researchers (Agrawal & Rao 1987; Tischler *et al* 1995; Mruthyunjaya 2003) in the past have successfully dealt this problem either at enumeration stage or a specific method has been developed for it. In addition to isomorphism identification, the proposed method is also able to classify multi degree of freedom kinematic chains on the basis of type of mobility. Five types of mobility are identified and discussed and the results are in accordance with Agrawal & Rao (1987).

In the present work, a loop-joint matrix is formulated with the help of a computer program which is capable (i) detecting isomorphism among kinematic chains with the help of chain string (CS), (ii) find DM derived from a kinematic chain with the help of link identification string (LIS), (iii) detect type of mobility with the help of loop participation of joints frequency string (LPOJFS) and loop size frequency string (LSFS).

2. Definitions

2.1 Loop

A loop is any closed path through a kinematic chain in which no link or joint is encountered more than once e.g., the loop (5,6,7,8,9,10) of figure 1a. The loop of a kinematic chain also represents the connectivity among the links.

2.2 Independent loop (IL)

The inner loop of kinematic chain inside which there is no other closed loop is known as independent loop. The numbers of independent loops of a kinematic chain are evaluated by the Euler's theorem

$$IL = m - n + 1,$$

where 'm' is the number of joints and 'n' is number of links.

For most of the planar kinematic chains the independent loops can be determined by visual inspection and are arbitrarily selected. Each independent loop must contain at least one link or joint which other independent loops do not contain and each link and/or joint must appear in at least on of the independent loops. For cross linked kinematic chains, the independent loops are selected by the same approach as mentioned by Rao & Raju (2000). The number of links or joints participating in a loop is known as loop size (LS).

2.3 Sub-Loops

The loops formed by the combination of independent loops are termed as sub-loops. For example, there are three independent loops in 10 link, $M = 3$ kinematic chain of figure 1a, namely loop-1, loop-2 and loop-3 (denoted by loop number inside the circle). The sub-loops formed by these loops will be loop (1-2), loop (1-3), loop (2-3) and loop (1-2-3). This theory has been explained by authors in their earlier work but all of the above said loops may or may not exists e.g., the kinematic chain of figure 1a, loop (1-3) does not exist. As it is not always easy to identify sub-loops by visual inspection especially for larger complex and cross linked

Figure 1. Two 10 link $M = 3$ kinematic chain.

kinematic chains therefore a methodology has been developed to identify them in the following sections.

2.4 Loop-Joint matrix

In the present method, a loop-joint matrix is prepared for identification of all sub-loops and some new invariants of kinematic chain. Previous work on loop identification (Rao & Raju 2000) is mainly based on loop-link relations where links associated with loop are identified. The major drawback of previous method is that it does not take into consideration all the joints of kinematic chain (figures 2 & 3). For example, the 11 link, $M = 4$, kinematic chain of figure 2 there are three independent loops namely, (1,2,3,4), (6,7,8,10,9) and (6,9,10,11). These three independent loops do not contain link '5' and joints (4-5) and (5-6). Now, if we develop a loop-link matrix, the link '5' will be included in loop-link matrix but the connectivity of the link '5' (i.e., joints (4-5) and (5-6)) will not be clear as it is not included in any of the independent

Figure 2. An 11 link $M = 4$ kinematic chain.

Figure 3. A 15 link $M = 4$ kinematic chain.

loops. But if a loop-joint matrix developed, it will include all the links and joints automatically. Hence the loop-link matrix representation of kinematic chain looks erroneous. In the present work, three parameter are included to represent a kinematic chain in the form of loop-joint matrix i.e., links, connectivity among the links (joints) and finally loops formed due to these connectivities.

The loop-joint matrix is prepared with the help of a computer program in the following steps and this can be understood with the help of examples.

- Step 1:** Identify all the independent loops of the given kinematic chain. For kinematic chain of figure 1a, the independent loops are (5,6,7,8,9,10), (1,4,9,10) and (1,4,9,3,2,10).
- Step 2:** Identify all the joints associated with the kinematic chain with the help of independent loops and loop-joint matrix is formed in such a way that if any joint is in the loop, it has a value equal to '1' otherwise '0'. Hence loop-joint matrix is a combination 1's and 0's. If any joint is not included in any of the independent loops, it is accommodated separately in the loop-joint matrix.
- Step 3:** As all the sub-loops of a kinematic chain do not exist, therefore the existence of a sub-loop is verified first by the following set of rules

- (a) For loop (1–2) to exist, there should be at least one common joint between loop-1 and loop-2. For example, in loop-joint matrix of figure 1a, loop (1–2) and loop (2–3) exists because joint (9–10) is common in loop-1 and loop-2 and joints (1–10), (1–4) and (1–9) are common in loop-2 and loop-3. But there is no common joint between loop-1 and loop-3; hence loop (1–3) does not exist. For loop (1–2–3) to exist, at least two loops from loop (1–2), loop (1–3) and loop (2–3) should exist. Similarly, for loop (1–2–3–4) to exist, at least two loops from loop (1–2–3), loop (1–2–4) and loop (2–3–4) should exist.

