The Dugdale solution for two unequal straight cracks weakening in an infinite plate

R R BHARGAVA¹ and SHEHZAD HASAN²

¹Department of Mathematics, Indian Institute of Technology Roorkee,
Roorkee 247 667

²Department of Mathematics, Jamia Millia Islamia, New Delhi 110 025
e-mail: rajrbfma@iitr.ernet.in; drshehzadhasan@gmail.com

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Abstract. A crack arrest model is proposed for an infinite elastic perfectly-plastic plate weakened by two unequal, quasi-static, collinear straight cracks. The Dugdale model solution is obtained for the above problem when the developed plastic zones are subjected to normal cohesive quadratically varying yield point stress. Employing complex variable technique and the superposition principle the desired solution is obtained. A qualitative study of load required to arrest the plastic zones opening with respect to affecting parameters viz. inter-crack distance, plastic zone and crack length is carried out. The results obtained are presented graphically.

Keywords. Crack arrest; Dugdale model; plastic zone.

1. Introduction

The study of collinear crack problems becomes imperative from the view point of its occurrence in many real life situations. Such situations are encountered in rocks science to material manufacturing and to mechanical engineering, etc. For example, rock media is practically always contains a network of cracks, defects in the form of crack or cavity or both nucleate in course of manufacturing process and for service life of mechanical components, etc. To be able to predict the integrity in such situation it is desired to model to such cases and investigate.

Dugdale (1960) proposed a simple model when a straight slit weakening a plate is opened in Mode-I type deformation under tensile force applied remotely. The rims of the plastic zones formed were taken to be closed by normal uniform constant yield point stress. This model was modified by Harrop (1978) when rims of the plastic zones were closed by cohesive normal yield point stress distribution varying as an arbitrary second order polynomial. Theocaris (1983) found Dugdale model solution for two collinear straight cracks. Bhargava & Hasan (2001) have proposed a crack closure model for a circular arc cracks weakening a plate. The model was modified by Wang *et al* (2002) for the case when applied load was characterized by the specific remote strain. Beretta & Carboni (2005) used enriched finite element penalty functions method for modelling interface crack with contact zones. An analytic solution

for two equal length collinear strip yield cracks weakening an unbounded plate has been obtained by Collins & Cartwright (2001) using complex variable technique developed by Muskhelishvili (1953). The problem was further extended by Nishimura (2002) for the case of two collinear unequal straight cracks weakening an infinite sheet. Li Jia (1997) established a mixed mode Dugdale–Barenblatt model for a semi-infinite crack in an ideally elastic-plastic thin plate which was loaded by a pair of self-equilibrating concentrated forces, at the crack rims. An analytic solution was obtained for infinite elastic perfectly-plastic plate with center crack loaded by two pairs of point tensile force, by Wang & Zhou (2004).

Present paper obtains a modified Dugdale model solution for two unequal asymmetric collinear hairline straight cracks weakening an unbounded plate. Remotely applied tensile forces open the faces of the cracks in mode-I deformation developing a plastic zone ahead of each of the tip of the cracks. These developed plastic zones rims are closed by prescribing normal cohesive quadratically varying yield point stress distribution, thus arresting crack from further opening.

2. Mathematical equations

According to Muskhelishvili's (1953) complex variable formulation, the stress components $P_{ij}(i, j = x, y)$ may be expressed in terms of two complex potentials $\Phi(z)$ and $\Omega(z)$ as

$$P_{yy} - iP_{xy} = \Phi(z) + \Omega(\bar{z}) - (z - \bar{z})\Phi'(z), \quad (1)$$

where $\Omega(z) = \Phi(z) + z\Phi'(z) + \bar{\Psi}(z)$ and a bar over the function denotes its complex conjugate. Prime after the function denotes its differentiation with respect to the argument of the functions.

Consider an unbounded plate weakened by n unequal collinear straight cracks $L_i (i = 1, 2, 3, \dots, n)$ with end points a_i, b_i . The configuration so obtained is subjected to following boundary conditions:

- (a) $P_{yy}^{\pm}$ act along the rims of the crack $L_i (i = 1, 2, 3, \dots, n)$.
- (b) No stresses act at infinite boundary of the plate.
- (c) Displacements are single-valued around the rims of cracks.

