Thermo mechanical buckling analysis of carbon nanotubes on winkler foundation using non-local elasticity theory and DTM

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Abstract. Thermo-mechanical buckling behaviour of Single Walled Carbon Nanotube (SWCNT) on Winkler foundation is reported. Non-local elasticity theory and differential transformation method are applied. Effects of (i) size of SWCNT, (ii) non-local parameter, (iii) Winkler modulus, (iv) boundary conditions and (v) temperature on buckling load are investigated.

Keywords. Nanotube; thermo-mechanical buckling; non-local elasticity theory; boundary condition.

1. Introduction

Carbon nanotubes are cylindrical macromolecules consisting of carbon atoms arranged in a periodic hexagonal structure and were invented by Iijima (1991). This started a new field of research for the accurate analysis of nano-size structures. Recently there has been growing interest for application of non-local continuum mechanics especially in the field of fracture mechanics, dislocation mechanics and micro/nano technologies (carbon nanotubes). Numerous studies showed that carbon nanotubes possess extraordinary physical properties such as high stiffness to weight and strength to weight ratios and enormous electrical and thermal conductivities known to any other material. Due to these superior properties CNTs are used in the emerging fields of nano electronics, nano devices. The CNT's also hold exciting promise in useful potential applications as electrodes in super capacitors, as cable material for space elevators. This gives CNT's unique properties and wide range of uses. Many believe that CNT's may provide the ultimate reinforcing materials for the development of a new class of nanocomposites (Thostenson *et al* 2001; Qian *et al* 2002). Their potential applications led to many investigations into measurements of mechanical properties of nanotubes, using techniques such as transmission electron microscopy (TEM) and atomic force microscopy (AFM) (Treacy *et al* 1996). At nanolength scales, size

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effects often become prominent, the cause of which need to be explicitly addressed with an increasing interest in the general area of nanotechnology (Sharma *et al* 2003). The classical theory of elasticity being the long wave limit of the atomic theory excludes these effects and thus would fail to analyse the structure with small scale effects accurately. Eringen (1983) proposed the non-local continuum theories for the analysis of small sized structures. In non-local elasticity theory, the small-scale effects are captured by assuming that the stress at a point is a function not only of the strain at that point but also a function of the strains at all other points of the domain. This is due to the scale effect of the nanostructures. Thereby size-dependent continuum-based methods (Zhou & Li 2001; Fleck & Hutchinson 1997) are becoming popular in modelling small sized structures as it offers much faster and accurate solutions. Drawbacks of the classical continuum theory could be efficiently avoided and size-dependent phenomena can be satisfactorily explained by the non-local theory. Sun & Zhang (2003) pointed out the limitation of the applicability of continuum models in nanotechnology and indicated the importance of a semi-continuum model in analysing nanostructured materials with plate-like geometry. Their results showed that material properties were found completely dependent on the thickness of the plate structure. Sudak (2003) carried out buckling analysis of multi-walled carbon nanotubes. Wang & Varadhan (2005) analysed the small scale effect of CNT and shell model. Yakobson *et al* (1996) introduced an atomistic model for axially compressed SWCNT and compared it to a simple continuum shell model. Sears & Batra (2006) proposed a comprehensive buckling analysis of single-walled and multi-walled CNT's by molecular mechanics simulations and continuum mechanics models, such as the Euler Bernoulli Beam theory and study of buckling analysis of carbon nanotubes is of interest for better understanding of mechanical responses of CNT's. A relevant reference concerning non-local theories for bending, buckling and vibration analysis of beams is reported by Reddy (2007). Work related to bending, vibration and buckling analyses of CNTs and graphene sheets using non-local elasticity are found in Pradhan & Murmu (2009a,b, 2010), Murmu & Pradhan (2009a,b,c,d,e, 2010), Pradhan (2009), Pradhan & Phadikar (2009a,b), Pradhan & Sarkar (2009) and Pradhan *et al* (2009). Limited amount of literature has been found on the thermo mechanical buckling analysis of carbon nanotubes on elastic foundation. In the present work, differential transformation method (DTM) has been employed to study the non-local effects on SWCNT. This method was first proposed by Zhou (1986) for solving both linear and nonlinear initial value problems of electrical circuits. With this technique it is possible to obtain highly accurate solutions for the differential or integral differential equations (Ayaz 2004; Arikoglu 2004). With the help of DTM we analyse the thermo mechanical buckling behaviour of SWCNT on Winkler foundation. Engineering structures are often found to be resting on elastic foundation. These structures are modelled as being supported on elastic foundation along their span. The elastic type foundation is modelled as Winkler medium. The Winkler type elastic foundation is approximated as a series of closely spaced, mutually independent, vertical linear elastic springs. The foundation modulus is represented by the stiffness of the spring.

