Some recent work on lattice structures for digital signal processing

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Abstract. This paper is concerned with a review of some recent work on derivation and synthesis of lattice structures for digital signal processing (DSP). In particular, synthesis of canonical structures for both finite impulse response (FIR) and infinite impulse response (IIR) transfer functions is presented in detail. This has been an outstanding problem in DSP, and I demonstrate here how the solution came through very simple ideas and reasoning. Besides a consolidation of results published earlier in various papers, some new results containing refinements and simplifications in the synthesis procedures have also been presented.

Keywords. Digital signal processing (DSP); lattice structures; finite impulse response (FIR); infinite impulse response (IIR).

1. Introduction

The lattice is one of the most important structures in digital signal processing (DSP), because of its robustness and modularity, and has many applications in digital filtering, signal modelling, spectral estimation and adaptive signal processing. Unfortunately, no procedure was known for making the structure canonical in terms of multipliers. Canonicity in multipliers is important not only because it minimizes the complexity of computation, and hence the hardware and the processing time, but also because it reduces the finite word length errors. The lattice structure has fascinated this author for a long time, and for the known non-canonical structures, much effort was devoted towards their derivation, exploring new structures for important applications, and finding procedures for synthesis for cases in which the conventional procedures fail. Through out these investigations, the question of obtaining canonical structures for both finite impulse response (FIR) and infinite impulse response (IIR) transfer functions remained a burning question, till finally, very simple ideas and reasoning led to a breakthrough. Starting with the initial work on non-canonical structures, the paper goes into the details of the evolution of the canonical structures. Problem cases have been considered and handled with ease. Limitations of the procedure have been pointed out, leading to unsolved problems which form the scope of further research on the subject.

In the process of obtaining the solution for a problem case in IIR lattice synthesis, a new general structure for obtaining a canonical realization for an arbitrary IIR transfer function was innovated. This new structure, which involves a simpler IIR transfer function in the forward path and a lower order FIR transfer function in the feedback path has also been discussed here.

While the paper is mostly a consolidation of the recent researches on lattice structures by the author, some new results, aimed at refinement and simplification of the published methods, have also been included.

2. The conventional FIR lattice

The conventional FIR lattice structure, first disclosed by Itakura & Saito (1971), is shown in figure 1a, where the i -th section is of the form shown in figure 1b. It realizes the two transfer functions

$$H_N(z) = X_N(z) / X_0(z) = 1 + a_{1(N)}z^{-1} + a_{2(N)}z^{-2} + \dots + a_{N(N)}z^{-N} \quad (1)$$

and

$$G_N(z) = X'_N(z) / X_0(z) = z^{-N} H_N(z^{-1}) = a_{N(N)} + a_{N-1(N)}z^{-1} + \dots + a_{1(N)}z^{-(N-1)} + z^{-N} \quad (2)$$

and also the intermediate transfer functions

$$H_i(z) = X_i(z) / X_0(z) = 1 + a_{1(i)}z^{-1} + a_{2(i)}z^{-2} + \dots + a_{i(i)}z^{-i}, i = 1 \text{ to } N - 1 \quad (3)$$

and

$$G_i(z) = X'_i(z) / X_0(z) = z^{-i} H_i(z^{-1}) = a_{i(i)} + a_{i-1(i)}z^{-1} + \dots + a_{1(i)}z^{-(i-1)} + z^{-i}, i = 1 \text{ to } N-1 \quad (4)$$

The parameters $k_i, i=1$ to N , are given by

$$k_i = a_{i(i)} \quad (5)$$

Figure 1. (a) The conventional FIR lattice structure, (b) the i -th section of (a).

while the coefficients of the $(i-1)$ -th transfer function are obtained from those of the i -th one by the algorithm

$$a_{n(i-1)} = [a_{n(i)} - k_i a_{i-n(i)}] / (1 - k_i^2), \quad n = 1 \text{ to } i - 1. \quad (6)$$

The realization is canonic in delays, but non-canonic in multipliers because it uses $2N$, instead of N multipliers needed for canonicity. Further, the transfer functions $H_i(z)$ and $G_i(z)$, $i = 1$ to N , are related to each other, one being the mirror image polynomial of the other. On the unit circle, their magnitudes are identical (as is the case with the numerator and denominator polynomials of a unity magnitude all-pass IIR filter), while the phases are equal and opposite to each other. Hence the structure is wasteful in multipliers, and the question naturally arises as to whether we can realize two arbitrary transfer functions, $H_N(z)$ and $G_N(z)$, which are not related to each other, by the same architecture and the same number of multipliers. The answer will be given in the affirmative in section 7 of this paper. First, we present a case in which the algorithm of Eq. (6) fails, and also a solution to the problem that arises.

3. FIR transfer function with $a_{N(N)} = \pm 1$

If in Eq. (1), $a_{N(N)} = \pm 1$, then $k_N = \pm 1$ and one cannot proceed to the $(N-1)$ -th stage because $a_{n(N-1)} \rightarrow \infty$ for all n . The problem has not been adequately addressed to in the standard literature, although it is known (Jackson 1989) that in this case, realization by a lattice structure is possible only if $H_N(z)$ is a linear phase polynomial i.e.,

$$a_{n(N)} = \pm a_{N-n(N)}. \quad (7)$$

This is easily proved by observing from Eq. (6) that with $i = N$,

$$a_{n(N)} - k_N a_{N-n(N)} = (1 - k_N^2) a_{n(N-1)} \quad (8)$$

and for $k_N = \pm 1$, this gives Eq. (7). A solution to the realization problem was given by this author (Dutta Roy 2000) by appealing to the relation

$$H_N(z) = H_{N-1}(z) - k_N z^{-N} H_{N-1}(z^{-1}) \quad (9)$$

as given in standard text books see e.g., Oppenheim & Schaffer (1989). Combining this with Eq. (3) with $i = N$ and $k_N = a_{N(N)} = \pm 1$, we get, after simplification,

$$a_{n(N)} = a_{n(N-1)} \pm a_{N-n(N-1)} \quad (10)$$

and

$$a_{N-n(N)} = a_{N-n(N-1)} \pm a_{n(N-1)}. \quad (11)$$

Four cases are to be considered depending on whether $a_{n(N)}$ is $+1$ or -1 and N is even or odd. We illustrate the solution by considering just one case in which $a_{n(N)}$ is $+1$ and N is odd. In this case, lattice realizability demands that $H_N(z)$ be of the form

$$H_N(z) = 1 + a_{1(N)} z^{-1} + a_{2(N)} z^{-2} + \dots + a_{(N-1)/2(N)} z^{-(N-1)/2} \\ + a_{(N-1)/2(N)} z^{-(N+1)/2} + \dots + a_{2(N)} z^{-(N-2)} + a_{1(N)} z^{-(N-1)} + z^{-N} \quad (12)$$

