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Vibration control of an elastic strip by a singular force

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Abstract. Vibration characteristics of an elastic plate in the shape of an infinite strip are changed by applying a lateral concentrated force to the plate. The homogeneous, isotropic, elastic plate is infinite in the x-direction and the sides are simply supported. The size of the force is changed in proportion to the displacement measured at a certain point of the plate. The proportionality constant serves as the control parameter. The mathematical formulation of this distributed control problem and its analytical solution in terms of the vibration frequencies of the plate are given. The vibration frequencies are plotted as a function of the control parameter.

Keywords. Control; vibration; distributed parameter system; plate.

1. Introduction

Predicting resonance frequency of plates is an important technological and scientific problem for which many fairly standard methods have been developed. Recently, controlling structural vibrations by various passive and active means is being vigorously investigated. Piezoelectric sensors and actuators have been chosen in most of the current studies, for example, Chandiramani *et al* (2004); Han & Lee (1999); Sadri *et al* (1999); Shete *et al* (2007), although there are other alternatives. Obviously, being able to control plate vibrations would have profound results especially in aerospace structures. Applying control theory to simpler models which use ordinary differential equations has been considered in a number of studies. The approximate ordinary differential equations have been obtained by finite element or Rayleigh–Ritz methods in a number of studies pertaining to control of flutter in aerospace structures and other vibration problems (Moon & Kim 2005; Frampton *et al* 1996; Döngi *et al* 1996; Forster & Yang 1998; Han *et al* 2006; Kwak & Heo 2007; Walker & Yaneske 1976; Bingham *et al* 2001; Young *et al* 2002; De Matteis *et al* 2008; Bos & Casagrande 2003; Luo *et al* 2008; Bolotin *et al* 2001; Shen & Homaifar 2001; Luo *et al* 2008; Ro & Baz 2002; Hurlebaus *et al* 2008) using piezoelectric actuators. (Benjeddou 2000; Sunar & Rao 1999) are surveys of piezoelectric control of flexible structures. (Yang & Ngoi 2000; Gaudenzi *et al* 1997; Cai & Yang 2006) use piezoelectric actuators in controlling vibration and shape of beams. Varadarajan *et al* (1998) applied piezoelectric materials in the shape control of composite plates. De Fonseca *et al* 1999; Halim *et al* (2008) studied to find optimal sensor and actuator locations in

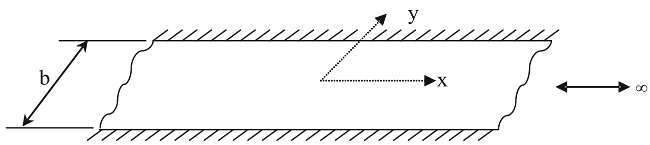

Figure 1. Problem geometry.

control of flexible structures. Chow & Maestrello (1997) considered control of nonlinear plate vibrations by an applied pressure, but the pressure is assumed to be present at every point on the surface of the plate. In the present paper, an elastic plate in the shape of an infinite strip is considered; but we assume that the measurement and actuation are at single points rather than distributed over the whole plate. Since this is a distributed parameter system from the control theory point of view, the standardised control techniques can not be used even for the linear problem. One approach has been discretizing the problem so that the resulting lumped parameter model is a good representation of the original structure. Then the standard methods of control can be applied. In this paper, we tried to solve the distributed parameter system directly.

2. Problem formulation and solution

The homogeneous, isotropic, elastic plate is infinite in the x-direction and has width b in y-direction (figure 1). The sides are simply supported. The eigenfrequencies in this case can be easily computed assuming plane waves propagating in the x-direction. Our purpose is to modify the eigenfrequencies by feedback control. Piezoelectric actuators are widely used for controlling vibrating systems; here, we propose a simpler approach. We assume that the deformation is measured at a point on the plate at all times and this information is then used to apply a concentrated force to the plate at another point. The magnitude of the force is chosen to be proportional to the measured displacement; but the solution to be presented below is sufficiently general so that it can easily be extended to forces proportional to velocity or acceleration or their linear combination. This control scheme can be applied by measuring the displacement directly by a laser range finder (or perhaps measuring the acceleration and obtaining the displacement) and applying the required control force by an electrodynamic shaker or similar linear actuator.