2. Formulation

According to Eringen (1983) the non-local constitutive behaviour of a Hookean solid is represented by the following differential constitutive relation

$$(1 - \mu \nabla^2) \sigma^{nl} = \sigma^l. \quad (1)$$

Here is the non-local parameter; σ^l is the local stress tensor at a point which is related to strain by generalized Hooke's law

$$\sigma^l(x) = S(x) : \varepsilon(x), \quad (2)$$

where S is the fourth order elasticity tensor and $:$ denotes the double dot product.

From the free body diagram of an infinitesimal element of a beam structure subjected to an axial loading, p and embedded in an elastic medium of winker constant k_w , the equilibrium equations regarding the force in vertical direction and moment on the one-dimensional structure can be shown as (Wang *et al* 2006):

$$\frac{dV}{dx} = k_w w, \quad (3)$$

$$V - \frac{dM}{dx} + p_{res} \frac{dw}{dx} = 0, \quad (4)$$

where p_{res} is the resultant loading due to the applied axial force p and axial loading N due to change in temperature. Where N is given in Zhang *et al* (2008) as

$$N = -\frac{EA}{(1-2\nu)}\alpha T, \quad (5)$$

$$p_{res} = p - N, \quad (6)$$

where α is the coefficient of thermal expansion, ν is the Poisson ratio and T is the change in temperature.

The non-local constitutive relation is given by:

$$M - \mu \frac{d^2 M}{dx^2} = -EI \frac{d^2 w}{dx^2}. \quad (7)$$

Thus, substituting the above expressions bending moment and shear force can be written as:

$$M = -EI \frac{d^2 w}{dx^2} + \mu \left[p_{res} \frac{d^2 w}{dx^2} + k_w w \right], \quad (8)$$

$$V = -EI \frac{d^3 w}{dx^3} + \mu \left[p_{res} \frac{d^2 w}{dx^3} + k_w \frac{dw}{dx} \right] - p_{res} \frac{dw}{dx}. \quad (9)$$

Further considering equations, we finally come to the following equation:

$$EI \frac{d^4 w}{dx^4} + \left(1 - \mu \frac{d^2}{dx^2} \right) \left[k_w w + p_{res} \frac{d^2 w}{dx^2} \right] = 0. \quad (10)$$

Which can be non-dimensionalized as

$$R_1 \frac{d^4 W}{dX^4} + R_2 \frac{d^2 W}{dX^2} + R_3 W = 0, \quad (11)$$

where

$$R_1 = \left(1 - \frac{p_{res} \mu}{EI} \right), \quad (12)$$

$$R_2 = \left(\frac{p_{res} - k_w \mu}{EI} \right), \quad (13)$$

$$R_3 = \left(\frac{k_w L^4}{EI} \right). \quad (14)$$

3. Differential transformation method and solution

In order to derive differential form of Eq (9) we refer table 1 and the following expression can be written easily:

$$W(k+4) = \frac{-R_2(k+1)(k+2)W(k+2) - R_3W(k)}{R_1(k+1)(k+2)(k+3)(k+4)}. \quad (15)$$

4. Boundary conditions

4.1 Clamped-clamped SWCNT

The boundary conditions for this case are defined as:

$$w(0) = 0, w'(0) = 0, w(L) = 0, w'(L) = 0. \quad (16)$$

From the above boundary conditions we refer table 1 to write the differential transformation boundary conditions as

$$W(0) = 0, W(1) = 0, \quad (17)$$

$$\sum_{k=0}^{\infty} W(k) = 0, \quad (18)$$

$$\sum_{k=0}^{\infty} kW(k) = 0. \quad (19)$$

By assuming $W(2) = c_1$ and $W(3) = c_2$ the Eq (13) can be calculated up to n terms and it will be substituted in Eqs (16) and (17) and solving this for non-trivial solutions we get p_{cr} . As the n value increases the accuracy of the solution increases.

4.2 Simply supported SWCNT

The boundary condition for this case are defined as

$$w(0) = 0, M(0) = 0, w(L) = 0, M(L) = 0. \quad (20)$$

From the above boundary conditions, we refer table 1 to write the differential transformation boundary conditions as

$$W(0) = 0, W(2) = 0, \quad (21)$$

$$\sum_{k=0}^{\infty} W(k) = 0, \quad (22)$$

$$\sum_{k=0}^{\infty} [k(k-1) - R_4L^2] W(k) = 0, \quad (23)$$

where

$$R_4 = \frac{\mu k_w}{(EI - \mu p_{res})}. \quad (24)$$

Table 1. DTM theorems used for equations of motion and boundary conditions.