Now put $n = 1, 2, \dots, (N-1)/2$ in equation (10) to get the following set of equations

$$\begin{aligned} a_{1(N-1)} + a_{N-1(N-1)} &= a_{1(N)} \\ a_{2(N-1)} + a_{N-2(N-1)} &= a_{2(N)} \\ &\dots \dots \dots \\ a_{(N-1)/2(N-1)} + a_{(N+1)/2(N-1)} &= a_{(N-1)/2(N)} \end{aligned} \quad (13)$$

If this process is carried out with Eq. (11), we get a repetition of Eq. (13). There are $(N - 1)/2$ equations from which $a_{i(N-1)}$, $i = 1$ to $N - 1$, are to be determined. Hence $(N-1)/2$ coefficients can be chosen arbitrarily; the simplest and the most economic choice would be:

$$a_{(N-1)(N-1)} = a_{(N-2)(N-1)} = \dots = a_{(N+1)/2(N-1)} = 0. \quad (14)$$

Then

$$H_{N-1}(z) = 1 + a_{1(N)}z^{-1} + a_{2(N)}z^{-2} + \dots + a_{(N-1)/2(N)}z^{-(N-1)/2} = H_{(N-1)/2}(z), \quad (15)$$

which is of half the order. This order reduction means that

$$k_{N-1} = k_{N-2} = \dots = k_{(N+1)/2} = 0. \quad (16)$$

$H_{N-1}(z)$ can now be developed into a lattice unless $a_{(N-1)/2(N)}$ is also $+1$ or -1 . In the latter case, if Eq. (15) is linear phase, then the same procedure is to be repeated; otherwise, use the procedure discussed later in this section. We now take an example to illustrate the procedure.

3.1 Example 1

Let (Dutta Roy 2002)

$$H_5(z) = 1 + a_{1(5)}z^{-1} + a_{2(5)}z^{-2} + a_{3(5)}z^{-3} + a_{4(5)}z^{-4} + z^{-5}. \quad (17)$$

Here $k_5 = 1$. By Eq. (15), therefore

$$H_4(z) = 1 + a_{1(5)}z^{-1} + a_{2(5)}z^{-2} = H_2(z). \quad (18)$$

Thus

$$k_4 = k_3 = 0, k_2 = a_{2(5)} \text{ and } k_1 = a_{1(5)}/(1 + a_{2(5)}), \quad (19)$$

the last relationship being obtained by applying Eq. (6). The synthesis is now complete and the resulting structure is shown in figure 2.

The three other cases, viz. $k_N = +1$, N even; $k_N = -1$, N odd; and $k_N = -1$, N even can be similarly dealt with, as shown in (Dutta Roy 2000) and illustrated in the tutorial paper (Dutta Roy 2002) published later on the same topic (the notations used here are different from those used in the two cited papers).

For a nonlinear phase $H_N(z)$ with $k_N = \pm 1$, one can obtain lattice realization in two ways. One way (Dutta Roy 2000) is to decompose $H_N(z)$ as

$$H_N(z) = H'_N(z) + H_M(z), \quad (20)$$

Figure 2. Lattice structure realizing Eq. (17).

where $H'_N(z)$ is a linear phase polynomial of order N incorporating all the higher order terms beyond the middle one in $H_N(z)$, and $H_M(z)$, of order $M = (N-1)/2$ if N is odd, and $N/2$ if N is even, is not necessarily linear phase. $H'_N(z)$ can be synthesized by the procedure already outlined while $H_M(z)$ can be synthesized similarly if it is linear phase or by the conventional method if it is not so. If $H_M(z)$ is not linear phase but $a_{M(M)}/a_{0(M)} = \pm 1$, then further decomposition will be necessary. Obviously, the resulting overall structure will be a parallel connection of lattices.

A simpler structure is obtained with less number of multipliers by decomposing $H_N(z)$ as in Eq. (20) but realizing the overall transfer function by tapping the linear phase lattice. We illustrate the procedure by taking an example, for which, to bring variety into experience, we assume $k_N = -1$ and N even.

3.2 Example 2

Let (Dutta Roy 2002)

$$H_4(z) = 1 + a_{1(4)}z^{-1} + a_{2(4)}z^{-2} + a_{3(4)}z^{-3} - z^{-4} \quad (21)$$

which we decompose as follows:

$$H_4(z) = [1 - a_{3(4)}z^{-1} + a_{3(4)}z^{-3} - z^{-4}] + [(a_{3(4)} + a_{1(4)})z^{-1} + a_{2(4)}z^{-2}] = H'_4(z) + H_2(z) \quad (22)$$

$H'_4(z)$ is linear phase and can be realized by a lattice with $k_4 = -1$, $k_3 = k_2 = 0$ and $k_1 = -a_{3(4)}$. $H_2(z)$ can be realized by taking weighted taps from the first and second delays, as shown in figure 3. The three outputs are then combined. The multipliers α and β are obtained from the following equation:

$$z^{-1}\alpha + (k_1 + z^{-1})z^{-1}\beta = (a_{3(4)} + a_{1(4)})z^{-1} + a_{2(4)}z^{-2}. \quad (23)$$

This gives

$$\beta = a_{2(4)} \text{ and } \alpha = a_{3(4)} + a_{1(4)} - a_{2(4)}a_{3(4)}. \quad (24)$$

As will be shown later, neither the linear phase nor the nonlinear phase case with $k_N = \pm 1$ requires any special consideration in the new canonic procedure described in section 7. Unfortunately, however, the linear phase case gives the same non-canonical structure as obtained by the procedure described in this section. This implies that the linear phase case is not yet amenable

Figure 3. Realization of Eq. (21).

to a canonical lattice realization; this remains an outstanding problem, and is further elaborated later.

4. Derivation of the conventional FIR lattice

It is the usual practice in textbooks to assume the FIR lattice structure shown in figure 1 and then to derive the synthesis procedure. Even Mitra (2001) who is credited with the formulation of the concept of the digital two-pair, which he used to derive the IIR lattice structure, did not use it to derive the FIR lattice structure. However, Vaidyanathan (1986) used the two-pair approach to derive a variety of FIR lattice structures for the so called ‘lossless bounded real transfer functions’. The present author and his student evolved a simple derivation of the FIR lattice structure using the constraints of Eqs. (3) and (9) and no others (Dutta Roy & Vishwanath 2004).