- (b) For any sub-loop (1-2-3-...-n) if (a) is satisfied and if any closed loop is completely surrounded by this combination of sub-loop, the loop will not exist e.g., for chain number '7' of 9 link, $M=2$, kinematic chains of Rao (2004), the sub-loop formed by the combination of loops (1, 3, 2, 6, 8, 7, 5) and (1, 5, 9, 6, 2, 4) will not exist even though the joints (2-6) and (1-5) are common in these loops as the closed loop (5, 7, 8, 6, 9) will be completely surrounded for this combination of loops.

- Step 4:** If a sub-loop exists, then the row of sub-loop of loop-joint matrix is filled by exclusive OR operations of respective independent loops (i.e., for combination of two loops, $0 + 0 = 0$, $0 + 1 = 1$ and $1 + 1 = 0$). When two independent loops are combined to form the sub-loop, the common joints between independent loops will not be there in the sub-loop and the exclusive OR operation will remove those common joints (as $1 + 1 = 0$) and those joints which are not common will remain in the sub-loop. For example; the sub-loop (1-2) of figure 1a is (5,6,7,8,9,4,1,10) and it does not contain joint (9,10) which is a common joint of independent loops 1 and 2. Similarly, if three independent loops are involved, this operation is performed for all the three loops for filling the corresponding row. All the sub-loops are thus determined and filled in loop-joint matrix.
- Step 5:** Loop size of a loop is calculated by summing the corresponding row values of loop and loop participation of joint (LPOJ) is found out by summing column values of corresponding joint. Total loop size (TLS) of kinematic chain is calculated by summing all the loop sizes.
- Step 6:** As each joint will participate in different combination of loops, the summation of loop sizes in which the joint is participating is called loop value of joint (LVOJ) and the sum of LVOJ value for all the joints is called chain loop value (CLV). The chain loop value can also be calculated by summing up the squared values of each loop size.
- Step 7:** The LPOJFS of kinematic chain is prepared by LPOJ values of joints in which At one's place: The number of joints which are not participating in any of the loops i.e., joints having $LPOJ = 0$.
At tenth place: The number of joints which are participating in only one loop i.e., joints having $LPOJ = 1$.
At hundredth place: The number of joints which are participating in two loops i.e., joints having $LPOJ = 2$ and so on.
- Step 8:** The LSFS of kinematic chain is prepared by all independent loops and sub-loops in which
At one's place: The number of loops having loop size = 4.
At tenth place: The number of loops having loop size = 5.
At hundredth place: The number of loops having loop size = 6 and so on.
- Step 9:** Adjacent links (AL) and degree of links (DL) i.e., number of links connected to each link is found for each link. The link value (LV) of each link is defined as the sum of LVOJ values of joints to which it is connected e.g., link '1' of kinematic chain of figure 1a is connected with joints (1-4) and (1-10) having a LVOJ values of 18 and 18, respectively (from figure 4). Therefore, the link value of link '1' is $18 + 18 = 36$.

Loop Joint Matrix													
Loops/Joints	5,6	6,7	7,8	8,9	9,10	5,10	1,4	4,9	1,10	3,9	2,3	2,10	LS
Loop-1	1	1	1	1	1	1	0	0	0	0	0	0	6
Loop-2	0	0	0	0	1	0	1	1	1	0	0	0	4
Loop-3	0	0	0	0	0	0	1	1	1	1	1	1	6
Loop (1-2)	1	1	1	1	0	1	1	1	1	0	0	0	8
Loop (1-3)	0	0	0	0	0	0	0	0	0	0	0	0	0
Loop (2-3)	0	0	0	0	1	0	0	0	0	1	1	1	4
Loop (1-2-3)	1	1	1	1	0	1	0	0	0	1	1	1	8
LPOJ	3	3	3	3	3	3	3	3	3	3	3	3	36
LVOJ	22	22	22	22	14	22	18	18	18	18	18	18	232
LPOJFS													
[0,0,0,0,12,0,0,0]													[0,0,2,0,2,0,2]
CS	{36,232[0,0,2,0,2,0,2] [4-72-188-520-1388,4-72-188-520-1388,2-44-88-204-480,2-44-88-204-480,2-44-116-276-724,2-44-116-276-724,2-36-108-296-816,2-36-108-296-816,2-36-108-296-816,2-36-108-296-816]}												
Link	AL	DL	LV	ALV 1	ALV 2	ALV 3	LIS						
1	4,10	2	36	108	296	816	2-36-108-296-816						
2	3,10	2	36	108	296	816	2-36-108-296-816						
3	9,2	2	36	108	296	816	2-36-108-296-816						
4	1,9	2	36	108	296	816	2-36-108-296-816						
5	6,10	2	44	116	276	724	2-44-116-276-724						
6	5,7	2	44	88	204	480	2-44-88-204-480						
7	6,8	2	44	88	204	480	2-44-88-204-480						
8	7,9	2	44	116	276	724	2-44-116-276-724						
9	8,10,4,3	4	72	188	520	1388	4-72-188-520-1388						
10	9,5,1,2	4	72	188	520	1388	4-72-188-520-1388						
							DM=4						

Figure 4. Loop-joint matrix for kinematic chain of figure 1a.