Original fn	Transformed fn	Original B.C. (for $x = 0$)	Transformed B.C. (for $x = 0$)	Original B.C. (for $x = 1$)	Transformed B.C. (for $x = 1$)
$f(x) = g(x) \pm h(x)$	$F(k) = G(k) \pm H(k)$	$f(0) = 0$	$F(0) = 0$	$f(1) = 0$	$\sum_{k=0}^{\infty} F(k) = 0$
$f(x) = \lambda g(x)$	$F(k) = \lambda G(k)$	$\frac{df}{dx}(0) = 0$	$F(1) = 0$	$\frac{df}{dx}(1) = 0$	$\sum_{k=0}^{\infty} kF(k) = 0$
$f(x) = g(x) h(x)$	$F(k) = \sum_{l=0}^k G(l)H(k-l)$	$\frac{d^2f}{dx^2}(0) = 0$	$F(2) = 0$	$\frac{d^2f}{dx^2}(1) = 0$	$\sum_{k=0}^{\infty} k(k-1)F(k) = 0$
$f(x) = \frac{d^n g(x)}{dx^n}$	$F(k) = \frac{(k+n)!}{k!} G(k+n)$	$\frac{d^3f}{dx^3}(0) = 0$	$F(3) = 0$	$\frac{d^3f}{dx^3}(1) = 0$	$\sum_{k=0}^{\infty} k(k-1)(k-2)F(k) = 0$

By assuming $W(1) = c_1$ and $W(3) = c_2$ the Eq (13) can be calculated up to n terms and similar procedure is applied as that of clamped–clamped boundary condition.

4.3 Clamped–hinged SWCNT

The boundary conditions for this case are defined as

$$w(0) = 0, w'(0) = 0, w(L) = 0, M(L) = 0. \quad (25)$$

From the above boundary conditions, we refer table 1 to write the differential transformation boundary conditions as

$$W(0) = 0, W(1) = 0, \quad (26)$$

$$\sum_{k=0}^{\infty} W(k) = 0, \quad (27)$$

$$\sum_{k=0}^{\infty} [k(k-1) - R_4 L^2] W(k) = 0. \quad (28)$$

By assuming $W(2) = c_1$ and $W(3) = c_2$ the Eq (13) can be calculated up to n terms and similar procedure is applied as that of clamped–clamped boundary condition.

4.4 Clamped–free SWCNT

The boundary conditions for this case are defined as

$$w(0) = 0, w'(0) = 0, M(L) = 0, V(L) = 0. \quad (29)$$

From the above boundary conditions, we refer table 1 to write the differential transformation boundary conditions as:

$$W(0) = 0, W(1) = 0, \quad (30)$$

$$\sum_{k=0}^{\infty} [k(k-1) - R_4 L^2] W(k) = 0, \quad (31)$$

$$\sum_{k=0}^{\infty} [k(k-1)(k-2) + (R_5 L^2)k] W(k) = 0, \quad (32)$$

$$R_5 = \frac{(p_{res} - \mu k_w) L^2}{(EI - \mu P)}. \quad (33)$$

By assuming $W(2) = c_1$ and $W(3) = c_2$ the Eq (13) can be calculated up to n terms and similar procedure is applied as that of clamped–clamped boundary condition.

5. Results and discussions

The governing equations for the SWCNT with non-local effects are written in Eq (8). It is interesting to note that by putting $\mu = 0$ we can obtain the corresponding local elasticity equation for SWCNT. Analytical solutions exist for SWCNT with non-local effects without any elastic medium and with out any temperature change. Euler Beam is considered in the analysis. The analytical solution (Wang *et al* 2006) is given by:

$$p_{exact} = \frac{(m\pi)^2 \left(\frac{EI}{L^2}\right)}{\left(1 + \mu \left(\frac{m\pi}{L}\right)^2\right)}, \quad (34)$$

where $m = 2$ for clamped–clamped case, $m = 1$ for simply supported case, $m = 1/0.7$ for clamped–hinged case and $m = 0.5$ for clamped-free. Similar analysis was done by Senthilkumar *et al* (2010). The analytical results from Eq (32) are compared with those obtained from DTM and are presented in table 2. Diameter is taken as 1 nm for this case. Present results and the corresponding results reported in reference Wang *et al* (2006) are found to be in good agreement.

The critical load ratio (critical load due to non-local effects/critical load due to local effects) vs Winkler modulus parameter for the first buckling mode are plotted for various boundary conditions. The critical load ratios vs length for the first buckling mode are plotted for various boundary conditions. While plotting $R_3 = \frac{k_w L^4}{EI}$ is defined as Winkler modulus (K). Load deviation per cent defined as:

Load deviation per cent = $\frac{(p_{T(k)}^{nl} - p_{0(k)}^{nl})}{p_{0(k)}^{nl}}$ i.e., the difference of non-local critical loads with temperature change $T(k)$ and $0(k)$ divide by non-local load without any temperature change. In figure 4 load deviation per cent is plotted across temperature change for various boundary conditions.