Referring to the block schematic in figure 1(b), the two pair concept of Mitra (2001) uses the fact that the output can be expressed as a linear combination of the inputs through four t -parameters as follows:

$$X_i(z) = t_{11}X_{i-1}(z) + t_{12}X'_{i-1}(z) \text{ and } X'_i(z) = t_{21}X_{i-1}(z) + t_{22}X'_{i-1}(z). \quad (25)$$

In terms of the transfer functions, with $X_0(z)$ as the input, these translate to the following

$$H_i(z) = t_{11}H_{i-1}(z) + t_{12}G_{i-1}(z) \text{ and } G_i(z) = t_{21}H_{i-1}(z) + t_{22}G_{i-1}(z). \quad (26)$$

Observe that t_{11} has to be unity in order to have the constant term in $H_i(z)$ as unity. Also, in order that $H_i(z)$ is a polynomial of order i in z^{-1} , t_{12} must be of the form $k_i z^{-1}$. To obtain t_{21} and t_{22} , note, from Eq. (4) that

$$G_i(z) = z^{-i}H_i(z^{-1}). \quad (27)$$

In the first equation of the set (26), put $t_{11} = 1$, $t_{12} = k_i z^{-1}$, and then replace z by z^{-1} . The result is

$$H_i(z^{-1}) = H_{i-1}(z^{-1}) + k_i z G_{i-1}(z^{-1}). \quad (28)$$

Substituting this in Eq. (27) and simplifying, we get

$$G_i(z) = z^{-1}G_{i-1}(z) + k_i H_{i-1}(z). \quad (29)$$

Comparing this equation with the second equation of (26), we get the values of t_{21} and t_{22} as k_i and z^{-1} , respectively. Finally, therefore,

$$t_{11} = 1, t_{12} = k_i z^{-1}, t_{21} = k_i \text{ and } t_{22} = z^{-1}. \quad (30)$$

The structure representing Eq. (30) is precisely that of figure 1(b).

5. Derivation of another lattice

It was stated in the concluding comments of Dutta Roy & Vishwanath (2004) that other lattice structures should be possible to derive by assuming some other relationship between $H_i(z)$ and $G_i(z)$, e.g.,

$$G_i(z) = \pm z^{-i}H_i(-z^{-1}). \quad (31)$$

The relationship given by Eq. (31) is of interest because if $H_i(z)$ is low-pass, then $G_i(z)$ is high-pass and vice versa. Such a pair of complementary transfer functions is useful in crossover networks as well as in multi-rate signal processing. Following the same two-pair approach, it was found that the required t parameters are given by (Dutta Roy & Vishwanath 2005)

$$t_{11} = 1, t_{12} = k_i z^{-1}, t_{21} = (-1)^i k_i \text{ and } t_{22} = z^{-1}. \quad (32)$$

These differ from those given by Eq. (30) only in t_{21} . When i is even, the two lattice sections are identical, while if i is odd, then the two t_{21} 's are equal and opposite. The recurrence relation for the new structure was derived to be the following:

$$a_{n(i-1)} = [a_{n(i)} - k_i (-1)^{i-n} a_{i-n(i)}] / 1 - (-1)^i k_i^2. \quad (33)$$

6. Towards canonical FIR lattice synthesis

We now ask the question: Is it possible to realize a general FIR transfer function by using one, instead of two equal multipliers in each lattice section? This is inspired by the fact that such single multiplier sections exist in the special case of all-pass IIR lattices, as discussed later in this paper. The answer was given in the affirmative by Makhoul as early as 1978. Surprisingly, however, this work, although published in a prominent IEEE journal, remained largely ignored by later workers (Dognata & Vaidyanathan 1987; Krishna 1989) and textbooks. This fact was brought out by this author in a tutorial paper (Dutta Roy 2006) which also contained a simple derivation of the Makhoul structures. We give the essence of this derivation here. Consider the i -th section of the two multiplier FIR lattice section of figure 1b. For convenience of notation, let $X_{i-1} = U_1$, $X_{i-1}' = U_2$, $X_i = Y_1$, $X_i' = Y_2$ and $k_i = k$. Then, by inspection, we can write

$$Y_1 = U_1 + k z^{-1} U_2 \text{ and } Y_2 = k U_1 + z^{-1} U_2. \quad (34)$$

The input variables occurring in Eq. (34) are U_1 and $z^{-1} U_2$. Clearly, in order to obtain single multiplier realizations, we require one or more additional variables involving a linear combination of U_1 and $z^{-1} U_2$ without any multiplication. The possibilities for a third variable are $\pm U_1 \pm z^{-1} U_2$, where any combination of signs can be taken, thus making a total of four possibilities. We shall consider only one of them, viz. the variable set U_1 , $z^{-1} U_2$ and $U_1 + z^{-1} U_2$. The other possibilities can be similarly explored, as in (Makhoul 1978; Dutta Roy 2006).

In terms of the selected set of variables, Eq. (34) can be rewritten as

$$Y_1 = (1 - k) U_1 + k(U_1 + z^{-1} U_2) \text{ and } Y_2 = (1 - k) z^{-1} U_2 + k(U_1 + z^{-1} U_2). \quad (35)$$

Divide both sides by $(1 - k)$; then Eq. (35) become

$$\begin{aligned} Y_1 / (1 - k) &= U_1 + [k / (1 - k)] (U_1 + z^{-1} U_2) \text{ and} \\ Y_2 / (1 - k) &= z^{-1} U_2 + [k / (1 - k)] (U_1 + z^{-1} U_2). \end{aligned} \quad (36)$$

Eq. (36) involve only one multiplier as shown in figure 4. However, each output is now scaled by the factor $1/(1 - k)$. They can of course be recovered by multiplying each output by $(1 - k)$. If each stage of figure 1(a) is thus converted into a single multiplier one, then all output multipliers can be clubbed into a single multiplier of value $\prod_{i=1}^N (1 - k_i)$ at the input of the overall lattice, thus reducing the total number of multipliers from $2N$ to $N + 1$. Instead of lumping the multipliers at

Figure 4. A single multiplier realization of the FIR lattice section of figure 1b (with changed notation).

the input, one can also distribute them appropriately in order to prevent overflow and/or minimize quantization errors. Note that the structure of figure 4 is the same as the one multiplier lattice form LF2/1(a) of Makhoul (1978).

