An extension of Mangler transformation to a 3-D problem

J DEY^{1,*} and A VASUDEVA MURTHY²

¹Department of Aerospace Engineering, Indian Institute of Science,
Bangalore 560012, India

²Tata Institute of Fundamental Research, Chikkabommasandra, GKVK Post,
Bangalore 560065, India
e-mail: jd@aero.iisc.ernet.in

MS received 29 October 2010; accepted 19 August 2011

Abstract. Considering the linearized boundary layer equations for three-dimensional disturbances, a Mangler type transformation is used to reduce this case to an equivalent two-dimensional one.

Keywords. Boundary layer; three-dimensional; Mangler transformation.

1. Introduction

The Mangler transformation reduces an axisymmetric laminar boundary layer on a body of revolution to an equivalent planar boundary layer flow (Schlichting 1968). This transformation is also useful in turbulent boundary layer flow over a body of revolution (Cebeci & Bradshaw 1968). Another application of this transformation is in the reduction of a laterally strained boundary layer to the Blasius flow (Ramesh *et al* 1997). In this case the span-wise velocity is zero along a streamline but its non-zero span-wise gradient appears as a source/sink term in the continuity equation (Schlichting 1968). In this paper, we show that a Mangler type transformation can reduce a specific three-dimensional flow considered here to an equivalent two-dimensional case.

2. Analysis

Let u^* , v^* and w^* denote the non-dimensional velocity components in the non-dimensional x , y and z directions, respectively. u_0 and v_0 will denote the Blasius velocity components. The governing equations considered here are the linearized boundary layer equations for two- and three-dimensional disturbances of Libby & Fox (1964) and Luchini (1996). These authors perturbed the Blasius boundary layer as: $u^* = u_0(x, y) + u_1(x, y)\exp(i\alpha z)$, $v^* = v_0(x, y) + v_1(x, y)\exp(i\alpha z)$, $w^* = w_1(x, y)\exp(i\alpha z)$; for 2-D flow ($z = 0$, $w = 0$), $u_1 = u$, $v_1 = v$. We first consider the two-dimensional case.

*For correspondence

2.1 2-D Case

In this case, the governing boundary layer equations are (Libby & Fox 1964),

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$u_o \frac{\partial u}{\partial x} + v_o \frac{\partial u}{\partial y} + u \frac{\partial u_o}{\partial x} + v \frac{\partial u_o}{\partial y} = \frac{\partial^2 u}{\partial y^2}. \quad (2)$$

The boundary conditions are: $u(x, 0) = v(x, 0) = u(x, \infty) = v(x, \infty) = 0$. The Blasius boundary layer equations are,

$$\frac{\partial u_o}{\partial x} + \frac{\partial v_o}{\partial y} = 0, \quad (3)$$

$$u_o \frac{\partial u_o}{\partial x} + v_o \frac{\partial u_o}{\partial y} = \frac{\partial^2 u_o}{\partial y^2}, \quad (4)$$

along with the boundary conditions, $u_0(y = 0) = v_0(y = 0) = 0$, $u_0(y \rightarrow \infty) \rightarrow 1$.

Adding and subtracting the quantity u/x in the continuity eq. (1), we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{u}{x} - \frac{u}{x} = 0. \quad (5)$$

Consider the Mangler transformation,

$$X = \frac{x^3}{3}, \quad Y = yx, \quad u(x, y) \rightarrow U(X, Y),$$

$$V(X, Y) = \frac{1}{x} \left[\frac{yu}{x} + v \right], \quad u_o(x, y) \rightarrow U_o(X, Y), \quad V_o = \frac{1}{x} \left[\frac{yu_o}{x} + v_o \right]. \quad (6)$$

The usual Mangler variables are X, Y, U and V . The variables U_o and V_o are additional here. The boundary layer equations for an axi-symmetric body of radius r differ from those for two-dimensional flows by the term $(u/r)(dr/dx)$ in the continuity equation, $\frac{\partial(ur)}{\partial x} + \frac{\partial(vr)}{\partial y} = 0$; for $r = x$, the term $(u/r)(dr/dx)$ becomes u/x , which acts as a source term in the continuity equation.

In terms of the variables in (6), the governing equations (1)–(4) become,

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} - \frac{U}{3X} = 0, \quad (7)$$

$$U_o \frac{\partial U}{\partial X} + V_o \frac{\partial U}{\partial Y} + U \frac{\partial U_o}{\partial X} + V \frac{\partial U_o}{\partial Y} = \frac{\partial^2 U}{\partial Y^2}, \quad (8)$$

$$\frac{\partial U_o}{\partial X} + \frac{\partial V_o}{\partial Y} - \frac{U_o}{3X} = 0, \quad (9)$$

$$U_o \frac{\partial U_o}{\partial X} + V_o \frac{\partial U_o}{\partial Y} = \frac{\partial^2 U_o}{\partial Y^2}, \quad (10)$$

respectively.

