A simplified four-unknown shear and normal deformations theory for bidirectional laminated plates

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Abstract. This paper presents a simplified 4-unknown shear and normal deformations theory for the bending analysis of cross-ply laminated plates. The present theory accounts for an adequate distribution of transverse shear strains through the plate thickness and tangential stress-free on the plate surfaces. The effect of normal strain is also included. The governing, equilibrium equations and boundary conditions are derived by employing the virtual work principle. Numerical results for stresses and displacements are compared well with those obtained using 3-D elasticity solution.

Keywords. Cross-ply laminates; four-unknown sinusoidal plate theory.

1. Introduction

The modifications of plate theories are still the most interesting topics in the literature. Most of these theories are concerned with the so called equivalent single-layer plate theory. It contains, for example, the classical (CPT), first-order (FPT), higher-order (HPT), and other shear deformation theories. These theories have been applied to a variety of mechanical problems related to composite plates. The CPT ignores the transverse shear deformation and gives reliable results only for thin structures. The FPT is simple to implement and applied for both moderately thick and thin laminated plates and gives acceptable results but depends on a shear correction factor which is hard to find as it depends on many parameters. However, there is no need of shear correction factors when using HPT or other shear deformation plate theories but governing equations are more complicated than those of the FSDT. In fact, the HPT and other refined shear deformation theories give more accurate and stable transverse shear stresses.

In this article, a simplified shear and normal deformations plate theory is presented. Just four unknown displacement functions are used. The effects due to transverse shear and normal deformations are both included. The effectiveness of the present theory is demonstrated

through illustrative problem and results are compared with the corresponding 3-D elasticity solution.

2. A simplified four-unknown plate theory

To describe various plate theories, we introduce the following coordinate system. The x , y and z coordinates are taken along the length, the width, and the thickness of the plate, respectively. All applied mechanical loads and geometry are such that the displacements u_i along the coordinate system are functions of x , y , z and time t . With respect to the equivalent single-layer theories, the displacements for static problems are expanded in terms of various powers of the thickness coordinate z and unknown functions of x and y :

$$U_i(x, y, z) = \sum_{j=0}^{\infty} z^j u_i^{(j)}(x, y), \quad i = 1, 2, 3, \quad (1)$$

where $u_i^{(j)}(x, y)$ are undetermined functions. The displacements u_i of a material point in the plate are assumed with the aid of a refined shear deformation plate theory (RPT) as (Zenkour 2004a, b, 2005a, b, 2006, 2007a)

$$\left. \begin{aligned} u_1 &= u(x, y) - z \frac{\partial w}{\partial x} + f(z) \psi_1(x, y), \\ u_2 &= v(x, y) - z \frac{\partial w}{\partial y} + f(z) \psi_2(x, y), \\ u_3 &= w(x, y) + f'(z) \varphi(x, y), \end{aligned} \right\} \quad (2)$$

where u , v and w are the displacements of the middle surface along the axes x , y and z , respectively, ψ_1 and ψ_2 are the rotations about the y and x axes and account for the effect of transverse shear and φ is an additional displacement account for the effect of normal stress. The coefficient of ψ_1 and ψ_2 is given by f and it should be added to function of z . The prime denotes differentiation with respect to z . The displacement field of classical and shear deformation theories are given by taking suitable forms for $f(z)$:

Classical plate theory (CPT) (Love 1944): $f(z) = 0$,

First-order plate theory (FPT) (Mindlin 1951; Reissner 1944, 1945): $f(z) = z$ and $\varphi = 0$,

Higher-order plate theory (HPT) (Reddy 1984): $f(z) = z \left[1 - \frac{4}{3} \left(\frac{z}{h} \right)^2 \right]$ and $\varphi = 0$,

Simple sinusoidal plate theory (SPT) (Zenkour 2004a, b; 2005a, b; 2006): $f(z) = \frac{h}{\pi} \sin\left(\frac{\pi z}{h}\right)$ and $\varphi = 0$.