The link value of two links will be same if the summation of loop sizes in which they are participating is same.

Step 10: In literature, the adjacency of links is invariably used to determine the distinct links of kinematic chain. Rao & Raju (2000) in their loop based methodology used the first adjacency of link to determine distinct link of kinematic chain. It has been observed that first adjacency is not always sufficient to distinguish among the links of a kinematic chain e.g., in chain number 61 of Mruthyunjaya (1984c) the first adjacency of link 7 and 8 is same but second adjacency is different, therefore the links are distinct. The loop based methodology proposed by Rao & Raju (2000) fails in such occasions. Hence there is a need to calculate higher order adjacencies for accurate and precise determination of distinct links of kinematic chain. The number of adjacencies required to distinguish among the links depends on the highest loop size of kinematic chain but highest loop size for a kinematic chain having same number of links and mobility will vary from kinematic chain to kinematic chain. Hence there is a need to generalize the number of adjacency to be checked to distinguish the links of kinematic chain. As the highest possible loop size of a kinematic chain having 'n' number of links is 'n', therefore a formula has been proposed for the degree of adjacency to be checked for determination of distinct links of a kinematic chain. The adjacency to be checked (ADJ)

$$ADJ = \frac{n}{2} - 2 \text{ (when } n \text{ is even)}$$

$$ADJ = \frac{(n-1)}{2} - 2 \text{ (when } n \text{ is odd) .}$$

This formula has a fundamental basis, for $n = 4$ and 5 , the first adjacency is always sufficient to distinguish the links of a kinematic chain. This formula is applicable to kinematic chains having two or more independent loops. Hence for seven link kinematic chain the first adjacency is calculated, for 8 and 9 link kinematic chain, second adjacency is calculated and so on.

Now first adjacency link value (ALV1) of a link is defined as summation of link values of its adjacent links e.g. for kinematic chain of figure 1a and from figure 4, the adjacent links of link '1' are '4' and '10' and link values of these links are 36 and 72, respectively. Hence the first adjacency link value of link '1' is $36 + 72 = 108$. Similarly second adjacency link value (ALV2) is defined as the summation of first adjacency link values of adjacent links of '1'. This process is repeated for calculating the higher order adjacencies of each link.

Step 11: The LIS of each link is formed by concatenating (DL, LV, ALV1, ALV2, ALVn). LIS represents the order of connectivity of a link in kinematic chain.

Step 12: The CS of kinematic chain is formed by concatenating TLS, CLV, LSFS and LIS of each link in decreasing order. The chain string of a kinematic chain possesses all its topological characteristics, namely all the loops and their sizes, degree of links, order of connectivity of links or joints and higher order adjacencies of all links.

Loop Joint Matrix														
Loops/Joints	5,6	6,7	7,8	5,8	1,2	2,3	3,4	1,4	1,9	3,10	9,10	1,5	LS	
Loop-1	1	1	1	1	0	0	0	0	0	0	0	0	4	
Loop-2	0	0	0	0	1	1	1	1	0	0	0	0	4	
Loop-3	0	0	0	0	0	0	1	1	1	1	1	0	5	
Loop (1-2)	0	0	0	0	0	0	0	0	0	0	0	0	0	
Loop (1-3)	0	0	0	0	0	0	0	0	0	0	0	0	0	
Loop (2-3)	0	0	0	0	1	1	0	0	1	1	1	0	5	
Loop (1-2-3)	0	0	0	0	0	0	0	0	0	0	0	0	0	
LPOJ	1	1	1	1	2	2	2	2	2	2	2	0	18	TLS
LVOJ	4	4	4	4	9	9	9	9	10	10	10	0	82	CLV
LPOJFS														LSFS
[0,0,0,0,0,7,4,1]														[0,0,0,0,0,2,2]
CS	[18,82[0,0,0,0,0,2,2] [4-28-64-204-448,3-8-44-96-324,3-28-56-160-344,2-8-16-60-128,2-8-16-60-128,2-8-16-32-120,2-20-48-112-308,2-20-48-104-272,2-18-56-120-364,2-18-56-120-364]]													
Link	AL	DL	LV	ALV 1	ALV 2	ALV 3	LIS							
1	2,4,9,5	4	28	64	204	448	4-28-64-204-448							
2	1,3	2	18	56	120	364	2-18-56-120-364							
3	2,4,10	3	28	56	160	344	3-28-56-160-344							
4	3,1	2	18	56	120	364	2-18-56-120-364							
5	6,8,1	3	8	44	96	324	3-8-44-96-324							
6	5,7	2	8	16	60	128	2-8-16-60-128							
7	6,8	2	8	16	32	120	2-8-16-32-120							
8	7,5	2	8	16	60	128	2-8-16-60-128							
9	1,10	2	20	48	112	308	2-20-48-112-308							
10	3,9	2	20	48	104	272	2-20-48-104-272							
							DM=8							

Figure 5. Loop-joint matrix for kinematic chain of figure 1b showing joint fractionation at joint (1,5).