Boundary condition (a) together with (1) yields following two Hilbert problems

$$P_{yy}^+ - iP_{xy}^+ = \Phi^+(t) + \Omega(t)^-; \quad P_{yy}^- - iP_{xy}^- = \Phi^-(t) + \Omega(t)^+, \quad (2)$$

under the assumption $\lim_{y \rightarrow 0} \{y\Phi'(t + iy)\} = 0$. Superscript $+$ denotes the value of the function when approached from $y > 0$ and the superscript $-$ denotes the value of the function when approached from $y < 0$.

Solution of (2) may be written, using Muskhelishvili's (1953), as

$$\Phi(z) = \frac{1}{2\pi i X(z)} \int_L \frac{X(t)p(t)}{t - z} dt + \frac{1}{2\pi i X(z)} \int_L \frac{q(t)}{t - z} dt + \frac{P_n(z)}{X(z)},$$

and

$$\Omega(z) = \frac{1}{2\pi i X(z)} \int_L \frac{X(t)p(t)}{t - z} dt - \frac{1}{2\pi i X(z)} \int_L \frac{q(t)}{t - z} dt + \frac{P_n(z)}{X(z)}, \quad (3)$$

where

$$X(z) = \prod_{k=1}^n (z - a_k)^{1/2} (z - b_k)^{1/2}, \quad L = \bigcup_{i=1}^n L_i,$$

$$2p(t) = (P_{yy}^+ + P_{yy}^-) - i(P_{xy}^+ + P_{xy}^-); \quad 2q(t) = (P_{yy}^+ - P_{yy}^-) - i(P_{xy}^+ - P_{xy}^-)$$

and

$$P_n(z) = C_0 z^n + C_1 z^{n-1} + \dots + C_n. \quad (4)$$

Arbitrary constants C_i ($i = 1, 2, 3, \dots, n$) are determined using boundary conditions (b) and (c), together with (1 and 5) and single-valuedness condition of displacement, taken from Muskhelishili (1953)

$$2(\kappa + 1) \int_L \frac{P_n(t)}{X(t)} dt + \kappa \int_L [\Phi_0^+(t) - \Phi_0^-(t)] dt + \int_L [\Omega_0^+(t) - \Omega_0^-(t)] dt = 0. \quad (5)$$

3. Statement of the problem

An infinite homogeneous elastic perfectly-plastic plate assumed to occupy xoy -plane. The plate is weakened by two unequal collinear hairline straight cuts L_1 and L_2 occupy the intervals $[d_1, c_1]$ and $[b_1, a_1]$, on ox -axis, respectively. A uniform constant tension, σ_∞ , is applied at infinite boundary in a direction perpendicular to the rims of the cracks. Consequently, the faces of the cracks open in Mode-I type deformations forming a plastic zones $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 ahead of each tip d_1, c_1, b_1 and a_1 of the cracks, respectively. Each of the plastic zone $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 occupy the interval $[d, d_1], [c_1, c], [b, b_1]$ and $[a_1, a]$, respectively, along the ox -axis. Rims of these plastic zones are subjected to the stress $P_{yy}^\pm = t^2 \sigma_{ye}$, $P_{xy}^\pm = 0$, where σ_{ye} denotes yield point stress of the plate and t is any point on any of the plastic zone, Γ_i ($i = 1, 2, 3, 4$). Figure 1 shows the configuration of the problem.

Figure 1. Schematic representation of the configuration of the problem.

4. Solution of the problem

The above problem is solved using principle of superposition of the stress intensity factors for two component problems contributing towards the stress singularity. These two problems are appropriately derived from the above stated problem and are termed as Problem-I and Problem-II.

4.1 Statement and solution of problem-I

An infinite homogeneous elastic perfectly-plastic plate is weakened by two unequal hairline collinear straight cracks Q_1 and Q_2 . The crack: $Q_1 (= \Gamma_1 U L_1 U \Gamma_2)$ occupies the interval $[d, c]$ and crack $Q_2 (= \Gamma_3 U L_2 U \Gamma_4)$ occupies the interval $[b, a]$ along ox -axis. Boundary conditions of the problems are

- (i) Rims of the cracks Q_1 and Q_2 are stress free.
- (ii) Infinite boundary of the plate is subjected to stress distribution $P_{yy}^\pm = \sigma_\infty$, $P_{xy}^\pm = 0$.
- (iii) Displacements are single-valued around the rims of cracks Q_1 and Q_2 .