Controlled plate vibrations are governed by

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+\rho_p h_p\frac{\partial^2 w}{\partial t^2}+Q\,w(x_1,0,t)\delta(x-x_2)\delta(y)=0, \tag{1}$$

$w(x, y, t)$ shows the plate deflection, ρ_p and h_p are plate's density and thickness, respectively, and $D = Eh_p^3/12(1-\nu^2)$ is the bending rigidity of the plate. The last term in Eq. (1) models the control action in which $w(x_1, 0, t)$ shows the plate deflection measured on the y-axis at the location $x = x_1$ at all times. The factors including the delta distributions $\delta(x-x_2)\delta(y)$ shows that control force is applied on the x-axis at the location $x = x_2$ and Q is the proportionality constant (dimension N/m) which is the design parameter from the control theory point of view. We choose the measurement and actuation on the x-axis but they can be at any point on the plate. The dynamics of the means by which the measurements are made and the control force is applied was neglected. Because of the form of the last term, this is also an interesting mathematical problem. The plate is assumed to be simply supported along the side walls, i.e.

$$w = \frac{\partial^2 w}{\partial y^2} = 0 \text{ at } y = \pm b/2. \tag{2}$$

We use the following dimensionless quantities

$$x^* = \frac{x}{b},\, y^* = \frac{y}{b},\, w^* = \frac{w}{b},\, t^* = \frac{t}{\sqrt{\frac{\rho_p h_p b^4}{D}}}. \tag{3}$$

Then equations (1) and (2) become, discarding the stars

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{\partial^2 w}{\partial t^2} + K\, w(x_1, 0, t)\, \delta(x - x_2)\, \delta(y) = 0, \tag{4}$$

$$w = \frac{\partial^2 w}{\partial y^2} = 0 \text{ at } y = \pm 1/2, \tag{5}$$

where

$$K = \frac{Q\, b^2}{D}, \tag{6}$$

is the non-dimensional control parameter.

In the absence of the control term, the solutions of equations (4) and (5) can be sought in the form of travelling waves in the x-direction. Due to the control term, we have to follow a different approach. Assuming plate deflections of the form

$$w(x, y, t) = W(x, y)\, e^{i\omega t}, \tag{7}$$

equation (4) becomes

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} - \omega^2 W + K W(x_1, 0)\delta(x - x_2)\delta(y) = 0. \tag{8}$$

Taking Fourier transform of this equation with respect to x, we obtain

$$\xi^4 \hat{W} - 2\xi^2 \frac{d^2 \hat{W}}{dy^2} + \frac{d^4 \hat{W}}{dy^4} - \omega^2 \hat{W} + K W(x_1, 0)\delta(y) e^{i\xi x_2} = 0. \tag{9}$$

Fourier transform pair with the following convention has been used

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{i\xi x} dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-i\xi x} d\xi.$$

Boundary conditions become

$$\hat{W} = \frac{d^2 \hat{W}}{dy^2} = 0 \text{ at } y = \pm 1/2. \tag{10}$$

The method of variation of parameters finds the general solution from varying the parameters of the homogeneous solution. $C_1(y)$, $C_2(y)$, $C_3(y)$ and $C_4(y)$ are unknown functions of y to be determined.

$$\hat{W} = C_1(y)e^{r_1 y} + C_2(y)e^{r_2 y} + C_3(y)e^{r_3 y} + C_4(y)e^{r_4 y}, \tag{11}$$

where r_1, r_2, r_3 and r_4 are the roots of the algebraic equation

$$r^4 - 2\xi^2 r^2 + (\xi^4 - \omega^2) = 0, \tag{12}$$

and $C_1(y), C_2(y), C_3(y), C_4(y)$ satisfy

$$C_1' e^{r_1 y} + C_2' e^{r_2 y} + C_3' e^{r_3 y} + C_4' e^{r_4 y} = 0, \tag{13a}$$

$$r_1 C_1' e^{r_1 y} + r_2 C_2' e^{r_2 y} + r_3 C_3' e^{r_3 y} + r_4 C_4' e^{r_4 y} = 0, \tag{13b}$$

$$r_1^2 C_1' e^{r_1 y} + r_2^2 C_2' e^{r_2 y} + r_3^2 C_3' e^{r_3 y} + r_4^2 C_4' e^{r_4 y} = 0, \tag{13c}$$

$$r_1^3 C_1' e^{r_1 y} + r_2^3 C_2' e^{r_2 y} + r_3^3 C_3' e^{r_3 y} + r_4^3 C_4' e^{r_4 y} = -K W(x_1, 0)\delta(y)e^{i\xi x_2}. \tag{13d}$$