The mean flow continuity eq. (9) now has an artificial sink term $U_o/3X$. However, the similarity variables $\eta = Y/\sqrt{3X}$, $U_o = f'(\eta)$ reduce the mean flow to the Blasius one; here, a prime denotes the derivative with respect to η . (This may be an interesting application of the Mangler transformation to the Blasius flow.)

In terms of the variables,

$$U = 3XU_1(X, Y), \quad V = 3XV_1(X, Y), \quad (11)$$

equations (7) and (8) become

$$\frac{\partial U_1}{\partial X} + \frac{\partial V_1}{\partial Y} + \frac{2U_1}{3X} = 0, \quad (12)$$

$$U_o \frac{\partial U_1}{\partial X} + V_o \frac{\partial U_1}{\partial Y} + \frac{U_o U_1}{X} + \left[U_1 \frac{\partial U_o}{\partial X} + V_1 \frac{\partial U_o}{\partial Y} \right] = \frac{\partial^2 U_1}{\partial Y^2}, \quad (13)$$

respectively.

2.2 3-D Case

The governing boundary layer equations in this case are the linearized disturbance equations of Luchini (1996; his equations 7a–c),

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + w = 0, \quad (14)$$

$$u_o \frac{\partial u}{\partial x} + v_o \frac{\partial u}{\partial y} + u \frac{\partial u_o}{\partial x} + v \frac{\partial u_o}{\partial y} = \frac{\partial^2 u}{\partial y^2}, \quad (15)$$

$$u_o \frac{\partial w}{\partial x} + v_o \frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial y^2}. \quad (16)$$

($i\alpha$ in Luchini's eq. (7a) is eliminated by taking $u_1 = i\alpha u$, $v_1 = i\alpha v$, $w = w_1$). The boundary conditions are: $u(x, 0) = v(x, 0) = w(x, 0) = u(x, \infty) = v(x, \infty) = w(x, \infty) = 0$. The third equation is the span-wise disturbance equation.

As in eq. (5), adding and subtracting the quantity u/x in the continuity equation (14), we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{u}{x} - \frac{u}{x} + w = 0. \quad (17)$$

We consider the Mangler type transformation (6), along with an additional variable W , below

$$X = \frac{x^3}{3}, \quad Y = yx, \quad u(x, y) \rightarrow U(X, Y), \quad V(X, Y) = \frac{1}{x} \left[\frac{yu}{x} + v \right]$$

$$W(X, Y) = \frac{1}{x^2} \left[w - \frac{u}{x} \right], \quad u_o(x, y) \rightarrow U_o(X, Y), \quad V_o = \frac{1}{x} \left[\frac{yu_o}{x} + v_o \right]. \quad (18)$$

This additional variable W is to relate the span-wise velocity component, w , to the stream-wise velocity component, u , as discussed below. In terms of these variables, the disturbance equations (14), (15) and (16) are,

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + W = 0, \quad (19)$$

$$U_o \frac{\partial U}{\partial X} + V_o \frac{\partial U}{\partial Y} + U \frac{\partial U_o}{\partial X} + V \frac{\partial U_o}{\partial Y} = \frac{\partial^2 U}{\partial Y^2}, \quad (20)$$

$$2U_o W + 3X \left[U_o \frac{\partial W}{\partial X} + V_o \frac{\partial W}{\partial Y} \right] + U_o \frac{\partial U}{\partial X} - \frac{UU_o}{3X} + V_o \frac{\partial U}{\partial Y} = 3X \frac{\partial^2 W}{\partial Y^2} + \frac{\partial^2 U}{\partial Y^2}, \quad (21)$$

respectively.