7. Truly canonical FIR lattice: A new development

As mentioned earlier, the conventional FIR lattice is wasteful in multipliers and realizes two transfer functions which have mirror image relationship with each other. Search was therefore on for realizing two arbitrary FIR transfer functions

$$H_N(z) = a_{0(N)} + a_{1(N)}z^{-1} + \dots + a_{N(N)}z^{-N} \quad (37)$$

and

$$G_N(z) = b_{0(N)} + b_{1(N)}z^{-1} + \dots + b_{N(N)}z^{-N}, \quad (38)$$

by the same structure but with only $2N+2$ multipliers to make it truly canonic. This search ended up in the desired result from a very simple idea, viz. using two different parameters $k_{a(i)}$ and $k_{b(i)}$, instead of two identical ones in each section of the lattice (Dutta Roy 2007a). Consider the general lattice of figure 1a again, but now $H_0(z)$ and $G_0(z)$ are constants, not necessarily unity, to take care of the fact that the constant terms in Eqs. (37) and (38) are in general not unity. Instead of figure 1b, the i -th section of the lattice is now of the form shown in figure 5. Consider the N -th section of this modified lattice; our primary task is to choose $k_{a(N)}$ and $k_{b(N)}$ such that the transfer functions

$$X_{N-1}/X_0 = H'(z) \text{ and } X'_{N-1}/X_0 = G'(z) \quad (39)$$

are of order $N-1$. If this can be done, then we can repeat the procedure to obtain the sections numbered $N-1, N-2, \dots, 1, 0$ to get the canonical structure. By analysis of figure 5 with $i = N$, it is easily shown that

$$H_N(z) = H'(z) + z^{-1}k_{a(N)}G'(z) \text{ and } G_N(z) = z^{-1}G'(z) + k_{b(N)}H'(z). \quad (40)$$

Figure 5. Modified i -th lattice section of figure 1a with two different lattice parameters.

By inverting Eqs. (40), we obtain

$$H'(z) = K_N [H_N(z) - k_{a(N)} G_N(z)] \text{ and } G'(z) = K_N z [G_N(z) - k_{b(N)} H_N(z)], \quad (41)$$

where

$$K_N = 1/(1 - k_{a(N)} k_{b(N)}). \quad (42)$$

Substituting for $H_N(z)$ and $G_N(z)$ from Eqs. (37) and (38), it is easily shown that by choosing

$$k_{a(N)} = a_{N(N)}/b_{N(N)} \text{ and } k_{b(N)} = b_{0(N)}/a_{0(N)} \quad (43)$$

both $H'(z)$ and $G'(z)$ become of order $N - 1$ and can be redesignated as $H_{N-1}(z)$ and $G_{N-1}(z)$, respectively. Also, the coefficients of the latter two transfer functions are given by

$$a_{n(N-1)} = K_N [a_{n(N)} - k_{a(N)} b_{n(N)}] \text{ and } b_{n(N-1)} = K_N [b_{n+1(N)} - k_{b(N)} a_{n+1(N)}], n = 0 \text{ to } N - 1. \quad (44)$$

We have now achieved our goal and the procedure can be repeated till we reach

$$H_1(z) = a_{0(1)} + a_{1(1)} z^{-1} \text{ and } G_1(z) = b_{0(1)} + b_{1(1)} z^{-1} \quad (45)$$

so that by applying equation (44), we get

$$k_{a(1)} = a_{1(1)}/b_{1(1)}, k_{b(1)} = b_{0(1)}/a_{0(1)}, H_0(z) = a_{0(0)} = a_{0(1)} \text{ and } G_0(z) = b_{0(0)} = b_{1(1)}. \quad (46)$$

A refinement is possible here by noting that

$$a_{0(i-1)} = (a_{0(i)} - k_{a(i)} b_{0(i)})/(1 - k_{a(i)} k_{b(i)}) = a_{0(i)} (1 - k_{a(i)} b_{0(i)}/a_{0(i)})/(1 - k_{a(i)} k_{b(i)}) = a_{0(i)}. \quad (47)$$

Thus

$$H_0(z) = a_{0(0)} = a_{0(1)} = a_{0(2)} = \dots = a_{0(N)}. \quad (48)$$

Similarly, it can be shown that

$$G_0(z) = b_{0(0)} = b_{1(1)} = b_{2(2)} = \dots = b_{N(N)}. \quad (49)$$

Thus $H_0(z)$ and $G_0(z)$ are determined directly from the given transfer functions. In summary, the design equations for the canonical FIR lattice are as follows:

$$\begin{aligned} k_{a(i)} &= a_{i(i)}/b_{i(i)} \text{ and } k_{b(i)} = b_{0(i)}/a_{0(i)}, i = N \text{ to } 1; \\ a_{n(i-1)} &= [a_{n(i)} - k_{a(i)}b_{n(i)}]/(1 - k_{a(i)}k_{b(i)}) \text{ and} \\ b_{n(i-1)} &= [b_{n+1(i)} - k_{b(i)}a_{n+1(i)}]/(1 - k_{a(i)}k_{b(i)}), n = 0 \text{ to } i - 1; \\ H_0(z) &= a_{0(N)} \text{ and } G_0(z) = b_{N(N)}. \end{aligned} \quad (50)$$

Note that in view of Eqs. (48) and (49), $a_{0(i-1)}$ and $b_{(i-1)(i-1)}$ need not be calculated from the formulae; they are known. We shall now illustrate the procedure by an example.

7.1 Example 3

Let it be required to realize (Dutta Roy 2007a)

$$H_3(z) = 1 + z^{-1} + z^{-2} + z^{-3} \text{ and } G_3(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} \quad (51)$$

Using Eq. (50), we get

$$k_{a(3)} = 1/4, k_{b(3)} = 1, k_{a(2)} = 1/12, k_{b(2)} = 4/3, k_{a(1)} = 1/8, k_{b(1)} = 2, H_0(z) = 1 \text{ and } G_0(z) = 4 \quad (52)$$

The realization diagram is shown in figure 6.

8. Modifications needed in special cases

The algorithm given in Eq. (50) fails in a few cases, but it will be shown that simple modifications can overcome the problems which arise.

8.1 Case I

Suppose $b_{N(N)} = 0$ and $a_{N(N)} \neq 0$; then $k_{a(N)} \rightarrow \infty$ and the procedure breaks down. The remedy is simply to interchange $H_N(z)$ and $G_N(z)$ (Dutta Roy 2008), i.e., let

$$H_N'(z) = G_N(z) = b_{0(N)} + b_{1(N)}z^{-1} + \dots + b_{N-1(N)}z^{-(N-1)} \quad (53)$$

Figure 6. Realization of the equation set (51).

and

$$G_N'(z) = H_N(z) = a_{0(N)} + a_{1(N)}z^{-1} + \dots + a_{N(N)}z^{-N}. \quad (54)$$

Now realize the primed transfer functions as a pair by the usual procedure. Here

$$k_{a(N)} = 0 \text{ and } k_{b(N)} = a_{0(N)}/b_{0(N)} \quad (55)$$

and there is no breakdown. Can this occur at an intermediate stage $i < N$? The answer is no, because of Eq. (49).

Alternatively, one can write

$$H_N(z) = a_{0(N)} + z^{-1}(a_{1(N)} + a_{2(N)}z^{-1} + \dots + a_{N(N)}z^{-(N-1)}) = a_{0(N)} + z^{-1}H_{N-1}(z) \quad (56)$$

and

$$G_N(z) = b_{0(N)} + b_{1(N)}z^{-1} + \dots + b_{N-1(N)}z^{-(N-1)} = G_{N-1}(z). \quad (57)$$

Realize $H_{N-1}(z)$ and $G_{N-1}(z)$ as a pair and then combine them, as in figure 7 to obtain the required transfer functions.