3. Isomorphism identification

The CS has potential to identify isomorphism among kinematic chains hence it is used as an invariant to detect the duplication or isomorphism among kinematic chains. If CS of two kinematic chains are same, then the chains are isomorphic otherwise non-isomorphic. To test the capability of chain string, two 15 link ($M = 4$) kinematic chains (Marin *et al* 2007) shown in figure 8 are tested for isomorphism.

The CS of kinematic chains of figures 8 is same and given as

(208, 2148 [0,1,2,2,4,5,2,0,2,0,2,2] [3-376-886-2356-6290-16394, 3-376-752-2102-4698-13100, 3-374-878-2442-6188-17276, 3-374-862-2306-5630-15382, 3-304-886-2268-6226-15676, 3-304-702-2236-5286-15324, 3-296-886-2442-6372-17264, 3-296-878-2252-6308-15828, 2-352-750-1614-4408-10328, 2-216-592-1406-3834-9964, 2-216-520-1478-3674-10060, 2-206-678-1580-4678-11474, 2-206-670-1748-4748-12002, 2-200-680-1454-4338-9984, 2-200-672-1630-4354-11006]).

Hence, it can be concluded that these chains are isomorphic, Marin *et al* (2007) have given the same result for these two kinematic chains.

	Loop Joint Matrix																
Loops/Joints	6,9	9,10	10,11	6,11	6,7	7,8	8,10	1,2	2,3	3,4	1,4	4,5	5,6	LS			
Loop-1	1	1	1	1	0	0	0	0	0	0	0	0	0	4			
Loop-2	1	1	0	0	1	1	1	0	0	0	0	0	0	5			
Loop-3	0	0	0	0	0	0	0	1	1	1	1	0	0	4			
Loop (1-2)	0	0	1	1	1	1	1	0	0	0	0	0	0	5			
Loop (1-3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0			
Loop (2-3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0			
Loop {1-2-3}	0	0	0	0	0	0	0	0	0	0	0	0	0	0			
LPOJ	2	2	2	2	2	2	2	1	1	1	1	0	0	18	TLS		
LVOJ	9	9	9	9	10	10	10	4	4	4	4	0	0	82	CLV		
LPOJFS																LSFS	
[0,0,0,0,0,7,4,2]																[0,0,0,0,0,2,2]	
CS	[18,82[0,0,0,0,0,2,2] [4-28-56-196-400,3-8-16-68-136,3-28-56-160-328,2-8-16-32-64,2-8-16-32-100,2-8-16-32-100,2-20-48-104-300,2-20-48-104-264,2-18-56-112-356,2-18-56-112-356,2-0-36-72-264]]																
Link	AL	DL	LV	ALV 1	ALV 2	ALV 3	LIS										
1	2,4	2	8	16	32	100	2-8-16-32-100										
2	1,3	2	8	16	32	64	2-8-16-32-64										
3	2,4	2	8	16	32	100	2-8-16-32-100										
4	3,1,5	3	8	16	68	136	3-8-16-68-136										
5	4,6	2	0	36	72	264	2-0-36-72-264										
6	9,11,7,5	4	28	56	196	400	4-28-56-196-400										
7	6,8	2	20	48	104	300	2-20-48-104-300										
8	7,10	2	20	48	104	264	2-20-48-104-264										
9	6,10	2	18	56	112	356	2-18-56-112-356										
10	9,11,8	3	28	56	160	328	3-28-56-160-328										
11	10,6	2	18	56	112	356	2-18-56-112-356										
							DM=9										

Figure 6. Loop–joint matrix for kinematic chain of figure 2 showing a fractionated into body and serial chain.

4. Identification of distinct mechanism (DM) from a given kinematic chain

The loop-joint matrix has potential to identify DM derived from a kinematic chain. The LIS of links of kinematic chain are used to determine DM derived from a kinematic chain. If LIS of any two or more links are same, then the links are identical otherwise the links are distinct.