Prime denotes differentiation with respect to y. Equations (13) are solved by Cramer's rule to give

$$C_1' = D_1(y) K W(x_1, 0)\delta(y)e^{i\xi x_2} \tag{14a}$$

$$C_2' = D_2(y) K W(x_1, 0)\delta(y)e^{i\xi x_2} \tag{14b}$$

$$C_3' = D_3(y) K W(x_1, 0)\delta(y)e^{i\xi x_2} \tag{14c}$$

$$C_4' = D_4(y) K W(x_1, 0)\delta(y)e^{i\xi x_2}, \tag{14d}$$

where $D_k(y)$ are formed using the coefficients in equations (13), as the ratio of two determinants. Integrating equations (14) and substituting in equation (11),

$$\begin{aligned} \hat{W} = {} & E_1 e^{r_1 y} + E_2 e^{r_2 y} + E_3 e^{r_3 y} + E_4 e^{r_4 y} \\ & + [D_1(0)e^{r_1 y} + D_2(0)e^{r_2 y} + D_3(0)e^{r_3 y} + D_4(0)e^{r_4 y}] K W(x_1, 0) e^{i\xi x_2} h(y), \end{aligned} \tag{15}$$

$h(y)$ denotes the unit step function and E_k are integration constants. Despite the appearance of the unit step function, $\hat{W}$ is continuous at $y = 0$ because the square bracket in equation (15) becomes zero there. This can easily be seen by setting $y = 0$ in equation (13a) and using equation (14). Note that equation (15) is not an explicit solution since the right-hand side includes $W(x_1, 0)$. Next, the solution, equation (15), is substituted in the boundary conditions equation (10) to determine the constants E_1, E_2, E_3 and E_4;

$$E_1 e^{-r_1/2} + E_2 e^{-r_2/2} + E_3 e^{-r_3/2} + E_4 e^{-r_4/2} = 0 \tag{16a}$$

$$r_1^2 E_1 e^{-r_1/2} + r_2^2 E_2 e^{-r_2/2} + r_3^2 E_3 e^{-r_3/2} + r_4^2 E_4 e^{-r_4/2} = 0 \tag{16b}$$

$$\begin{aligned} & E_1 e^{r_1/2} + E_2 e^{r_2/2} + E_3 e^{r_3/2} + E_4 e^{r_4/2} \\ & \quad + [D_1(0)e^{r_1/2} + D_2(0)e^{r_2/2} + D_3(0)e^{r_3/2} + D_4(0)e^{r_4/2}] K W(x_1, 0) e^{i\xi x_2} = 0 \end{aligned} \tag{16c}$$

$$\begin{aligned} & r_1^2 E_1 e^{r_1/2} + r_2^2 E_2 e^{r_2/2} + r_3^2 E_3 e^{r_3/2} + r_4^2 E_4 e^{r_4/2} \\ & \quad + [r_1^2 D_1(0)e^{r_1/2} + r_2^2 D_2(0)e^{r_2/2} + r_3^2 D_3(0)e^{r_3/2} + r_4^2 D_4(0)e^{r_4/2}] K W(x_1, 0) e^{i\xi x_2} = 0. \end{aligned} \tag{16d}$$

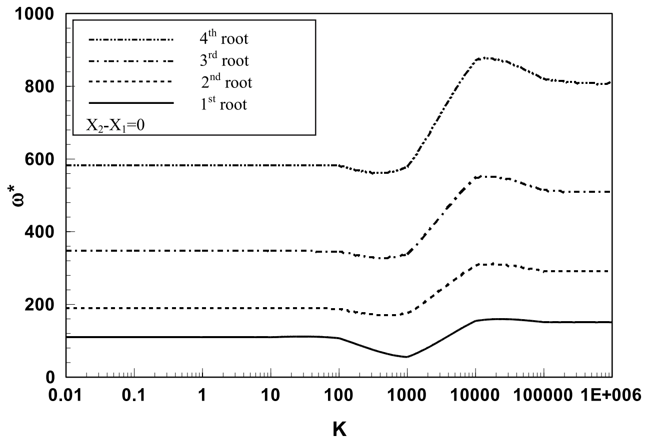

Figure 2. Change in the first four eigenvalues as function of the control parameter K for $x_2 - x_1 = 0$.

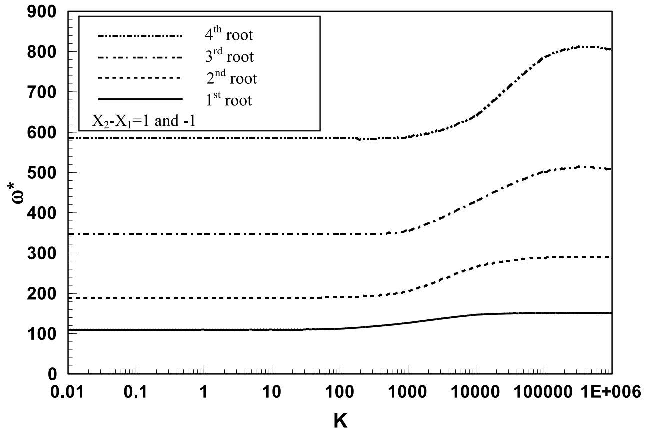

Figure 3. Change in the first four eigenvalues as function of the control parameter K for $x_2 - x_1 = 1$.