8.2 Case II

Suppose $a_{0(N)} = 0$. As in case I, this cannot occur at an intermediate stage because of Eq. (48). In this case, $k_{b(N)} \rightarrow \infty$. Here also, the solution is to interchange $H_N(z)$ and $G_N(z)$ (Dutta Roy 2008), that is, realize the pair $H_N'(z) = G_N(z)$ and $G_N'(z) = H_N(z)$; for this modification, $k_{b(N)}$ would be zero. Also, alternatively, one can write $H_N(z) = z^{-1}H_{N-1}(z)$ and $G_N(z) = b_{0(N)} + z^{-1}G_{N-1}(z)$, realize the pair $H_{N-1}(z)$ and $G_{N-1}(z)$, and combine them in the manner shown in figure 8.

8.3 Case III

Let $a_{0(N)} = b_{0(N)} = 0$. Then both $k_{a(N)}$ and $k_{b(N)}$ tend to infinity. In this case, write $H_N(z) = z^{-1}H_{N-1}(z)$ and $G_N(z) = G_{N-1}(z)$, realize the pair $H_{N-1}(z)$ and $G_{N-1}(z)$, and obtain $H_N(z)$ by adding a delay in the upper line output.

8.4 Case IV

If $k_{a(N)}k_{b(N)} = 1$, i.e. $a_{N(N)}b_{0(N)} = a_{0(N)}b_{N(N)}$, then $K_N \rightarrow \infty$ and one cannot proceed to the next stage. Can this occur at an intermediate stage? The answer is yes, it can. The solution in this case

Figure 7. Realization of Eqs. (56) and (57).

Figure 8. Realization for case II.

is simply to realize the pair $H_N'(z) = 1 + H_N(z)$ and $G_N(z)$ as a pair and then obtain $H_N(z)$ as $H_N'(z) - 1$. This modification can also be done to $G_N(z)$ instead of $H_N(z)$. Also, addition can be replaced by subtraction and the constant 1 can be replaced by any suitable power of z so that no extra multiplication is needed.

9. Canonical FIR lattice for a single transfer function

When it is required to realize a single transfer function of the form of Eq. (37), split it into two parts. When N is odd, one decomposition can be

$$\begin{aligned} H_N(z) &= [a_{0(N)} + a_{1(N)}z^{-1} + \dots + a_{(N-1)/2(N)}z^{-(N-1)/2}] + z^{-(N+1)/2}[a_{(N+1)/2(N)} \\ &\quad + a_{(N+3)/2(N)}z^{-1} + \dots + a_{N(N)}z^{-(N-1)/2}] \\ &= H_{(N-1)/2}(z) + z^{-(N+1)/2}G_{(N-1)/2}(z). \end{aligned} \quad (58)$$

Now realize the two $(N-1)/2$ -th order transfer functions and combine them as per Eq. (58) to get $H_N(z)$. We illustrate this by an example.

9.1 Example 4

Let (Dutta Roy 2008)

$$H_3(z) = 1 + z^{-1} + 2z^{-2} + 3z^{-3}. \quad (59)$$

We can write this as

$$H_3(z) = (1 + z^{-1}) + z^{-2}(2 + 3z^{-1}) = H_1(z) + z^{-2}G_1(z). \quad (60a)$$

For the pair $H_1(z)$ and $G_1(z)$, Eq. (50) give

$$k_{a(1)} = 1/3, k_{b(1)} = 2, H_0(z) = 1 \text{ and } G_0(z) = 3 \quad (60b)$$

and the realization takes the form of figure 9.

Clearly, the decomposition of $H_N(z)$ is not unique. For the example under consideration, we could also have taken (Dutta Roy 2008)

$$H_3(z) = (1 + 2z^{-2}) + z^{-1}(2 + 3z^{-2}) = H_2(z) + z^{-1}G_2(z). \quad (61)$$

Figure 9. Realization of Eq. (59).

The design parameters for the pair $H_2(z)$ and $G_2(z)$ can be easily found to be

$$k_{a(2)} = 2/3, k_{b(2)} = 1, k_{a(1)} = k_{b(1)} = 0, H_0(z) = 1 \text{ and } G_0(z) = 3 \quad (62)$$

and the realization is as shown in figure 10.

Now consider the case of even N . One solution is to split $H_N(z)$ as follows:

$$H_N(z) = H_{(N-2)/2}(z) + z^{-N/2}G_{N/2}(z), \quad (63)$$

where

$$H_{(N-2)/2}(z) = a_{0(N)} + a_{1(N)}z^{-1} + \dots + a_{(N-2)/2(N)}z^{-(N-2)/2} \quad (64)$$

and

$$G_{N/2}(z) = a_{N/2(N)} + a_{(N+2)/2(N)}z^{-1} + \dots + a_{N(N)}z^{-N/2}. \quad (65)$$

Let us illustrate the procedure with the help of an example.

9.2 Example 5

Let (Dutta Roy 2008)

$$H_4(z) = 1 + z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} \quad (66)$$

which we can decompose as

$$H_4(z) = (1 + z^{-1}) + z^{-2}(2 + 3z^{-1} + 4z^{-2}) = H_1(z) + z^{-2}G_2(z). \quad (67)$$

Using Eq. (50), the parameters for the pair $H_1(z)$ and $G_2(z)$ can be found as

$$k_{a(2)} = 0, k_{b(2)} = 2, k_{a(1)} = 1/4, k_{b(1)} = 1, H_0(z) = 1, G_0(z) = 4. \quad (68)$$

The resulting realization is shown in figure 11.

Figure 10. Alternative realization of Eq. (59).

Figure 11. A realization of Eq. (66).

An alternate decomposition of Eq. (66) would be

$$H_4(z) = (1 + z^{-1} + z^{-2}) + z^{-2}(1 + 3z^{-1} + 4z^{-2}). \quad (69)$$

Proceeding in the usual manner, it can be shown that the resulting realization is inferior to the previous one.

10. Limitations of the canonical FIR lattice synthesis procedure

The canonical lattice synthesis procedure given here has the limitation that if $H_N(z)$ and $G_N(z)$ are specially related, then, in general, it does not give canonical realization. For example, if

$$G_N(z) = z^{-N}H_N(z^{-1}) \quad (70)$$

as in the conventional lattice realization, all intermediate pairs obey Eq. (70), with N replaced by i , $i=N-1$ to 1, so that

$$b_{n(i)} = a_{i-n(i)}. \quad (71)$$

Thus by applying Eq. (50), we get

$$k_{a(i)} = k_{b(i)} = a_{i(i)}/a_{0(i)} = k_i. \quad (72)$$

This gives $2N+2$ number of multipliers instead of $N+1$ required for canonicity.