Using boundary conditions (i), (iii) and (3), the desired potential $\Phi^I(z)$ may be written as

$$\Phi^I(z) = \frac{C_0 z^2 + z C_1 + C_2}{X(z)} - \frac{\sigma_\infty}{4}, \quad (6)$$

where superscript I denotes that the function refers to Problem-I

$$C_0 = \frac{\sigma_\infty}{2}, \quad C_1 = -\frac{\sigma_\infty}{2} \frac{\begin{vmatrix} Y_2 & Y_0 \\ W_2 & W_0 \end{vmatrix}}{\begin{vmatrix} Y_1 & Y_0 \\ W_1 & W_0 \end{vmatrix}} = \frac{\sigma_\infty}{2} K_1, \quad C_2 = -\frac{\sigma_\infty}{2} \frac{\begin{vmatrix} Y_1 & Y_2 \\ W_1 & W_2 \end{vmatrix}}{\begin{vmatrix} Y_1 & Y_0 \\ W_1 & W_0 \end{vmatrix}} = \frac{\sigma_\infty}{2} K_2$$

$$X(z) = \sqrt{z-a} \sqrt{z-b} \sqrt{z-c} \sqrt{z-d} \quad (7)$$

$$W_0 = g V_0, \quad W_1 = g[a V_0 + (d-a) V_1], \quad W_2 = g[a^2 V_0 + 2a(d-a) V_1 + (d-a)^2 V_2]$$

$$V_0 = F(k), \quad V_1 = \Pi(\alpha^2, k),$$

$$V_2 = \frac{1}{2(\alpha^2 - 1)(k^2 - \alpha^2)} [\alpha^2 E(k) + (k^2 - \alpha^2) F(k) + (2k^2 \alpha^2 + 2\alpha^2 - \alpha^4 - 3k^2) \Pi(\alpha^2, k)]$$

$$g = \frac{2}{\sqrt{a-c} \sqrt{b-d}}, \quad \alpha^2 = \frac{d-c}{a-c} < 0, \quad k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}$$

$$Y_2 = g[c^2 U_0 + 2c(b-c) U_1 + (b-c)^2 U_2], \quad Y_1 = g[c U_0 + (b-c) U_1], \quad Y_0 = g U_0,$$

$$\beta^2 = \frac{a-b}{a-c} < 1, \quad \varphi = \frac{\pi}{2}, \quad U_0 = F(k), \quad U_1 = \Pi(\beta^2, k),$$

$$U_2 = \frac{1}{2(\beta^2 - 1)(k^2 - \beta^2)} [\beta^2 E(k) + (k^2 - \beta^2) F(k) + (2k^2 \beta^2 + 2\beta^2 - \beta^4 - 3k^2) \Pi(\beta^2, k)]$$

and $F(k)$, $\Pi(\alpha^2, k)$ are the complete elliptic integral of first and third kind, respectively.

4.2 Statement and solution of problem-II

Configuration of the problem remains the same as stated in section 4.1. The boundary conditions for this problem are as follows:

- (p) Segments $\Gamma_i (i = 1, 2, 3, 4)$ of cracks Q_1 and Q_2 are subjected to stress distribution $P_{yy}^\pm = t^2 \sigma_{ye}$, $P_{xy}^\pm = 0$.
- (q) Segments $L_i (i = 1, 2)$ of cracks Q_1 and Q_2 are stress free.
- (r) Infinite Boundary of the plate is stress free.
- (s) Displacements are single-valued around cracks Q_1 and Q_2 .

Boundary conditions (p) and (q) when substituted in (1) yield following two Hilbert problems

$$\begin{aligned} \{\Phi^{II}(t) + \Omega^{II}(t)\}^+ + \{\Phi^{II}(t) + \Omega^{II}(t)\}^- &= 2t^2 \sigma_{ye}, \\ \{\Phi^{II}(t) - \Omega^{II}(t)\}^+ - \{\Phi^{II}(t) - \Omega^{II}(t)\}^- &= 0, \end{aligned} \quad (8)$$

where superscript II denotes that the function refers to Problem II. Desired potential $\Phi^{II}(z)$ may be written using (3) putting $p(t) = t^2 \sigma_{ye}$ and $q(t) = 0$ and integrating

$$\Phi^{II}(z) = \frac{\sigma_{ye}}{2\pi i X(z)} \int_{\bigcup_{i=1}^4 \Gamma_i} \frac{t^2 X(t)}{t - z} dt + \frac{D_1 z + D_2}{X(z)}, \quad (9)$$

where $X(z)$ is same as given by (7). Integral is evaluated using Byrd & Friedman (1954). D_1 and D_2 are obtained using boundary conditions (r) and (s) together with (1, 5, 9).