The displacement field given in Eq. (2) with $\varphi = 0$ was presented by Touratier (1991) and extended to functionally graded plates by Zenkour (2006). In recent years, many investigators have used the same displacement field with the same or different forms of the function $f(z)$. Ferreira *et al* (2005) have used a trigonometric shear deformation theory for modelling symmetric composite plates discretized by a meshless method based on global multi-quadric radial basis functions. Khabbaz *et al* (2009) have used the energy concept along with the first- and third-order shear deformation theories based upon Eq. (2) to predict the large deflection and through the thickness stress of FGM plates. Aydogdu (2009) has proposed a new higher-order shear deformable laminated composite plate theory from the 3-D elasticity bending solutions by

using an inverse method. Jun & Hongxing (2009) have developed the exact dynamic stiffness matrix of a uniform laminated composite beam based on trigonometric shear deformation theory. Xiang *et al* (2009) have used various shear deformation theories for modelling isotropic, sandwich and laminated plates. A generalized higher-order theory describing the mechanical behaviour of multi-layered composite plates with arbitrary lamination scheme was proposed by Fiedler *et al* (2010). Batista (2010) has derived the equations of a moderately thick plate by the method of successive approximations. Ferreira & Roque (2011) have used a higher-order shear deformation theory and a radial basis function collocation technique for predicting the static deformations and free vibration behaviour of thick plates. Neves *et al* (2012a, b) have presented an original hyperbolic sine shear deformation theory for the bending and free vibration analysis of functionally graded plates. Mantari *et al* (2012a, b) have presented an analytical solution to the static analysis of functionally graded plates and sandwich and composite plates, using a recently developed higher-order shear deformation theory. Additional 3-D elasticity theory may be found in the literature (see, for instance, Pagano (1970), Pagano & Hatfield (1972), Kant *et al* (2008) and Kapuria & Kumari (2010)).

3. Basic equations

Consider a composite rectangular plate of length a , width b and thickness h made of an arbitrary lamination material (see figure 1). The plate is subjected to a transverse distributed load $q(x, y)$. The plate is referred to as coordinate system (x, y, z) with the co-ordinates x and y along the in-plane directions and z along the thickness direction. The plate composed of orthotropic layers oriented at angles $\theta_1, \theta_2, \dots, \theta_n$. The material of each layer is assumed to possess one plane of elastic symmetry parallel to the $x - y$ plane. Perfect bonding between the orthotropic layers and temperature-independent mechanical and thermal properties may be assumed.

The refined shear deformation plate theory that includes the effect of both transverse shear and normal deformations was first given by Zenkour (2007a). It is based upon the displacements u_i with $\varphi \neq 0$ as mentioned in Eq. (2). It contains only one additional dependent unknown of other shear deformation theories, but takes into accounts of the normal deformation through the thickness of the plate using a sine-law distribution. No transverse shear correction factors are needed for the RPT, as well as HPT and SPT, because a correct representation of the transverse

Figure 1. Schematic diagram for the laminated plate.

shearing strain is given. We noticed that there is an interesting relation between the unknown functions φ in one side and ψ_1 and ψ_2 the other side. This relation may be summarized by

$$\psi_1(x, y) = \alpha_1 \frac{\partial \varphi}{\partial x}, \quad \psi_2(x, y) = \alpha_2 \frac{\partial \varphi}{\partial y}, \quad (3)$$

where α_1 and α_2 are arbitrary parameters that may be given in terms of the plate dimensions. So, the refined four-unknown shear deformation plate theory is given by

$$\left. \begin{aligned} u_1 &= u(x, y) - z \frac{\partial w}{\partial x} + \alpha_1 f(z) \frac{\partial \varphi}{\partial x}, \\ u_2 &= v(x, y) - z \frac{\partial w}{\partial y} + \alpha_2 f(z) \frac{\partial \varphi}{\partial y}, \\ u_3 &= w(x, y) + f'(z) \varphi(x, y) \end{aligned} \right\}. \quad (4)$$

The above displacement field contains only four unknown functions u , v , w , and φ . In addition, the effects due to transverse shear strain and normal deformation are both included.

The stress-strain relations for a linear elastic plate for the k th layer are given by treating each layer as an individual homogeneous plate and they are written in the form:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix}^k = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{Bmatrix},$$

Figure 2. The in-plane longitudinal stress $\bar{\sigma}_1$ versus the side-to-thickness ratio a/h of a $(0^\circ/90^\circ/90^\circ/0^\circ)$ square plate.