		Loop Joint Matrix																																							
Loops/Joints	1,2	2,3	3,4	1,4	5,6	6,7	7,8	5,8	8,9	9,10	5,10	10,11	11,12	12,13	10,13	12,14	14,15	10,15	3,5	LS																					
Loop-1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4																					
Loop-2	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	4																					
Loop-3	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	4																					
Loop-4	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	4																					
Loop-5	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	5																					
Loop (1-2)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																					
Loop (1-3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																					
Loop (1-4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																					
Loop (1-5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																					
Loop (2-3)	0	0	0	0	1	1	1	0	1	1	1	0	0	0	0	0	0	0	0	6																					
Loop (2-4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																					
Loop (2-5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																					
Loop (3-4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																					
Loop (3-5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																					
Loop (4-5)	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	1	0	5																					
Loop (1-2-3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																					
Loop (1-2-4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																					
Loop (1-2-5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																					
Loop (1-2-3-4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																					
Loop (1-2-3-5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																					
Loop (1-2-4-5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																					
Loop (1-3-4-5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																					
Loop (2-3-4-5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																					
LPOJ	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	0	32	TLS																				
LVOJ	4	4	4	4	10	10	10	8	10	10	10	9	9	9	9	10	10	10	0	150	CLV																				
LPOJFS																					LSFS																				
[0,14,4,1]																					[0,0,0,0,0,0,0,0,1,2,4]																				
CS		[32,150][0,0,0,0,0,0,0,0,1,2,4] [5-48-104-400-908-3268,4-28-104-264-916-2224,3-8-44-136-384-1252,3-28-68-228-552-1924,3-28-56-200-444-1552,2-8-16-60-168-504,2-8-16-60-168-504,2-8-16-32-120-336,2-20-76-172-628-1460,2-20-68-152-524-1260,2-20-48-152-380-1296,2-20-48-124-352-968,2-20-48-116-380-932,2-18-76-160-600-1352,2-18-76-160-600-1352]]																																							
Link	AL	DL	LV	ALV 1	ALV 2	ALV 3	ALV 4	LIS																																	
1	2,4	2	8	16	32	120	336	2-8-16-32-120-336																																	
2	1,3	2	8	16	60	168	504	2-8-16-60-168-504																																	
3	2,4,5	3	8	44	136	384	1252	3-8-44-136-384-1252																																	
4	3,1	2	8	16	60	168	504	2-8-16-60-168-504																																	
5	6,8,10,3	4	28	104	264	916	2224	4-28-104-264-916-2224																																	
6	5,7	2	20	48	152	380	1296	2-20-48-152-380-1296																																	
7	6,8	2	20	48	116	380	932	2-20-48-116-380-932																																	
8	7,5,9	3	28	68	228	552	1924	3-28-68-228-552-1924																																	
9	8,10	2	20	76	172	628	1460	2-20-76-172-628-1460																																	
10	9,5,11,13,15	5	48	104	400	908	3268	5-48-104-400-908-3268																																	
11	10,12	2	18	76	160	600	1352	2-18-76-160-600-1352																																	
12	11,13,14	3	28	56	200	444	1552	3-28-56-200-444-1552																																	
13	12,10	2	18	76	160	600	1352	2-18-76-160-600-1352																																	
14	12,15	2	20	48	124	352	968	2-20-48-124-352-968																																	
15	14,10	2	20	68	152	524	1260	2-20-68-152-524-1260																																	
DM=13																																									

Figure 7. Loop-joint matrix for kinematic chain of figure 3 showing combination of body and joint fractionation.

5. Identification of type of mobility for kinematic chains having $M > 1$

In conceptual design of modern robots, the kinematic chains having $M > 1$ are invariably used and these chains are required to be classified on the basis of type of mobility. The LPOJFS and LSFS have a potential to identify the type of mobility of kinematic chains having $M > 1$ and is explained in the following sections. The kinematic chains having $M > 1$ can possess following types of mobility.

5.1 Total freedom mobility (TF)

For a kinematic chain, the absence of any loop having loop size equal to 4 (for $M = 2$), 4 and 5 (for $M = 3$), 4, 5 and 6 (for $M = 4$) and so on, then the kinematic chain has said to have total freedom mobility. If there is zero value at one's place of LSFS of kinematic chain, it indicates an absence of four bar loop. Similarly, zero value at tenth and hundredth place indicates the absence of five and six bar loop, respectively.

5.2 Partial freedom mobility (PF)

For a kinematic chain, the presence of any loop having loop size equal to 4 (for $M = 2$), 4 and/or 5 (for $M = 3$), 4 and/or 5 and/or 6 (for $M = 4$) and so on then the kinematic chain has said to have partial freedom mobility. If there is any value other than zero at one's, tenth and hundredth place of LSFS of kinematic chain, it indicates the presence of four, five and six bar loop, respectively.

5.3 Body or link fractionated freedom mobility (BFF)

If there is zero at one's place and any value other than zero at tenth place of LPOJFS, then the kinematic chain possess body fractionated freedom mobility. Body fractionated kinematic chains having a mobility of ' M ' can be fractionated into two closed loop kinematic chains by cutting a specific link of it. The two fractionated kinematic chains have a combined mobility equal to ' M ' e.g., the kinematic chain of figure 3 can be converted into two closed loop kinematic chains by cutting link 10. The value at tenth place of LPOJFS may be 4 (for kinematic chains having $M = 2$), may be a multiple of 4 and/or 5 and their combination (for kinematic chains having $M = 3$), may be a multiple of 4 and/or 5 and/or 6 and their combination (for kinematic chains having $M = 4$) and so on. If there is 4 at tenth place of LPOJFS, it indicates that one of the fractionated chains

Figure 8. Two 15 link, $M = 4$, isomorphic kinematic chains.

has 4 links. In this manner the number of links of another kinematic chain can be calculated and mobility of both the fractionated kinematic chains can be evaluated.