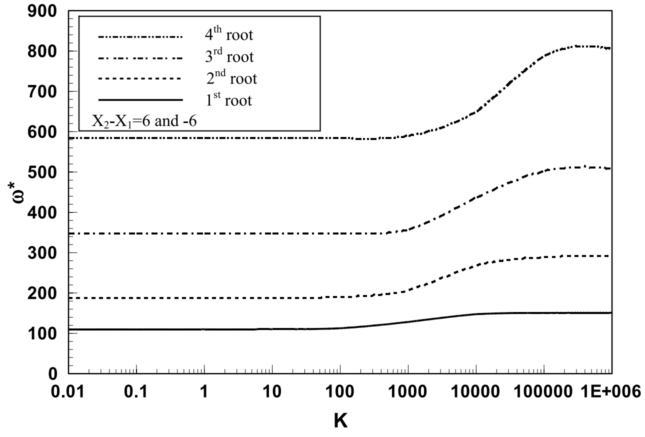

Figure 4. Change in the first four eigenvalues as function of the control parameter K for $x_2 - x_1 = 6$.

Here, E_k are again found by Cramer's rule and found like $E_1 = H_1 K W(x_1, 0)e^{i\xi x_2}$, $E_2 = H_2 K W(x_1, 0)e^{i\xi x_2}$, $E_3 = H_3 K W(x_1, 0)e^{i\xi x_2}$ $E_4 = H_4 K W(x_1, 0)e^{i\xi x_2}$, where H_k are formed using the coefficients in equations (16), as the ratio of two determinants.

When the constants E_k are substituted back into the solution, the result is

$$\hat{W} = K W(x_1, 0)e^{i\xi x_2}\{[H_1 + h(y)D_1(0)]e^{r_1 y} + [H_2 + h(y)D_2(0)]e^{r_2 y} + [H_3 + h(y)D_3(0)]e^{r_3 y} + [H_4 + h(y)D_4(0)]e^{r_4 y}\}. \tag{17}$$

We can simply write

$$\hat{W} = K W(x_1, 0)e^{i\xi x_2}\hat{J}(\xi, y), \tag{18}$$

where $\hat{J}(\xi, y) = e^{i\xi x_2}\{H_1 e^{r_1 y} + H_2 e^{r_2 y} + H_3 e^{r_3 y} + H_4 e^{r_4 y}\}$.

Taking inverse Fourier transform of equation (18), we obtain

$$W(x, y) = K W(x_1, 0)\, J(x, y), \tag{19}$$

where $J(x, y)$ is the inverse Fourier transform of the factors involving ξ in equation (18). Writing $x = x_1$ and $y = 0$ in equation (18), the terms $W(x_1, 0)$ cancel and we obtain

$$1 = J(x_1, 0)K, \tag{20}$$

as the eigenvalue equation. It involves, in addition to the eigenvalue ω, control gain constant K, and measurement and actuation point locations on the y-axis, x_1 and x_2. Thus, we are able to retrieve the equation satisfied by the controlled eigenvalues although we did not solve the problem explicitly. Furthermore, written explicitly

$$J(x_1, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi(x_2 - x_1)}(H_1 + H_2 + H_3 + H_4)\, d\xi,$$

depends only on the difference $x_2 - x_1$ and not on x_1 and x_2 separately. This is also reasonable since the plate is infinite in the x-direction.

3. Conclusions

Equation (19) gives the eigenvalue ω as a function of the control parameter K and the difference between measurement and actuation locations $x_2 - x_1$. It was verified that equation (19) gives the eigenvalues without any control when K approaches zero.

Figures 2, 3, 4 show the first four eigenvalues as a function of the control parameter K for various values of $x_2 - x_1$. It is to be noted that $H_1 + H_2 + H_3 + H_4$ is an even function of ξ, therefore the results are independent of the sign $x_2 - x_1$ which is also reasonable.

It appears that the (first four) eigenvalues do not change appreciably until the control parameter reaches the value of 100. When measurement and actuation are at the same point $(x_2 - x_1 = 0)$ the eigenvalues decrease slightly after $K = 100$ and then they increase. For large values of K the eigenvalues become independent of K and each eigenvalue achieves a constant value regardless of the value of $x_2 - x_1$. The control force required of the actuator is $Q\,w = \frac{KD}{b^2}w$, where w is the measured displacement. The results presented show that the non-dimensional control constant should be larger than about 100 to have any effect on the plate eigenfrequencies. The actual force also depends on the plate bending rigidity, plate width and the measured displacement. The resulting force is within common linear actuators for typical cases.

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