$$\tau_{yz}^{(k)} = c_{44}^{(k)} \gamma_{yz}, \quad \tau_{xz}^{(k)} = c_{55}^{(k)} \gamma_{xz}, \quad \tau_{xy}^{(k)} = c_{66}^{(k)} \gamma_{xy}, \quad (5)$$

where $c_{ij}^{(k)}$ are the transformed elastic coefficients. The strain components ε_i and γ_{ij} are expressed in terms of the displacement quantities as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} + f(z) \begin{Bmatrix} \eta_x \\ \eta_y \\ \eta_{xy} \end{Bmatrix},$$

$$\varepsilon_z = f''(z) \varepsilon_z^0, \quad \gamma_{xz} = f'(z) \gamma_{xz}^0, \quad \gamma_{yz} = f'(z) \gamma_{yz}^0, \quad (6)$$

where

$$\begin{aligned} \varepsilon_x^0 &= \frac{\partial u}{\partial x}, \quad \varepsilon_y^0 = \frac{\partial v}{\partial y}, \quad \gamma_{xy}^0 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \quad \varepsilon_z^0 = \varphi, \quad \gamma_{yz}^0 = (1 + \alpha_2) \frac{\partial \varphi}{\partial y}, \\ \gamma_{xz}^0 &= (1 + \alpha_1) \frac{\partial \varphi}{\partial x}, \quad \kappa_x = -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \quad \kappa_{xy} = -2 \frac{\partial^2 w}{\partial x \partial y}, \\ \eta_x &= \alpha_1 \frac{\partial^2 \varphi}{\partial x^2}, \quad \eta_y = \alpha_2 \frac{\partial^2 \varphi}{\partial y^2}, \quad \eta_{xy} = (\alpha_1 + \alpha_2) \frac{\partial^2 \varphi}{\partial x \partial y}. \end{aligned} \quad (7)$$

Figure 3. The in-plane normal stress $\bar{\sigma}_2$ versus the side-to-thickness ratio a/h of a $(0^\circ/90^\circ/90^\circ/0^\circ)$ square plate.

The stress, moment, and additional moment resultants of a laminated composite plate made up of n layers of orthotropic laminas can be obtained by integrating Eq. (5) over the thickness, and are written as

$$\begin{Bmatrix} N_x \\ N_y \\ M_x \\ M_y \\ S_x \\ S_y \\ S_z \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & B_{11} & B_{12} & B_{11}^a & B_{12}^a & G_{13}^a \\ A_{12} & A_{22} & B_{12} & B_{22} & B_{12}^a & B_{22}^a & G_{23}^a \\ B_{11} & B_{12} & D_{11} & D_{12} & D_{11}^a & D_{12}^a & H_{13}^a \\ B_{12} & B_{22} & D_{12} & D_{22} & D_{12}^a & D_{22}^a & H_{23}^a \\ B_{11}^a & B_{12}^a & D_{11}^a & D_{12}^a & F_{11}^a & F_{12}^a & L_{13}^a \\ B_{12}^a & B_{22}^a & D_{12}^a & D_{22}^a & F_{12}^a & F_{22}^a & L_{13}^a \\ G_{13}^a & G_{23}^a & H_{13}^a & H_{23}^a & L_{13}^a & L_{23}^a & P_{33}^a \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \kappa_x \\ \kappa_y \\ \eta_x \\ \eta_y \\ \varepsilon_z^0 \end{Bmatrix},$$

$$\begin{Bmatrix} N_{xy} \\ M_{xy} \\ S_{xy} \end{Bmatrix} = \begin{bmatrix} A_{66} B_{66} B_{66}^\alpha \\ B_{66} D_{66} D_{66}^\alpha \\ A_{66}^\alpha D_{66}^\alpha F_{66}^\alpha \end{bmatrix} \begin{Bmatrix} \gamma_{xy}^0 \\ \kappa_{xy} \\ \eta_{xy} \end{Bmatrix}, \quad Q_{yz} = P_{44}^\alpha \gamma_{yz}^0, \quad Q_{xz} = P_{55}^\alpha \gamma_{xz}^0, \quad (8)$$

$\bar{\sigma}_6$

Figure 4. The in-plane tangential stress $\bar{\sigma}_6$ versus the side-to-thickness ratio a/h of a $(0^\circ/90^\circ/90^\circ/0^\circ)$ square plate.

where A_{ij} and $B_{ij}; \dots$ etc., are the laminate stiffness coefficients and they are defined in terms of the reduced stiffness coefficients $c_{ij}^{(k)}$ for the layers $k = 1, 2, \dots, n$, as:

$$\{A_{ij}, B_{ij}, D_{ij}\} = \sum_{k=1}^n \int_{z_k}^{z_{k+1}} c_{ij}^{(k)} \{1, z, z^2\} dz, \quad i, j = 1, 2, 6,$$

$$\{B_{ij}^a, D_{ij}^a, F_{ij}^a\} = \sum_{k=1}^n \int_{z_k}^{z_{k+1}} c_{ij}^{(k)} f(z) \{1, z, f(z)\} dz, \quad i, j = 1, 2, 6,$$

$$\{G_{i3}^a, H_{i3}^a, L_{i3}^a\} = \sum_{k=1}^n \int_{z_k}^{z_{k+1}} c_{ij}^{(k)} f''(z) \{1, z, f(z)\} dz, \quad i = 1, 2,$$

$$P_{ii}^a = \sum_{k=1}^n \int_{z_k}^{z_{k+1}} c_{ii}^{(k)} [f'(z)]^2 dz, \quad i = 4, 5,$$