5.4 Joint or edge fractionated freedom mobility (EFF)

If value at one's place of LPOJFS is 1, it indicates that the kinematic chain possesses joint fractionated freedom mobility. In joint fractionated kinematic chains, a joint divides the kinematic chain into two closed kinematic chains having a combined mobility of "M-1" (where 'M' is the mobility of original kinematic chain). The kinematic chain of figure 1 (b) and its corresponding loop-joint matrix in figure 5 indicates that it can be fractionated into two closed kinematic chains by the joint (1-5) as its LPOJFS [0,0,0,0,0,7,4,1].

5.5 Fractionated into open chains

The kinematic chains having $M \geq 4$ possess more complex type of fractionation i.e., either the fractionation is a combination of body and joint fractionation or a combination of closed and serial kinematic chains. The kinematic chain of figure 2 and its corresponding loop-joint matrix shown in figure 6 indicates that it contains a combination of body and joint fractionation as the LPOJFS of kinematic chain is [0,0,0,0,0,7,4,2]. This kinematic chain can be fractionated into one closed loop and another serial chain by either of the joint (4-5) or (5-6). The LPOJFS of kinematic chain of figure 3 is [0,0,0,0, ..., 0,14,4,1] (from figure 7). It indicates that this kinematic chain is a combination of body and joint fractionation.

6. Conclusions

- (i). Although no mathematical proof has been presented, the authors believe that the proposed method is unique and may be applicable to planar kinematic chains of any size and mobility as it is utilizing all basic parameters of kinematic chain namely links, joints and all combination of loops.
- (ii). The proposed method is capable of testing isomorphism among kinematic chains, finding distinct mechanism derived from kinematic chain and the type of mobility of multi degree of freedom kinematic chains from a single loop-joint matrix.
- (iii). A computer program is developed for formulating the loop-joint matrix of a kinematic for which the inputs are independent loops, degree of mobility and additional joint (if any). The code of the program is written in C#, is available with the corresponding author. The authors have tested all the available $M = 1$ kinematic chains up to 12 links, $M = 2$ kinematic chains up to 11 links, $M = 3$ kinematic chains up to 12 links and $M = 4$ kinematic chains up to 11 links and found no counter example. The results of 9 links, $M = 2$ (chain numbers are taken from Rao (2004)) and 10 links, $M = 3$ kinematic chains (97 chains are taken from Mruthyunjaya (1984c) and 98th chain is from Tischler *et al* (1995)) is given in appendix I and appendix II. The time taken to develop a loop-joint matrix for kinematic chains having 2 to 5 independent loops roughly varies from 4 to 8 seconds in Intel(R) Core(TM) Duo @ 1.66GHz computer having 512 MB RAM.
- (iv). It is concluded that single adjacency is not enough to test the equivalence of links as proposed by authors in previous loop based methods in literature and a generalized formula is proposed for number of adjacency to be checked for finding the equivalence of links.