Similarly, if

$$G_N(z) = z^{-N}H_N(-z^{-1}) \quad (73)$$

then, as already mentioned, the two transfer functions are complementary. By application of Eq. (50), it can be shown that all intermediate transfer function pairs also obey Eq. (73) with N replaced by i , $i = N-1$ to 1, with

$$b_{n(i)} = (-1)^{i-n} a_{i-n(i)} \quad (74)$$

so that

$$k_{a(i)} = (-1)^i k_{b(i)}. \quad (75)$$

Again, the realization is not canonic.

It should be obvious that a single linear phase transfer function cannot be realized canonically by the procedure given here because the two component transfer functions are specially related to each other by Eq. (70).

11. A new canonical realization of an arbitrary IIR transfer function

As a prelude to IIR canonical lattice development, we first present a new canonical realization (Dutta Roy & Tholeti 2008) of an arbitrary IIR transfer function

$$H_N(z) = [a_{0(N)} + a_{1(N)}z^{-1} + \dots + a_{N(N)}z^{-N}] / [b_{0(N)} + b_{1(N)}z^{-1} + \dots + b_{N(N)}z^{-N}]. \quad (76)$$

Many canonical realizations exist in the literature, for example, direct, parallel, cascade, ladder, etc. In the development of a true canonical lattice structure for Eqs. (37) and (38), a problem case was encountered, the solution of which needed one or more extra delays (Dutta Roy 2007b), thus making the structure non-canonical in delays. To remove this difficulty, a new canonical IIR structure was innovated as a by-product; it uses feedback and is of the form shown in figure 12. The forward path has the transfer function $F(z)$, while the feedback path has the transfer function $G(z)$. The overall transfer function is

$$H(z) = F(z) / [1 + F(z) G(z)]. \quad (77)$$

Let

$$H(z) = P(z) / Q(z) = [a_0 + a_1z^{-1} + \dots + a_Mz^{-M}] / [1 + b_1z^{-1} + \dots + b_Nz^{-N}] \quad (78)$$

where $M < N$ i.e., unlike Eq. (76), $H(z)$ is a proper rational function (If $M = N$, as in Eq. (76), one can always take out a constant to make the numerator of degree $N-1$). Also notice that we have taken the constant term in the denominator as unity, this causes no loss of generality because any IIR transfer function can always be written in this form by appropriate scaling.

Assume

$$F(z) = P(z) / (1 + c_1z^{-1} + \dots + c_Rz^{-R}) = P(z) / D(z), \quad (79)$$

where $R \geq M$. Then Eq. (77) becomes

$$H(z) = P(z) / [D(z) + P(z) G(z)] = P(z) / Q(z). \quad (80)$$

Since both $D(z)$ and $P(z)$ are polynomials, of degrees R and M , respectively, it is clear that $G(z)$ is required to be a polynomial of degree $N-M$. Thus $G(z)$ is an FIR transfer function. Further, since $Q(z)$ has N known coefficients and $D(z)$ has R unknown coefficients, one is required to have $N-R$ unknown coefficients in $G(z)$. This means that the first $R - M + 1$ coefficients in $G(z)$ can

Figure 12. The new configuration for the realization of an arbitrary IIR transfer function of the form of Eq. (78).

be put equal to zero. Explicitly, then, equating the denominator polynomials in equation (80) gives the following:

$$1 + b_1 z^{-1} + \dots + b_N z^{-N} = 1 + c_1 z^{-1} + \dots + c_R z^{-R} + (a_0 + a_1 z^{-1} + \dots + a_M z^{-M}) \cdot (g_{R-M+1} z^{-(R-M+1)} + \dots + g_{N-M} z^{-(N-M)}). \quad (81)$$

Equating coefficients on both sides gives a set of linear equations which can be solved to get the c_i 's, $i = 1 \rightarrow R$ and g_i 's, $i = R - M + 1 \rightarrow N - M$.

Clearly, $F(z)$ and $G(z)$ are simpler than the given transfer function and can be realized by any existing method, including lattice, as discussed later. We take an example to illustrate the procedure.

11.1 Example 6

Let it be required to realize

$$H(z) = (a_0 + a_1 z^{-1}) / (1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4}). \quad (82)$$

The denominator polynomial of $F(z)$ can be chosen to be any of the following:

$$D_1(z) = 1 + c_1 z^{-1}; D_2(z) = 1 + c_1 z^{-1} + c_2 z^{-2}; D_3(z) = 1 + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3}. \quad (83)$$

Accordingly, $G(z)$ has to be of the form:

$$G_1(z) = g_1 z^{-1} + g_2 z^{-2} + g_3 z^{-3}; G_2(z) = g_2 z^{-2} + g_3 z^{-3}; G_3(z) = g_3 z^{-3}. \quad (84)$$

In the second and third choices, the feedback configuration can be applied to $F(z)$ as well, which would result in a nested configuration. The constants c_i and g_i can be found out in the manner already mentioned. For the first set of choice,

$$1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4} = 1 + c_1 z^{-1} + (a_0 + a_1 z^{-1})(g_1 z^{-1} + g_2 z^{-2} + g_3 z^{-3}). \quad (85)$$

This gives rise to the following set of equations:

$$c_1 + a_0 g_1 = b_1; a_0 g_2 + a_1 g_1 = b_2; a_0 g_3 + a_1 g_2 = b_3; \text{ and } a_1 g_3 = b_4. \quad (86)$$

Solving Eqs. (86) gives

$$g_3 = b_4 / a_1, g_2 = (b_3 - a_0 g_3) / a_1; g_1 = (b_2 - a_0 g_2) / a_1; \text{ and } c_1 = b_1 - a_0 g_1. \quad (87)$$

The resulting realization is shown in figure 13.

As mentioned earlier, this new canonic IIR realization for an arbitrary transfer function will be useful in canonical IIR lattice realization, as discussed later in section 17 of this paper.

12. Canonical all-pass lattice realization

As shown in Mitra (2001), the all-pass transfer function

$$T_N(z) = z^{-N} D_N(z^{-1}) / D_N(z) = [b_{N(N)} + b_{N-1(N)} z^{-1} + \dots + b_{0(N)} z^{-N}] / [1 + b_{1(N)} z^{-1} + \dots + b_{N(N)} z^{-N}] \quad (88)$$

Figure 13. Realization of Eq. (82) by the new method.

can be realized by the lattice structure of figure 14a, where the i -th section is of the form shown in figure 14b. The algorithm used is as follows:

$$\begin{aligned} k_i &= b_{i(i)}, i = N \rightarrow 1 \\ b_{n(i-1)} &= [b_{n(i)} - b_{i(i)}b_{i-n(i)}]/[1 - (b_{i(i)})^2], n = i - 1 \rightarrow i, i = N \rightarrow 1. \end{aligned} \quad (89)$$

Clearly, the realization uses $2N$ multipliers instead of N , required for canonicity. It has been shown in Mitra (2001) that a canonical structure is possible by using the alternative structure shown in figure 15 for the i -th section, where k_i 's are obtained by the same algorithm as given

Figure 14. (a) IIR all-pass lattice structure, (b) i -th section of figure 14a.