Figure 5. The in-plane displacement $\bar{u}_1$ versus the side-to-thickness ratio a/h of a $(0^\circ/90^\circ/90^\circ/0^\circ)$ square plate.

$$P_{33}^a = \sum_{k=1}^n \int_{z_k}^{z_{k+1}} c_{33}^{(k)} [f''(z)]^2 dz. \quad (9)$$

4. Governing equations

The governing equations can be carried out using the principle of virtual displacements. First, the equilibrium equations associated with the present theory are

$$\left. \begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0, \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0, \quad \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0, \\ S_z + \alpha_1 \frac{\partial^2 S_x}{\partial x^2} + (\alpha_1 + \alpha_2) \frac{\partial^2 S_{xy}}{\partial x \partial y} + \alpha_{21} \frac{\partial^2 S_y}{\partial y^2} - (1 + \alpha_1) \frac{\partial Q_{xy}}{\partial x} - (1 + \alpha_2) \frac{\partial Q_{yz}}{\partial y} &= 0. \end{aligned} \right\}. \quad (10)$$

Figure 6. The in-plane displacement $\bar{u}_2$ versus the side-to-thickness ratio a/h of a $(0^\circ/90^\circ/90^\circ/0^\circ)$ square plate.

Substituting the stress resultants given in Eq. (8) into the above equations, one obtains the following operator equation,

$$[J]\{\delta\} = \{Q\}, \quad (11)$$

where $\{\delta\} = \{u, v, w, \varphi\}^T$ is the displacement field, $\{Q\} = \{0, 0, -q(x, y), 0\}^T$ is the load vector, and $[J]$ is the symmetric matrix of differential operators and its elements are given in Appendix A.

Second, the principle of virtual displacements also gives the requirements of the kinematic and dynamic boundary conditions as follows.

Figure 7. The out-of-plane displacement $\bar{u}_3$ versus the side-to-thickness ratio a/h of a $(0^\circ/90^\circ/90^\circ/0^\circ)$ square plate.

At edge $x = 0, a$	At edge $y = 0, b$
$u = \text{prescribed; otherwise } N_x = 0$	$u = \text{prescribed; otherwise } N_{xy} = 0$
$v = \text{prescribed; otherwise } N_{xy} = 0$	$v = \text{prescribed; otherwise } N_y = 0$
$w = \text{prescribed; otherwise } \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = 0$	$w = \text{prescribed; otherwise } \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = 0$
$\frac{\partial w}{\partial x} = \text{prescribed; otherwise } M_x = 0$	$\frac{\partial w}{\partial x} = \text{prescribed; otherwise } M_{xy} = 0$
$\frac{\partial w}{\partial y} = \text{prescribed; otherwise } M_{xy} = 0$	$\frac{\partial w}{\partial y} = \text{prescribed; otherwise } M_y = 0$
$\varphi = \text{prescribed; otherwise}$	$\varphi = \text{prescribed; otherwise}$
$\alpha_1 \frac{\partial S_x}{\partial x} + \frac{\alpha_1 + \alpha_2}{2} \frac{\partial S_{xy}}{\partial y} - (1 + \alpha_1) Q_{xz} = 0$	$\frac{\alpha_1 + \alpha_2}{2} \frac{\partial S_{xy}}{\partial x} + \alpha_2 \frac{\partial S_y}{\partial y} - (1 + \alpha_2) Q_{yz} = 0$
$\frac{\partial \varphi}{\partial x} = \text{prescribed; otherwise } \alpha_1 S_x = 0$	$\frac{\partial \varphi}{\partial x} = \text{prescribed; otherwise } \frac{\alpha_1 + \alpha_2}{2} S_{xy} = 0$
$\frac{\partial \varphi}{\partial y} = \text{prescribed; otherwise } \frac{\alpha_1 + \alpha_2}{2} S_{xy} = 0$	$\frac{\partial \varphi}{\partial y} = \text{prescribed; otherwise } \alpha_2 S_y = 0$

5. Closed-form solution

The determination of displacements and stresses is of fundamental importance in the design of many structural components. The present boundary value problem associated with the equilibrium of laminated plates involves solving the operator equation, Eq. (11), subjected to the above given set of boundary conditions. An exact closed-form solution to Eq. (11) can be constructed

Figure 8. The transverse shear stress $\bar{\sigma}_4$ through-the-thickness of a $(0^\circ/90^\circ/90^\circ/0^\circ)$ plate ($a/h = 10$).

when the plate is of a rectangular geometry (figure 1) with the following edge conditions, loading and plate constructions.