- ## Appendix I

Ch. No.	Chain String (CS)	DM	LPOJFS	LSFS	Type of Freedom
KC1	(44,284)[0,1,2,2,0][3-78-156-464,3-78-156-464,3-76-180-488,3-76-180-488,2-52-156-312,2-52-154-336,2-52-154-336,2-52-154-336]	4	[0,0,0,11,0,0,0,0]	[0,1,2,2,2,0]	TF
KC2	(36,228)[0,0,4,0,0,2][3-64-136-392,3-64-136-392,3-64-128-384,3-64-128-384,2-56-128-256,2-56-128-264,2-56-128-264,2-36-128-264]	4	[0,0,0,3,8,0,0,0]	[0,0,4,0,2]	PF
KC3	(44,288)[0,1,3,0,1][3-78-184-470,3-78-184-470,3-76-182-400,3-76-182-400,2-56-134-318,2-56-134-318,2-54-366,2-52-154-366,2-52-152-360]	4	[0,0,0,11,0,0,0,0]	[1,0,1,3,2,0]	TF
KC4	(44,288)[0,2,0,4,0,1][3-80-160-448,3-76-208-492,3-76-180-516,3-76-180-516,2-56-136-292,2-56-132-344,2-52-156-340,2-52-152-360]	4	[0,0,0,11,0,0,0,0]	[0,2,0,4,0,1]	PF
KC5	(44,288)[0,1,1,3,0][3-78-182-498,3-78-182-498,3-78-156-464,3-74-208-516,2-56-134-316,2-56-134-316,2-52-156-338,2-52-156-338,2-52-152-360]	6	[0,0,0,11,0,0,0,0]	[0,2,1,1,3,0]	TF
KC6	(44,288)[0,1,3,1,1,1][3-80-160-446,3-78-182-494,3-76-178-516,3-74-206-514,2-56-136-294,2-56-134-318,2-52-156-338,2-52-154-366,2-52-154-366]	4	[0,0,0,11,0,0,0,0]	[0,1,3,1,1,1]	PF
KC7	(35,210)[0,2,0,2,1,1][3-60-184-508,3-62-132-354,3-60-130-372,3-60-130-372,2-38-100-232,2-38-100-232,2-34-124-264,2-34-120-260,2-34-120-260]	5	[0,0,0,2,9,0,0,0]	[0,0,2,2,1,1]	PF
KC8	(36,232)[0,0,2,0,2,1][3-64-168-400,3-64-160-400,3-64-136-424,3-64-128-384,2-56-128-288,2-56-104-272,2-40-104-264,2-36-128-264,2-36-128-264]	8	[0,0,0,3,8,0,0,0]	[0,2,0,2,0,2]	PF
KC9	(44,292)[1,1,0,3,0][3-78-184-500,3-78-184-500,3-76-210-526,3-76-210-526,2-56-134-316,2-56-134-316,2-52-156-342,2-56-132-344,2-52-156-368]	5	[0,0,0,11,0,0,0,0]	[1,1,0,2,3,0]	TF
KC10	(44,292)[1,0,2,2,1,1][3-80-186-480,3-78-184-494,3-76-208-524,3-74-208-546,2-56-136-292,2-56-136-318,2-56-134-320,2-56-132-344,2-52-152-392]	5	[0,0,0,11,0,0,0,0]	[1,0,2,2,1,1]	PF
KC11	(44,292)[0,2,0,2,1,1][3-80-184-508,3-78-180-526,3-78-180-526,3-72-156-544,2-56-136-318,2-56-136-318,2-56-134-316,2-56-134-316,2-52-156-368]	6	[0,0,0,11,0,0,0,0]	[0,2,0,2,1,1]	PF
KC12	(35,215)[0,1,1,2,1,1][3-62-132-384,3-62-132-384,3-60-160-390,3-60-160-390,2-38-98-258,2-38-98-258,2-38-100-232,2-38-100-232,2-34-124-264]	5	[0,0,0,2,9,0,0,0]	[0,1,1,2,1,1]	PF
KC13	(44,296)[1,1,1,2,1][3-80-186-506,3-78-210-530,3-76-182-516,3-74-210-546,2-60-140-306,2-60-138-300,2-60-120-278,2-52-156-368,2-52-156-392]	9	[0,0,0,11,0,0,0,0]	[1,1,1,2,1,1]	PF
KC14	(44,296)[1,0,3,0,6,2][3-80-186-506,3-80-186-500,3-74-206-546,3-74-206-546,2-60-140-306,2-60-140-306,2-60-120-280,2-52-154-392,2-52-154-392]	6	[0,0,0,11,0,0,0,0]	[1,0,3,1,0,2]	PF
KC15	(44,296)[0,2,0,2,1,1][3-80-184-532,3-80-184-532,3-76-180-548,3-72-156-544,2-60-140-304,2-60-140-304,2-60-120-280,2-52-156-360,2-52-156-360]	6	[0,0,0,11,0,0,0,0]	[0,3,0,2,0,2]	PF
KC16	(35,219)[0,2,0,3,1,1][3-62-164-398,3-62-164-398,3-60-130-404,3-60-130-404,2-42-84-208,2-42-104-248,2-42-104-248,2-34-120-260,2-34-120-260]	5	[0,0,0,2,9,0,0,0]	[0,2,0,3,1]	PF
KC17	(36,236)[1,0,2,0,1,2][3-64-168-432,3-64-168-432,3-64-160-400,3-64-160-400,2-56-128-320,2-56-104-272,2-40-104-272,2-40-104-264,2-40-104-264]	5	[0,0,0,3,8,0,0,0]	[1,0,2,0,1,2]	PF
KC18	(44,300)[0,0,2,2,1,1][3-78-212-562,3-78-212-562,3-76-212-556,3-76-212-556,2-60-138-332,2-60-138-332,2-60-120-276,2-56-134-324,2-56-132-344]	5	[0,0,0,11,0,0,0,0]	[0,2,0,2,1,1]	PF