Figure 15. Single multiplier i -th section for figure 14(a).

by Eq. (89). It must be pointed out, however, that the structures of figure 15 and figure 14b are not equivalent to each other because their t -parameters are different. Only $T_i(z) = Y_i(z) / X_i(z)$ in both the two-pairs are equal when terminated in $T_{i-1}(z) = Y_{i-1}(z) / X_{i-1}(z)$. The overall structure obtained with single multiplier lattice sections is indeed canonical.

As we shall see later, one limitation of the new canonical IIR lattice synthesis is that applied to all-pass transfer functions, it gives the same non-canonic structure as that obtained by the conventional method discussed in this section.

13. Conventional IIR lattice for arbitrary transfer functions

Consider the general IIR transfer function

$$H_N(z) = A_N(z)/B_N(z) = [a_{0(N)} + a_{1(N)}z^{-1} + \dots + a_{N(N)}z^{-N}] / [1 + b_{1(N)}z^{-1} + \dots + b_{N(N)}z^{-N}]. \quad (90)$$

The existing procedure for lattice realization of Eq. (90), first formulated by Gray & Markel (1973) is to realize the all-pass function of Eq. (88), then tap the signals $Y_0, Y_1, \dots, Y_N$, and finally take the weighted sum of them. For example, for $N = 3$, the structure will be of the form shown in figure 16.

The weights α_i are calculated by writing a set of linear equations, which can be easily derived. If each section of the lattice is realized by a single multiplier of the form shown in figure 15, then

Figure 16. Gray-Markel structure for the transfer function (90) with $N = 3$.

the total number of multipliers will be $2N+1$ so that the structure qualifies to be called canonic. However, it is no longer a true lattice, rather it is a lattice-ladder structure. As will be shown in the next section, the new canonical lattice synthesis gives rise to a true lattice, and in that sense it is indeed a breakthrough.

14. Deriving the new IIR canonic lattice structure

The key to the new structure is the same as that used in the FIR lattice discussed in section 7. Instead of using two equal and opposite lattice parameters in each section, we use two different lattice parameters to reduce the order of the transfer function by one (Dutta Roy 2007b). Let it be required to realize Eq. (90) with $b_{0(N)}$ not necessarily equal to unity. Consider the lattice section shown in figure 17 where $Y(z)/X(z)$ is required to be $H_N(z)$ and the terminating transfer function $G(z)$ is to be determined. If, by appropriate choice of $k_a^{(N)}$ and $k_b^{(N)}$, we can find a $G(z)$ of order $N-1$, then our purpose will be served and we can repeat the procedure till we get a terminating transfer function of order zero, which is, of course, a constant.

Analysing figure 17, we obtain

$$Y(z)/X(z) = H_N(z) = [k_a^{(N)} + z^{-1}G(z)]/[1 - k_b^{(N)}z^{-1}G(z)]. \quad (91)$$

One can verify that if $k_a^{(N)} = -k_b^{(N)}$ and $G(z) = 1$, then we get a first order all-pass function, as expected (this also indicates a limitation of this procedure, as mentioned in section 18). Putting $H_N(z) = A_N(z)/B_N(z)$ in Eq. (91) and simplifying, we get

$$G(z) = z[A_N(z) - k_a^{(N)}B_N(z)]/[B_N(z) + k_b^{(N)}A_N(z)] = C(z)/D(z), \text{ say.} \quad (92)$$

Substituting for $A_N(z)$ and $B_N(z)$ from Eq. (90), we get

$$\begin{aligned} C(z) &= z \left\{ [a_{0(N)} - k_a^{(N)}b_{0(N)}] + [a_{1(N)} - k_a^{(N)}b_{1(N)}]z^{-1} + \dots + [a_{N(N)} - k_a^{(N)}b_{N(N)}]z^{-N} \right\} \text{ and} \\ D(z) &= [b_{0(N)} + k_b^{(N)}a_{0(N)}] + [b_{1(N)} + k_b^{(N)}a_{1(N)}]z^{-1} + \dots + [b_{N(N)} + k_b^{(N)}a_{N(N)}]z^{-N}. \end{aligned} \quad (93)$$

Clearly, if we choose

$$k_a^{(N)} = a_{0(N)}/b_{0(N)} \text{ and } k_b^{(N)} = -b_{N(N)}/a_{N(N)}, \quad (94)$$

Figure 17. N -th lattice section with two different parameters.

then $C(z)$ becomes a polynomial of order $N-1$ and so does $D(z)$. Thus $G(z)$ of figure 17 becomes of order $N-1$ and can be redesignated as $H_{N-1}(z)$. It has the same form as that of Eq. (90) with N replaced by $N-1$ and its coefficients are related to those of $H_N(z)$ as follows:

$$a_{n(N-1)} = a_{n+1(N)} - k_{a(N)} b_{n+1(N)}, b_{n(N-1)} = b_{n(N)} + k_{b(N)} a_{n(N)}, n = 0 \text{ to } N-1. \quad (95)$$

As mentioned earlier, this step can be repeated $N-1$ times to end up in a terminating transfer function equal to a constant. The necessary design equations are obtained by replacing N by a general index i in Eqs. (94) and (95) where $i = (N-1) \rightarrow 1$. Specifically,

$$\begin{aligned} k_{a(i)} &= a_{0(i)}/b_{0(i)} \text{ and } k_{b(i)} = -b_{1(i)}/a_{1(i)} \\ a_{n(i-1)} &= a_{n+1(i)} - k_{a(i)} b_{n+1(i)} \text{ and } b_{n(i-1)} = b_{n(i)} + k_{b(i)} a_{n(i)} \\ n &= 0 \rightarrow (i-1), i = (N-1) \rightarrow 1 \\ H_0(z) &= a_{N(N)}/b_{0(N)}. \end{aligned} \quad (96)$$

The last equation in the set (96) needs justification and is a refinement on the original expression. The terminating transfer function $H_0(z)$ is actually $H_0(z) = a_{0(0)}/b_{0(0)}$, but note from the third equations in the set (96) that

$$a_{i(i-1)}/b_{0(i-1)} = [a_{i(i)} - k_{a(i)} b_{1(i)}] / [b_{0(i)} + k_{b(i)} a_{0(i)}] = a_{i(i)}/b_{0(i)}, \quad (97)$$

the last expression being obtained by substituting for $k_{a(i)}$ and $k_{b(i)}$ from the first two equations in the set (96). Thus

$$a_{N(N)}/b_{0(N)} = a_{N-1(N-1)}/b_{0(N-1)} \dots = a_{0(0)}/b_{0(0)} = H_0(z). \quad (98)$$

Note that we have not taken $b_{0(N)} = 1$, as is the usual practice, because $b_{0(N-1)}$, obtained by putting $n = 0$ and $i = N$ in the second equation of the set (96) may not necessarily be unity. We shall now illustrate the procedure by a number of examples which will include some problem cases.