The following set of simply supported boundary conditions along the edges of the plate is considered:

$$\begin{aligned} v &= w = \varphi = N_1 = M_1 = S_1 = 0 & \text{at } x = 0, a, \\ u &= w = \varphi = N_2 = M_2 = S_2 = 0 & \text{at } y = 0, b. \end{aligned} \quad (12)$$

Under a general co-ordinate transformation, an initially orthotropic material becomes generally anisotropic. However, there are three specific coordinate transformations under which an orthotropic material retains monoclinic symmetry, namely, rotations about the axes x , y , or z . If the plate construction is cross-ply, the orthotropic material with respect to the old coordinate system follows under rotation through an angle θ_k ($= 0^\circ$ or 90°) about the z -axis that the transformation formulae for the stiffnesses $c_{ij}^{(k)}$ are of the form (Bogdanovich & Pastore 1996):

$$\begin{Bmatrix} c_{11} \\ c_{12} \\ c_{22} \\ c_{66} \end{Bmatrix}^k = \begin{bmatrix} c^4 & 2c^2s^2 & s^4 & 4c^2s^2 \\ c^2s^2 & c^4 + s^4 & c^2s^2 & -4c^2s^2 \\ s^4 & 2c^2s^2 & c^4 & 4c^2s^2 \\ c^2s^2 & -2c^2s^2 & c^2s^2 & (c^2 - s^2)^2 \end{bmatrix} \begin{Bmatrix} c_{11} \\ c_{12} \\ c_{22} \\ c_{66} \end{Bmatrix},$$

Figure 9. The transverse shear stress $\bar{\sigma}_5$ through-the-thickness of a $(0^\circ/90^\circ/90^\circ/0^\circ)$ plate ($a/h = 10$).

$$c_{33}^{(k)} = c_{33}, \quad \left(\left\{ \begin{matrix} c_{13} \\ c_{23} \end{matrix} \right\}^k, \left\{ \begin{matrix} c_{44} \\ c_{55} \end{matrix} \right\}^k \right) = \begin{bmatrix} c^2 & s^2 \\ s^2 & c^2 \end{bmatrix} \left(\left\{ \begin{matrix} c_{13} \\ c_{23} \end{matrix} \right\}, \left\{ \begin{matrix} c_{44} \\ c_{55} \end{matrix} \right\} \right), \quad (13)$$

where $c = \cos \theta_k$, $s = \sin \theta_k$ and c_{ij} are the material stiffness of the lamina:

$$\begin{aligned} c_{11} &= \frac{E_1 (1 - v_{23}v_{32})}{\Delta}, \quad c_{12} = \frac{E_1 (v_{21} + v_{23}v_{31})}{\Delta} = \frac{E_2 (v_{12} + v_{32}v_{13})}{\Delta}, \quad c_{22} = \frac{E_2 (1 - v_{13}v_{31})}{\Delta}, \\ c_{13} &= \frac{E_1 (v_{31} + v_{21}v_{32})}{\Delta} = \frac{E_3 (v_{13} + v_{12}v_{23})}{\Delta}, \quad c_{23} = \frac{E_3 (v_{23} + v_{13}v_{21})}{\Delta} = \frac{E_2 (v_{32} + v_{12}v_{31})}{\Delta}, \\ c_{33} &= \frac{E_3 (1 - v_{12}v_{21})}{\Delta}, \quad c_{44} = G_{23}, \quad c_{55} = G_{13}, \quad c_{66} = G_{12}, \\ \Delta &= 1 - v_{12}v_{21} - v_{23}v_{32} - v_{31}v_{13} - 2v_{21}v_{32}v_{13}, \end{aligned} \quad (14)$$

in which E_i are Young's moduli in the material principal directions, v_{ij} are Poisson's ratios, and G_{ij} are shear moduli.