KC19	(35,223)[0,1,1,2,1,1][3-62-184-528,3-62-184-528,3-60-160-422,3-60-160-422,2-42-84-208,2-42-104-248,2-42-104-248,2-38-98-258,2-38-98-258]	5	[0,0,0,2,9,0,0,0]	[1,0,1,1,2,1]	PF
KC20	(35,211)[0,0,2,1,1,1][4-70-140-500,3-62-124-354,3-60-120-382,2-52-122-244,2-38-108-240,2-38-100-232,2-34-132-264,2-34-130-260,2-34-130-260]	8	[0,0,0,2,9,0,0,0]	[1,1,1,1,2,1]	PF
KC21	(36,228)[0,0,4,0,0,2][4-72-144-544,3-64-128-392,3-64-128-392,2-56-120-248,2-56-120-248,2-36-136-272,2-36-136-272,2-36-136-272,2-36-136-272]	4	[0,0,0,3,8,0,0,0]	[0,0,4,0,2]	PF
KC22	(34,196)[0,0,5,0,1,1][4-68-136-456,3-60-128-316,3-56-124-376,2-36-96-232,2-36-96-232,2-36-104-252,2-36-104-252,2-32-124-260,2-32-124-260]	9	[0,0,0,1,10,0,0,0]	[0,0,5,0,1]	PF
KC23	(34,196)[0,0,1,2,3,0][4-68-136-460,3-58-126-346,3-58-126-346,2-36-94-230,2-36-94-230,2-36-104-230,2-36-104-230,2-32-126-262,2-32-126-262]	5	[0,0,0,1,10,0,0,0]	[0,0,1,2,3,0]	TF
KC24	(18,82)[0,0,0,2,2,1][4-28-64-192,3-28-64-176,3-28-56-168,2-36-80,2-36-80,2-8-16-72,2-8-16-72,2-8-16-72,2-8-16-72]	7	[0,0,0,0,0,7,4,0]	[0,0,0,2,2]	BFF
KC25	(34,200)[0,0,2,1,2,1][4-68-136-484,3-60-128-352,3-56-124-376,2-40-80-208,2-40-108-216,2-40-100-208,2-32-128-264,2-32-124-260,2-32-124-260]	8	[0,0,0,1,10,0,0,0]	[0,0,2,1,2,1]	PF
KC26	(35,215)[0,1,1,2,1,1][4-70-170-508,3-62-124-354,3-60-130-390,2-52-122-284,2-38-98-268,2-38-108-272,3-64-108-268,2-38-100-252,2-34-132-264]	9	[0,0,0,2,9,0,0,0]	[0,1,1,2,1,1]	PF
KC27	(36,232)[0,0,2,0,2,1][4-72-144-552,3-64-108-400,3-64-128-392,2-56-120-288,2-56-120-248,2-40-112-280,2-40-104-280,2-36-136-304,2-36-136-304]	8	[0,0,0,3,8,0,0,0]	[0,2,0,2,0,2]	PF
KC28	(18,84)[0,0,0,1,0,3,1][4-28-64-204,3-28-76-180,3-28-68-180,2-36-80,2-36-80,2-8-16-72,2-8-16-72,2-8-16-72,2-8-16-72]	8	[0,0,0,0,0,7,4,0]	[0,0,0,1,3,0]	BFF
KC29	(35,219)[0,2,0,3,1,1][4-70-172-536,3-62-164-398,3-60-120-382,2-52-122-284,2-42-84-216,2-42-112-262,2-42-104-248,2-34-130-292,2-34-130-292]	8	[0,0,0,2,9,0,0,0]	[0,2,0,3,1]	PF
KC30	(34,204)[0,1,1,0,3,1][4-68-166-504,3-58-136-392,3-58-128-386,2-40-98-246,2-40-80-206,2-40-108-246,2-36-94-230,2-36-104-260,2-32-126-292]	9	[0,0,0,1,10,0,0,0]	[0,1,1,0,3,1]	PF
KC31	(18,82)[0,0,0,2,2,3,0][4-28-64-198,3-28-64-176,3-28-64-176,2-36-84,2-36-84,2-8-16-72,2-8-16-72,2-8-16-72,2-8-16-72]	6	[0,0,0,0,0,7,4,0]	[0,0,0,2,2]	BFF
KC32	(34,200)[0,0,2,1,2,1,1][4-68-164-476,3-60-128-358,3-56-164-384,2-36-96-232,2-36-96-232,2-36-104-260,2-36-104-260,2-32-124-260]	7	[0,0,0,1,10,0,0,0]	[0,0,2,1,2,1]	PF
KC33	(18,82)[0,0,0,2,2,1,1][5-36-72-272,2-38-56-176,2-38-44-88,2-38-44-88,2-16-88-2,2-16-88-2,2-16-88-2,2-16-88-2]	7	[0,0,0,0,0,7,4,0]	[0,0,0,2,2]	BFF
KC34	(34,204)[0,1,0,3,0,2][4-68-164-504,3-60-128-392,3-56-164-384,2-40-80-208,2-40-108-244,2-40-100-252,2-36-92-256,2-36-104-256,2-32-128-292]	9	[0,0,0,1,10,0,0,0]	[0,1,0,3,0,2]	PF
KC35	(36,232)[0,0,2,0,2,1,1][4-72-144-552,3-64-108-400,3-64-128-392,2-56-120-288,2-56-120-288,2-40-112-280,2-40-112-280,2-36-136-304,2-36-136-304]	9	[0,0,0,3,8,0,0,0]	[1,0,2,0,1,2]	PF
KC36	(44,304)[0,1,1,1,2,1,1][4-106-274-760,3-60-118-572,3-60-118-572,2-60-166-394,2-60-140-338,2-60-120-306,2-56-162-408,2-56-134-376,2-52-158-432]	9	[0,0,0,0,0,11,0,0,0]	[2,0,1,2,1]	PF
KC37	(33,183)[0,0,3,0,3,0,1][4-66-132-432,4-66-132-432,2-34-100-23				

Figure 9. Nine link. $M = 2$. Kinematic chains.

[illegible]

Figure 10. Ten link. $M = 3$. Kinematic chains.

[illegible]

Figure 11. Ten link. $M = 3$. Kinematic chains.

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