15. Examples of canonic IIR lattice synthesis, including problem cases

For simplicity, we shall confine ourselves to a third order transfer function, which, in general, is of the form

$$H_3(z) = (1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}) / (1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}). \quad (99)$$

The realization of Eq. (99) will in general give the structure shown in figure 18.

Figure 18. General canonical realization of Eq. (99).

15.1 Example 7

If $a_3 = 0$, then from Eq. (96), $k_{b(3)} \rightarrow \infty$, and we cannot proceed further. The solution is to realize $H_3'(z) = 1 - H_3(z)$, which presents no such difficulty and then obtain $H_3(z)$ by subtracting it from unity. The parameters for $H_3'(z)$ are as follows:

$$\begin{aligned} k_{a(3)} &= 0, k_{b(3)} = -1, k_{a(2)} = b_1 - a_1, k_{b(2)} = -a_2/b_3, \\ k_{a(1)} &= b_3 [b_2 - a_2 - a_1 (b_1 - a_1)] / [b_3 - a_2 (b_1 - a_1)], \\ k_{b(1)} &= [a_1 b_3 - a_2 (b_2 - a_2)] / \{b_3 [b_3 - a_2 (b_1 - a_1)]\}, \text{ and } H_0(z) = b_3. \end{aligned} \quad (100)$$

The parameters for the special cases of $a_3 = a_2 = a_1 = 0$ and $a_3 = a_2 = 0$ can be derived from Eq. (100).

15.2 Example 8

For the case in which the numerator in Eq. (99) contains only the $a_3 z^{-3}$ term, no problem arises, and the parameters can be calculated by applying Eq. (96) as:

$$k_{a(3)} = 0, k_{b(3)} = -b_3/a_3, k_{a(2)} = 0, k_{b(2)} = -b_2/a_3, k_{a(1)} = 0, k_{b(1)} = -b_1/a_3, \text{ and } H_0(z) = a_3. \quad (101)$$

15.3 Example 9

Now consider the case in which the numerator of $H_3(z)$ is $a_1 z^{-1} + a_2 z^{-2}$. It is easily verified that if we proceed with $H_3'(z) = 1 - H_3(z)$, as in the previous three cases in which a_3 was equal to zero, we get $k_{a(2)} \rightarrow \infty$ so that we cannot proceed further. In general, one can observe from Eq. (96) that

$$b_{0(N-1)} = b_{0(N)} - [b_{N(N)}/a_{N(N)}]a_{0(N)} \quad (102)$$

which will vanish, leading to $k_{a(N-1)} \rightarrow \infty$ whenever

$$a_{0(N)}/b_{0(N)} = a_{N(N)}/b_{N(N)}. \quad (103)$$

This is a case where the new feedback realization of an arbitrary IIR transfer function, as discussed in section 11 works without any difficulty. We write

$$H_3(z) = (a_1 z^{-1} + a_2 z^{-2}) / (1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}) = F(z) / [1 + G(z) F(z)], \quad (104)$$

where

$$F(z) = (a_1 z^{-1} + a_2 z^{-2}) / (1 + c_1 z^{-1} + c_2 z^{-2}) \text{ and } G(z) = g_1 z^{-1}. \quad (105)$$

Following the procedure in section 11, we can find the constants c_1 , c_2 and g_1 , and hence the complete realization shown in figure 19, where $F(z)$ has been realized by a canonic lattice and the vertical line at the left shows the feedback transfer function $G(z)$. The parameters $k_{b(2)}$ and $k_{b(1)}$ are given by

$$k_{b(2)} = (a_1 b_3 - a_2 b_2) / a_2^2 \text{ and } k_{b(1)} = - \left\{ (b_1/a_1) + \left[(a_1 b_3 - a_2 b_2) / a_2^2 \right] \right\}. \quad (106)$$

Figure 19. Realization of Eq. (99) with numerator = $a_1z^{-1} + a_2z^{-2}$.

15.4 Example 10

As another example, in which the feedback realization gives a canonic structure, let the numerator of Eq. (99) be simply a_1z^{-1} . In this case, we take

$$F(z) = (a_1z^{-1})/(1 + c_1z^{-1}) \text{ and } G(z) = g_1z^{-1} + g_2z^{-2}. \quad (107)$$

Proceeding as in the previous case, we get the complete realization as shown in figure 20.

15.5 Example 11

As yet another example of the application of the feedback method, we consider a case in which Eq. (103) is valid. Let

$$H_3(z) = (a_0 + a_1z^{-1} + a_2z^{-2} + b_3a_0z^{-3})/(1 + b_1z^{-1} + b_2z^{-2} + b_3z^{-3}). \quad (108)$$

The solution is to subtract a_0 from $H_3(z)$ and realize the resulting transfer function by the feedback method as in the previous two examples.

In the last three examples, the required feedback transfer function $G(z)$ was simple, of either first or second order. For higher order $G(z)$, one can use the canonic FIR lattice structure.

Figure 20. Realization of Eq. (99) with numerator = a_1z^{-1} .

16. Limitations of the canonical IIR lattice synthesis procedure

For the Gray–Markel structure, the stability of the given transfer function is automatically tested for; this is not so in the new procedure. Neither is it possible to incorporate the stability testing in the algorithm; stability has to be tested separately before starting the design. Also, the Gray–Markel structure, being derived from a basic all-pass filter, remains stable even under finite word length errors. This is not so in the new procedure.

Another limitation is that if $A_N(z)$ and $B_N(z)$ are related to each other, as in all-pass filters, then canonicity cannot be achieved. In fact, for all-pass transfer functions, the new procedure gives the same non-canonical structure as in the conventional synthesis procedure. Canonical lattice for all-pass functions using figure 17 thus remains an outstanding problem.

17. Conclusion

This paper is a review of the work done by the author in exploring some of the gaps in the conventional lattice synthesis procedures, which, in general, yield non-canonical structures. Canonical lattice synthesis, formulated by the author for FIR as well as IIR transfer functions, are then discussed in detail along with their limitations. In particular, it has been pointed out that linear phase FIR and all-pass IIR are not amenable to canonical lattice synthesis by the new procedures, and these remain open problems for future investigations. In the process of development of canonical IIR lattice, a new feedback method for realization of an arbitrary IIR transfer function has been innovated, which uses IIR in the forward path and FIR in the feedback path.

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