Figure 10. The in-plane longitudinal stress $\bar{\sigma}_1$ versus the side-to-thickness ratio a/h of a $(0^\circ/90^\circ/90^\circ)$ rectangular plate ($b = 3a$).

To solve this problem, Navier presented the applied transverse sinusoidal load in the form:

$$q = q_0 \sin(\lambda x) \sin(\mu y), \quad (15)$$

where $\lambda = \pi/a$, $\mu = \pi/b$, and q_0 is the intensity of the transverse load at the plate center. In addition, the displacements take the forms

$$\begin{Bmatrix} u \\ v \\ w \\ \varphi \end{Bmatrix} = \begin{Bmatrix} U \cos(\lambda x) \sin(\mu y) \\ V \sin(\lambda x) \cos(\mu y) \\ W \cos(\lambda x) \sin(\mu y) \\ \Phi \sin(\lambda x) \sin(\mu y) \end{Bmatrix}, \quad (16)$$

where U , V , W and Φ are arbitrary parameters. The above solution form for $\{u, v, w, \varphi\}$ satisfies the simply supported boundary conditions and the parameters U , V , W , and Φ will be

Figure 11. The in-plane normal stress $\bar{\sigma}_2$ versus the side-to-thickness ratio a/h of a $(0^\circ/90^\circ/90^\circ)$ rectangular plate ($b = 3a$).

determined, subjected to the condition that the solution satisfies the differential equations given in Eq. (11). Substituting Eqs. (15) and (16) into Eq. (11), one obtains

$$[\bar{J}] \{\Delta\} = \{\bar{Q}\}, \quad (17)$$

where $\{\Delta\} = \{U, V, W, \Phi\}^t$ and $\{\bar{Q}\} = \{0, 0, q_0, 0\}^t$. Also, the elements of the symmetric matrix $[\bar{J}]$ are given in Appendix B.

6. Numerical examples and discussion

Numerical results for displacements and stresses are presented for symmetric cross-ply ($0^\circ/90^\circ/90^\circ/0^\circ$) square and ($0^\circ/90^\circ/0^\circ$) rectangular plates subjected to sinusoidal distributed loading. The results are compared with the most famous 3-D solution of Pagano (1970) and Pagano & Hatfield (1972) (see also Zenkour (2007b)). The plate laminae are assumed to be of the

Figure 12. The in-plane tangential stress $\bar{\sigma}_6$ versus the side-to-thickness ratio a/h of a ($0^\circ/90^\circ/90^\circ$) rectangular plate ($b = 3a$).

same thickness and made of the same orthotropic material. The lamina properties are assumed to be:

$$\begin{aligned} E_1 &= 25\text{Msi}, & E_2 &= E_3 = 1\text{Msi}, & G_{12} &= G_{13} = 0.5\text{Msi}, \\ G_{23} &= 0.2\text{Msi}, & \nu_{12} &= \nu_{13} = \nu_{23} = 0.25. \end{aligned} \quad (18)$$

The dimensionless displacements and stresses are given by

$$\begin{aligned} \bar{u}_1 &= -\frac{100E_3h^2}{q_0a^3}u_1\left(0, \frac{b}{z}, \bar{z}\right), & \bar{u}_2 &= -\frac{100E_3h^2}{q_0a^3}u_2\left(\frac{a}{2}, 0, \bar{z}\right), & \bar{u}_3 &= \frac{\pi^4Qh^3}{q_0a^4}u_3\left(\frac{a}{2}, \frac{b}{2}, \bar{z}\right), \\ \bar{w} &= \frac{100E_3h^3}{q_0a^4}u_3\left(\frac{a}{2}, \frac{b}{2}, \bar{z}\right), & \bar{\sigma}_4 &= \frac{h}{q_0a}\tau_{yz}\left(\frac{a}{2}, 0, \bar{z}\right), & \bar{\sigma}_5 &= \frac{h}{q_0a}\tau_{yz}\left(0, \frac{b}{2}, \bar{z}\right), \\ \bar{\sigma}_1 &= \frac{h^2}{q_0a^2}\sigma_x\left(\frac{a}{2}, \frac{b}{2}, \bar{z}\right), & \bar{\sigma}_2 &= \frac{h^2}{q_0a^2}\sigma_y\left(\frac{a}{2}, \frac{b}{2}, \bar{z}\right), & \bar{\sigma}_6 &= -\frac{100h^2}{q_0a^2}\tau_{xy}\left(0, 0, \bar{z}\right), \end{aligned} \quad (19)$$

where

$$\bar{z} = \frac{z}{h}, \quad Q = 4G_{12} + \frac{E_1 + E_2(1 + 2\nu_{23})}{1 - \nu_{12}\nu_{21}}. \quad (20)$$

Figure 13. The in-plane displacement $\bar{u}_1$ versus the side-to-thickness ratio a/h of a $(0^\circ/90^\circ/90^\circ)$ rectangular plate ($b = 3a$).