Figure 1 shows a schematic diagram of SWCNT on Winkler foundation with simply supported boundary conditions. In figure 2 load ratio's for Winkler Modulus for various boundary conditions are plotted. The material properties for this case are taken as $E = 1.06$ TPa, diameter = 1 nm, length = 10 nm, non-local parameter is taken as 1 nm^2 and temperature change of 50 k is considered. It is observed that for clamped–clamped, simply supported, clamped–hinged the critical load ratios increased as Winkler modulus K is varied from 0 to 200 as predicted. Because increasing Winkler modulus increases the stiffness of the beam. But anomaly was observed

Table 2. Critical buckling load vs length for various boundary conditions ($T = 0(k)$ and $K = 0$).

Length	Clamped–clamped		Simply supported		Clamped–hinged		Clamped-free	
	P(exact ³²)	P(DTM)	P(exact ³²)	P(DTM)	P(exact ³²)	P(DTM)	P(exact ³²)	P(DTM)
10	13.8939	13.8939	4.4095	4.4095	8.2295	8.2295	1.182	1.182
12	10.652	10.652	3.1486	3.1486	6.0235	6.0235	0.8269	0.8269
14	8.2296	8.2296	2.3533	2.3533	4.5744	4.5744	0.6103	0.6103
16	6.5585	6.5585	1.8222	1.8222	3.584	3.584	0.4686	0.4686
18	5.3375	5.3375	1.4511	1.4511	2.873	2.873	0.371	0.371
20	4.4095	4.4095	1.182	1.182	2.3533	2.3533	0.3009	0.3009

Figure 1. Single-walled carbon nanotube with simply supported boundary conditions on Winkler foundation.

in the case of clamped-free case where load ratios were seen to be decreasing with increase in K .

In figure 3 the critical load ratios are plotted for different boundary conditions and different sizes of SWCNT's. The material properties are taken as $E = 1.06$ Tpa, diameter = 1 nm, Winkler modulus (K) = 50, change in temperature as 50 k and non-local parameter as 1 nm^2 . In the present study coefficient of thermal expansion for temperature change at lower temperature or room temperature is considered to be $-1.6\text{E-}06$ (Yao & Han 2006). From these figures one can observe that non-local effects are dominant up to a length of (i) 25 nm for clamped-clamped, (ii) 10 nm for simply supported, (iii) 15 nm for clamped-hinged and (iv) 10 nm for clamped-free boundary conditions, respectively. The critical load ratio's of clamped-free were found to be

Figure 2. Critical load ratio vs Winkler modulus for various boundary conditions.

Figure 3. Critical load ratio vs length with Winkler modulus parameter 50 for various boundary conditions.

Figure 4. Critical load deviation percent vs temperature change for various boundary conditions at $K = 50$.

slightly below simply supported, this is consistent with the results of figure 2 where the graphs of clamped-free and simply supported were found to cross each other around $K = 50$.

In figure 4 percentage load deviation is plotted for various boundary conditions. The material properties are same as above and Winkler modulus is taken as 50. We notice that percentage deviation is least for clamped–clamped followed by clamped–hinged, simply supported and clamped-free. This is as predicted as for clamped–clamped boundary condition SWCNT buckles for higher critical load compared to other boundary conditions therefore deviation is supposed to be the least.

6. Conclusions

Non-local elasticity equations of Eringen are employed and the buckling analysis of SWCNT with elastic foundation and temperature effect are carried out. Effects of (i) size of beam, (ii) Winkler modulus (K), (iii) non-local parameter and (iv) change in temperature of SWCNT's on non-local critical buckling load are investigated and discussed. Non-local effects increase with decrease in the size of SWCNT. Load ratio's decrease with increase in non-local parameter. Load ratio's increase with increase in Winkler modulus K for clamped–clamped, simply supported, clamped–hinged boundary conditions but an anomaly was observed in the case of clamped-free boundary condition, where load ratio is observed to decrease with increase in K . Further, load deviation percentage is plotted for various boundary conditions. Deviation is observed to be the least for clamped–clamped and highest for clamped-free boundary conditions.

Nomenclature

- L Length of SWCNT
- E Young's modulus of SWCNT
- M Bending moment
- V Shear force
- $S(x)$ Fourth order elasticity tensor
- σ^l Macroscopic local stress
- W Deflections of SWCNT at any point x
- μ Non-local parameter
- ν Poisson ratio
- α Coefficient of thermal expansion
- I Moment of inertia
- p_{res} Resultant critical load
- k Winkler constant
- p^{nl} Non-local load

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