Following Squire (1933), we add (20) and (21) to obtain

$$\begin{aligned} 2U_o W + 3X \left[U_o \frac{\partial W}{\partial X} + V_o \frac{\partial W}{\partial Y} \right] + 2 \left[U_o \frac{\partial U}{\partial X} + V_o \frac{\partial U}{\partial Y} \right] \\ - \frac{UU_o}{3X} + U \frac{\partial U_o}{\partial X} + V \frac{\partial U_o}{\partial Y} = 3X \frac{\partial^2 W}{\partial Y^2} + 2 \frac{\partial^2 U}{\partial Y^2}. \end{aligned} \quad (22)$$

In terms of the variables in (11), the continuity equation (19) and the momentum equation (22) become,

$$\frac{\partial U_1}{\partial X} + \frac{\partial V_1}{\partial Y} + \frac{U_1}{X} + \frac{W}{3X} = 0, \quad (23)$$

$$\begin{aligned} U_o \frac{\partial W}{\partial X} + V_o \frac{\partial W}{\partial Y} - \frac{U_1 U_o}{3X} + \frac{2U_o W}{3X} + 2 \left[U_o \frac{\partial U_1}{\partial X} + \frac{U_o U_1}{X} + V_o \frac{\partial U_1}{\partial Y} \right] \\ + U_1 \frac{\partial U_o}{\partial X} + V_1 \frac{\partial U_o}{\partial Y} = \frac{\partial^2 W}{\partial Y^2} + 2 \frac{\partial^2 U_1}{\partial Y^2}, \end{aligned} \quad (24)$$

respectively.

By letting $W = aU_1$ the disturbance equations (23) and (24) become,

$$\frac{\partial U_1}{\partial X} + \frac{\partial V_1}{\partial Y} + \frac{(3+a)U_1}{3X} = 0, \quad (25)$$

$$U_o \frac{\partial U_1}{\partial X} + V_o \frac{\partial U_1}{\partial Y} + \frac{(2a+5)U_o U_1}{3(2+a)X} + \frac{1}{(2+a)} \left[U_1 \frac{\partial U_o}{\partial X} + V_1 \frac{\partial U_o}{\partial Y} \right] = \frac{\partial^2 U_1}{\partial Y^2}, \quad (26)$$

respectively. Comparing these with the two-dimensional equations (12) and (13), we find them similar. We may note that the span-wise velocity component, w , is $w=(1+a)u/x$. For $a = -1$, $w=0$, equations (25) and (26) are exactly the same as (12) and (13), as it should be. Also, u/x ($\sim w$) acts as a source term in the continuity eq. (14). Thus enabling the use of the Mangler type transformation.

By letting $U_1 = X^N g'(\eta)$, and satisfying the continuity equation (25), the similarity form of (26) is readily obtained as,

$$g''' + \frac{fg''}{2} + \frac{gf''}{2} + gf'' \left[\frac{(7+a)}{2(2+a)} + \frac{3N}{(2+a)} \right] - f'g' \left[3N + \frac{(5+2a)}{(2+a)} \right] = 0. \quad (27)$$

In terms of the similarity variables, the perturbed velocity components are: $u = 3^{-N} x^{3(N+1)} g'$, $w = (1+a)u/x$.

For both (i) $a = -1$, $N = -4/3$ and (ii) $a = -8/3$, $N = -1/2$, eq. (27) reduces to the two-dimensional equation of Libby & Fox (1964)

$$g''' + \frac{fg''}{2} + \frac{gf''}{2} + f'g' - gf'' = 0. \quad (28)$$

The solution (Libby & Fox 1964) of this equation is $g = f - \eta f'$. The first case of $w = 0$ ($a = -1$) is obvious. In the second case, the perturbed velocities are: $u = 3^{-1/2}x^{3/2}g'$, and $w = -(5/3)(x/3)^{1/2}g'$. That is, the proposed Mangler type transformation (18) could reduce the three-dimensional problem considered here to a two-dimensional equivalent one.

3. Conclusion

A three-dimensional boundary layer flow is considered here. A Mangler type of transformation is proposed to reduce this flow to an equivalent two-dimensional one. This reduction has been possible by relating the span-wise velocity component to the stream-wise velocity component leading to an equivalent source term in the continuity equation.

References

- Cebeci T, Bradshaw P 1968 *Momentum transfer in boundary layers*. Hemisphere, p 112
 Libby P A, Fox H 1964 Some perturbation solutions in laminar boundary-layer theory, *J. Fluid Mech.* 17: 433
 Luchini P 1996 Reynolds-number-independent instability of the boundary layer over a flat plate, *J. Fluid Mech.* 327: 101
 Ramesh O N, Dey J, Prabhu A 1997 Transformation of a laterally diverging boundary layer flow to a two-dimensional boundary layer flow, *Zeit. Angew. Math. Phys* 48: 694
 Schlichting H 1968 *Boundary layer theory*. McGraw Hill, p 605
 Squire H B 1933 On the stability of three-dimensional disturbances of viscous fluid flow between parallel walls, *Proc. Royal Soc. London A* 142: 621–629