It is found that the two parameters α_1 and α_2 should be given in terms of the side-to-thickness ratios as

$$\alpha_1 = \frac{50b_2}{h^2 \left(\frac{b}{h} + \frac{1}{4} \right)}, \quad \alpha_2 = \frac{50a^2}{h^2 \left(\frac{115}{100} \sqrt{\left(\frac{a}{h} \right)^3 + \frac{1}{4}} \right)}. \quad (21)$$

The above expressions for α_1 and α_2 give accurate results when compared the present model with the 3-D one. Note that, the quantities $\bar{\sigma}_1$, $\bar{\sigma}_6$, $\bar{u}_1$, and $\bar{u}_2$ are computed at $\bar{z} = 1/2$ while the normal displacements $\bar{w}$ and $\bar{u}_3$ are computed at $\bar{z} = 1/6$. In addition, the in-plane normal stress $\bar{\sigma}_2$ is computed in the $(0^\circ/90^\circ/90^\circ/0^\circ)$ square plate at $\bar{z} = 1/4$ while it is computed in the $(0^\circ/90^\circ/0^\circ)$ rectangular plate at $\bar{z} = 1/6$. Dimensionless displacements of and stresses in symmetric $(0^\circ/90^\circ/90^\circ/0^\circ)$ square plates are presented in figures 2–9. All stresses and displacements are compared well with the corresponding 3-D solutions (Pagano 1970; Pagano & Hatfield 1972). Low sensitivity relative errors occur only at $a/h = 2$. In addition, the transverse shear stresses are plotted through-the-thickness of the $(0^\circ/90^\circ/90^\circ/0^\circ)$ square and rectangular

Figure 14. The in-plane displacement $\bar{u}_2$ versus the side-to-thickness ratio a/h of a $(0^\circ/90^\circ/90^\circ)$ rectangular plate ($b = 3a$).

Figure 15. The out-of-plane displacement $\bar{w}$ versus the side-to-thickness ratio a/h of a $(0^\circ/90^\circ/90^\circ)$ rectangular plate ($b = 3a$).

plates. For the sake of completeness and discussion, the stresses in and displacements of symmetric $(0^\circ/90^\circ/0^\circ)$ rectangular plates are also presented in figures 10–15. The results are more accurate and compared well with the 3-D solution.

7. Conclusions

The explicit governing equations of a simplified 4-unknown sinusoidal shear and normal deformations plate theory are presented for the cross-ply laminated plates. This paper addresses the bending analysis and accounts for the through-the-thickness deformations. Numerical results demonstrate the efficiency of the present approach. The displacement field of the classical thin plate theory (CPT) is obtained easily from Eq. (3) by setting $f(z) = 0$. However, the displacement field of the first-order shear deformation plate theory (FPT) is obtained by setting

$$f(z) = z, \quad w \rightarrow w - \varphi, \quad \psi_1 = (\alpha_1 + 1) \frac{\partial \varphi}{\partial x}, \quad \psi_2 = (\alpha_2 + 1) \frac{\partial \varphi}{\partial y}. \quad (22)$$

We found that the two parameters α_1 and α_2 should not be equal. The author is looking forward to that all interested researchers could generate other forms of the coefficients α_1 and α_2 in order to get results with high accuracy. An extension to the present simplified 4-uknwon theory will be presented for the bending of functionally graded plates in future works.

Appendix A

The elements $J_{ij} = J_{ji}$ of the matrix elements $[J]$ appeared in Eq. (11) are given by