The function $\Phi^{II}(z)$

$$\Phi^{II}(z) = \frac{\sigma_{ye}}{2\pi X(z)} \{M + zN + z^2 F + z^3 G + X^2(z)H + zX^2(z)J_0\} + f(z) \quad (10)$$

where

$$M = -Q - H_4 E - H_1 H_4 J_0, \quad N = -P + H_3 E - H_4 J_0 + H_1 H_3 J_0,$$

$$F = C - H_2 E + H_3 J_0 - H_1 H_2 J_0, \quad G = D + H_1 E - H_2 J_0 + H_1^2 J_0,$$

$$H = E + H_1 J_0 \quad A = \{J_5 - H_1 J_4 + H_2 J_3 - H_3 J_2 + H_4 J_1\}$$

$$B = \{J_4 - H_1 J_3 + H_2 J_2 - H_3 J_1 + H_4 J_0\}$$

$$C = J_3 - H_1 J_2 + H_2 J_1 - H_3 J_0, \quad D = J_2 - H_1 J_1 + H_2 J_0, \quad E = J_1 - H_1 J_0,$$

$$H_1 = a + b + c + d, \quad H_2 = ab + ac + ad + bc + bd + cd,$$

$$H_3 = abc + abd + acd + bcd, \quad H_4 = abcd$$

$$J_k = (R_k + S_k - T_k - U_k), \quad k = 0, 1, 2, 3, 4, 5$$

$$R_k = \int_d^{d_1} \frac{t^k}{\sqrt{a-t}\sqrt{b-t}\sqrt{c-t}\sqrt{t-d}} dt, \quad S_k = \int_{c_1}^c \frac{t^k}{\sqrt{a-t}\sqrt{b-t}\sqrt{c-t}\sqrt{t-d}} dt$$

$$T_k = \int_b^{b_1} \frac{t^k}{\sqrt{a-t}\sqrt{t-b}\sqrt{t-c}\sqrt{t-d}} dt, \quad U_k = \int_{a_1}^a \frac{t^k}{\sqrt{a-t}\sqrt{t-b}\sqrt{t-c}\sqrt{t-d}} dt$$

$$\begin{aligned}
 P &= C \frac{\left[\frac{W_2}{W_0} - \frac{Y_2}{Y_0} \right]}{\left[\frac{W_1}{W_0} - \frac{Y_1}{Y_0} \right]} + D \frac{\left[\frac{W_3}{W_0} - \frac{Y_3}{Y_0} \right]}{\left[\frac{W_1}{W_0} - \frac{Y_1}{Y_0} \right]} + E \frac{\left[\frac{W_4}{W_0} - \frac{Y_4}{Y_0} \right]}{\left[\frac{W_1}{W_0} - \frac{Y_1}{Y_0} \right]} + J_0 \frac{\left[\frac{W_5}{W_0} - \frac{Y_5}{Y_0} \right]}{\left[\frac{W_1}{W_0} - \frac{Y_1}{Y_0} \right]} \\
 Q &= C \frac{\left[\frac{W_2}{W_1} - \frac{Y_2}{Y_1} \right]}{\left[\frac{W_0}{W_1} - \frac{Y_0}{Y_1} \right]} + D \frac{\left[\frac{W_3}{W_1} - \frac{Y_3}{Y_1} \right]}{\left[\frac{W_0}{W_1} - \frac{Y_0}{Y_1} \right]} + E \frac{\left[\frac{W_4}{W_1} - \frac{Y_4}{Y_1} \right]}{\left[\frac{W_0}{W_1} - \frac{Y_0}{Y_1} \right]} + J_0 \frac{\left[\frac{W_5}{W_1} - \frac{Y_5}{Y_1} \right]}{\left[\frac{W_0}{W_1} - \frac{Y_0}{Y_1} \right]} \\
 f(z) &= \frac{X(z)\sigma_{ye}}{2\pi i} \int_{\bigcup_{i=1}^4 \Gamma_i} \frac{z^2}{(t-z)X(t)} dt.
 \end{aligned} \tag{11}$$

5. Plastic zone length

Lengths of each plastic zone are obtained using the Dugdale hypothesis that stresses remain finite at every point of the body yields

$$\frac{\sigma_\infty}{\sigma_{ye}} = - \frac{(M + zN + z^2F + z^3G)}{\pi(z^2 + zK_1 + K_2)}. \tag{12}$$

Four non-linear equations are obtained from the above conditions which enables to determine a , b , c and d for prescribed values of a_1 , b_1 , c_1 , d_1 , and $\frac{\sigma_\infty}{t^2\sigma_{ye}}$. Plastic zones are then calculated using the relation $|d_1 - d|$, $|c - c_1|$, $|b_1 - b|$ and $|a - a_1|$.