$$\begin{aligned}
 J_{11} &= A_{11}d_{11} + A_{66}d_{22}, & J_{12} &= (A_{12} + A_{66})d_{12}, \\
 J_{13} &= -B_{11}d_{111} - (B_{12} + 2B_{66})d_{122}, \\
 J_{14} &= \alpha_3 G_{13}^a d_1 + \alpha_1 B_{11}^a d_{111} + [\alpha_2 B_{12}^a + (\alpha_1 + \alpha_2) B_{66}^a]d_{122}, \\
 J_{22} &= A_{66}d_{11} + A_{22}d_{22}, & J_{23} &= -(B_{12} + 2B_{66})d_{122} - B_{22}d_{222}, \\
 J_{24} &= \alpha_3 G_{23}^a d_2 + \alpha_2 B_{22}^a d_{222} + [\alpha_1 B_{12}^a + (\alpha_1 + \alpha_2) B_{66}^a]d_{122}, \\
 J_{33} &= -D_{11}d_{1111} - 2(D_{12} + 2D_{66})d_{1122} - D_{22}d_{2222}, \\
 J_{34} &= \alpha_3 (H_{23}^a d_{11} + H_{23}^a d_{22}) + \alpha_1 D_{11}^a d_{1111} + (\alpha_1 + \alpha_2) [D_{12}^a + 2D_{66}^a]d_{1122} \\
 &\quad + \alpha_2 D_{22}^a d_{2222}, \\
 J_{44} &= \alpha_3^2 P_{33}^a + [2\alpha_1 \alpha_3 L_{13}^a - (\alpha_1 + \alpha_3)^2 A_{55}^a]d_{11} + [2\alpha_2 \alpha_3 L_{23}^a - (\alpha_2 + \alpha_3)^2 A_{44}^a]d_{22} \\
 &\quad + \alpha_1^2 F_{11}^a d_{1111} + [2\alpha_1 \alpha_2 F_{12}^a + (\alpha_1 + \alpha_2)^2 F_{66}^a]d_{1122} + \alpha_2^2 F_{22}^a d_{2222},
 \end{aligned}$$

where $d_i, d_{ij}, d_{ijl}, d_{ijlm}$ are the following differential operators:

$$d_i = \frac{\partial}{\partial x_i}, \quad d_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \quad d_{ijl} = \frac{\partial^3}{\partial x_i \partial x_j \partial x_l}, \quad d_{ijlm} = \frac{\partial^4}{\partial x_i \partial x_j \partial x_l \partial x_m}.$$

Appendix B

The elements $\bar{J}_{ij} = \bar{J}_{ji}$ of the matrix elements $[\bar{J}]$ appeared in Eq. (17) are given by

$$\begin{aligned}
 \bar{J}_{11} &= A_{11}\lambda^2 + A_{66}\mu^2, & \bar{J}_{12} &= (A_{12} + A_{66})\lambda\mu, \\
 \bar{J}_{13} &= -\lambda [B_{11}\lambda^2 + (B_{12} + 2B_{66})\mu^2], \\
 \bar{J}_{14} &= \lambda \left\{ \alpha_1 B_{11}^a \lambda^2 + [\alpha_2 B_{12}^a + (\alpha_1 + \alpha_2) B_{66}^a] \mu^2 - \alpha_3 G_{13}^a \right\}, \\
 \bar{J}_{22} &= A_{66}\lambda^2 + A_{22}\mu^2, & \bar{J}_{23} &= -\mu [(B_{12} + 2B_{66})\lambda^2 + B_{22}\mu^2], \\
 \bar{J}_{24} &= \mu \left\{ \alpha_2 B_{22}^a \mu^2 + [\alpha_1 B_{12}^a + (\alpha_1 + \alpha_2) B_{66}^a] \lambda^2 - \alpha_3 G_{23}^a \right\}, \\
 \bar{J}_{33} &= D_{11}\lambda^4 + 2(D_{12} + 2D_{66})\lambda^2\mu^2 + D_{22}\mu^4, \\
 \bar{J}_{34} &= \alpha_3 (H_{13}^a \lambda^2 + H_{23}^a \mu^2) - \alpha_1 D_{11}^a \lambda^4 - (\alpha_1 + \alpha_2) [D_{12}^a + 2D_{66}^a] \lambda^2\mu^2 - \alpha_2 D_{22}^a \mu^4, \\
 \bar{J}_{44} &= \alpha_3^2 P_{33}^a - [2\alpha_1 \alpha_3 L_{13}^a - (\alpha_1 + \alpha_3)^2 A_{55}^a] \lambda^2 - [2\alpha_2 \alpha_3 L_{23}^a - (\alpha_2 + \alpha_3)^2 A_{44}^a] \mu^2 \\
 &\quad + \alpha_1^2 F_{11}^a \lambda^4 + [2\alpha_1 \alpha_2 F_{12}^a + (\alpha_1 + \alpha_2)^2 F_{66}^a] \lambda^2\mu^2 + \alpha_2^2 F_{22}^a \mu^4.
 \end{aligned}$$

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