Figure 2. Variation between load ratio and a ratio of plastic zone length at the outer tip d_1 of the smaller crack.

Figure 3. Variation between load ratio and a ratio of plastic zone length at the inner tip c_1 of the smaller crack.

6. Case study

An illustrative example is presented to study the variation of applied load ratio $\frac{\sigma_\infty}{t^2\sigma_{ye}}$ (load applied at infinite boundary to load applied on the developed plastic zone) versus ratio of developed plastic zone to relevant crack length $\frac{\Gamma_i}{L_i}$. The investigations are carried out for different crack length ratios, $\frac{L_2}{L_1}$, starting from the case of two cracks being of equal in length to the case when the bigger crack length being four times that of the smaller crack length. For the case when crack lengths are equal, the results match with the results of Theocaris (1983).

Figure 4. Variation between load ratio and a ratio of plastic zone length at the inner tip b_1 of the bigger crack.

Figure 5. Variation between load ratio and a ratio of plastic zone length the outer tip a_1 of the bigger crack.

Figure 2 depicts the variations of $\frac{\Gamma_1}{L_1}$ with respect to load ratio $\frac{\sigma_\infty}{t^2\sigma_{ye}}$, when it is varied from zero to one at the tip d_1 of smaller crack L_1 . A parabolic increased is observed in load required to arrest crack opening as the plastic zone is increased. If the lengths of cracks are made unequal then comparatively less load is required to crack arrest although the behaviour remains the same.

Now, for this variation of plastic zone at the crack tip d_1 (is the maximum load sustainability at tip d_1) the variation plastic zone length at the three remaining tips of the cracks is investigated

Figure 6. Variation of load required with respect to inter crack distance at the exterior tip d_1 of the smaller.

Figure 7. Variation of load required with respect to inter crack distance at the interior tip c_1 of the smaller crack.

with respect to $\frac{\sigma_{\infty}}{t^2 \sigma_{ye}}$. The inner tip c_1 of the smaller crack is most distressed situation as shown in figure 3. The load bearing capacity at this tip is almost sixty percent less than at the tip d_1 and the plastic zone developed at tip d_1 is almost half the size of the plastic zone developed at the tip c_1 . As plastic zone size is increased the load required to arrest the plastic zones increased exponentially.

Figure 8. Variation of load required with respect to inter crack distance at the inner tip b_1 of the bigger crack.

Figure 9. Variation of load required with respect to inter crack distance at the interior tip c_1 of the smaller crack.

Figure 4 shows the variation of $\frac{\sigma_{\infty}}{t^2 \sigma_{ye}}$ against $\frac{\Gamma_3}{L_1}$ (plastic zone increases at the inner tip b_1 of the bigger crack). For the maximum load bearing capacity at the tip d_1 , it is observed that almost half the load is required to arrest the crack at this inner tip of the bigger crack as compared to that at the exterior tip d_1 of the smaller crack. As crack length ratio $\frac{L_2}{L_1}$ is increased, the developed crack zone size is much reduced and load required to arrest the developed plastic zone also reduced drastically.

While figure 5 shows that the smallest plastic zone develops at this exterior tip of the bigger crack tip. This far end tip remains most unaffected. As the length of the bigger crack is increased a very small plastic zone develops at this tip, as expected.

Figures 6, 7, 8 and 9 show the variation of load required to arrest the crack opening as the inter crack distance between the two cracks is increased. It may be noted that the behaviour of load ratio to inter-crack distance/crack length at the tip d_1 , b_1 is similar, although the plastic zone is bigger at the tip of smaller crack. It is important to point out that the behaviour of the load required to arrest the crack at the interior tips of the bigger and smaller cracks is opposite to each other, as expected.

Comparing the behaviour at the interior tips of the bigger and smaller crack it is observed that the interior tip of the smaller crack is most stressed and shows a parabolic behaviour while the interior tip of the bigger crack shows a closing behaviour but the variation is comparatively much linear.

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