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State-of-the-art review on fracture analysis of concrete structural components

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Abstract. This paper presents a critical review of literature on fracture analysis of concrete structural components. Review includes various fracture models, tension softening models, methodologies for crack growth analysis and remaining life prediction. The widely used fracture models which are based on fictitious crack approach and effective elastic crack approach have been explained. Various tension softening models such as linear, bi-linear, tri-linear, etc. have been presented with appropriate expressions. From the critical review of models, it has been observed that some of the models have complex expressions involving many parameters. There is a need to develop some more generalised models. Studies have been conducted on crack growth analysis and remaining life prediction using linear elastic fracture mechanics (LEFM) principles. From the studies, it has been observed that there is significant difference between predicted and experimental observations. The difference in the values is attributed to not considering the tension softening effect in the analysis.

Keywords. Concrete fracture; concrete fracture models; tension softening models; crack growth; remaining life.

1. Introduction

Concrete is a widely used material that is required to withstand a large number of cycles of repeated loading in structures such as highways, airports, bridges and ocean structures. The present state-of-the-art of designing such structures against the distress due to fatigue loading is largely empirical, gained by many years of experience. As long as the designer is dealing with structures made of similar materials to those for which the relationships were derived, the performance can be reasonably well predicted. However, as conditions change, a need exists for a rational approach. Concrete generally contains numerous flaws, such as holes or air pockets, pre-cracked aggregates, lack of bond between aggregate and matrix, etc. from

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which cracking may originate. The words ‘crack’ and ‘flaw’ tend to be used interchangeably. But, while all cracks can be considered flaws, all flaws cannot be considered as cracks. The distinction is the sharpness of the crack tip, a crack being with a very small radius of curvature. When the tensile strength of a material is reached in a structure, cracking will occur. During fatigue cyclic loading, the flaw is blunted and re-sharpened and it is reasonable to assume that the crack so formed will be the nucleus of crack propagation that may lead to failure and that the crack will initiate after the first loading cycle. Cracks generally propagate in a direction, which is perpendicular to the maximum tensile stress. In heterogeneous materials, crack tends to follow the weakest path in the material. While the shape of the crack is likely to be highly irregular, it is expected that the irregularities will be smoothed out and the cracks will grow in a slow manner to a simple shape along which the stress intensity factor (SIF) is nearly uniform. Fracture mechanics is a rapidly developing field that has great potential for application to concrete structural design (Karihaloo 1995; Shah *et al* 1995; Van 1997; Bazant 1998, 2002).

For an ideally brittle material, the stress–strain curve is linearly elastic up to the maximum stress, at which point an initial flaw catastrophically propagates, leading to failure. A typical tensile stress-elongation curve for an ideally brittle material for which linear elastic fracture mechanics (LEFM) is valid is shown in figure 1a. For a quasi-brittle material like concrete, a substantial non-linearity exists before the maximum stress. The mechanisms of deformation beyond the proportional limit f_y (refer figure 1b) are not clearly understood. Initially, randomly distributed micro-cracks are formed. At some point before the peak stress, micro-cracks begin to localize into a macro-crack that critically propagates at the peak stress. Strain softening is observed under steady-state propagation of this crack. Under a closed-loop displacement-controlled testing condition, it is observed that the displacement during the post-peak stage consists of opening of the major crack accompanied by unloading of the rest of the specimen.

The fracture behaviour of concrete is greatly influenced by the Fracture Process Zone (FPZ). FPZ, defined as the zone in which the material undergoes softening damage (tearing), is quite small, in concrete and rock fracture the plastic flow is next to non-existent and the non-linear zone is almost entirely rolled by FPZ. Such materials are now commonly called quasi-brittle. The variation of the along the structure thickness or width is usually neglected. The inelastic fracture response due to the presence of FPZ may then be taken into account by a cohesive pressure acting on the crack faces. Figure 2 shows FPZ in brittle-ductile materials and quasi-brittle materials (Bazant 2002).

To model this behaviour using discrete crack fracture mechanics, it is assumed that an initial crack begins to propagate at the proportional limit f_y and continues to propagate in a stable manner until the peak stress. When the crack extends in concrete, new crack surfaces are formed along the path of the initial crack tip. The newly formed crack surfaces may be in contact and this leads to toughening mechanisms in FPZ such as aggregate bridging. Further, they may continue to sustain some normal tensile stress that is characterized by a material tensile stress-separation relationship.

Distribution of the tensile stresses on the newly formed crack surfaces depends on the definition of FPZ ahead of the initial crack tip. If FPZ does not include the effect of micro-cracks ahead of the newly formed crack tip as shown in figure 3a, the normal tensile stress gradually increases from the initial (open) crack tip and reaches the tensile strength of the material, f_t, at the end of FPZ (Bazant 1998).

It is noted that the tensile strength, f_t is different than the conventional concrete tensile strength obtained from a regular tensile test. The former is regarded as a material fracture

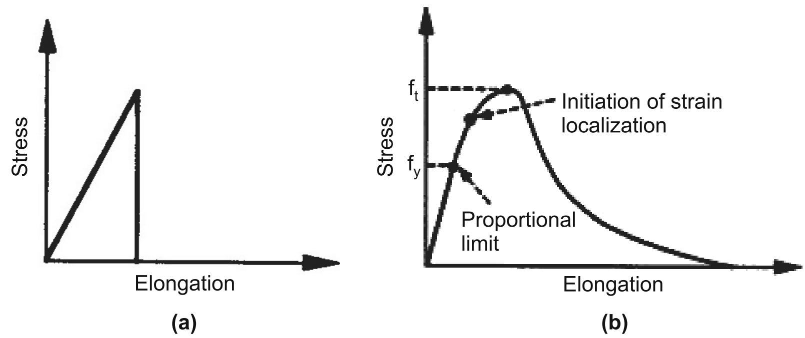


Figure 1. Tensile stress-elongation curves for **(a)** linear elastic material, **(b)** quasi-brittle material.

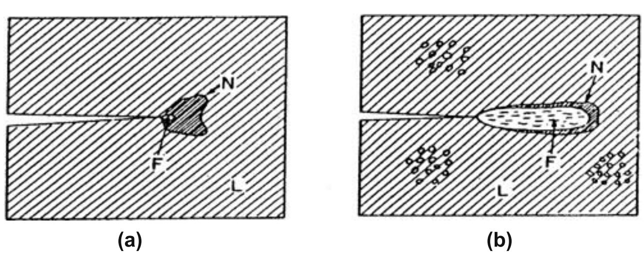


Figure 2. FPZ in brittle-ductile materials. **(a)** Ductile-brittle (metals), **(b)** Quasi-brittle (concrete).

parameter, whereas the latter depends on the material as well as the size and geometry of the tested specimen and the testing procedure. If FPZ is defined such that it includes effects of micro-cracks ahead of the newly formed crack tip as shown in figure 3b, the normal tensile stress gradually increases from the initial crack tip and reaches its maximum value (the tensile strength of the material) and drops to the proportional limit f_y at the end of FPZ, as shown in figure 3b, where the proportional limit f_y corresponds to initiation of micro-cracks in the material.

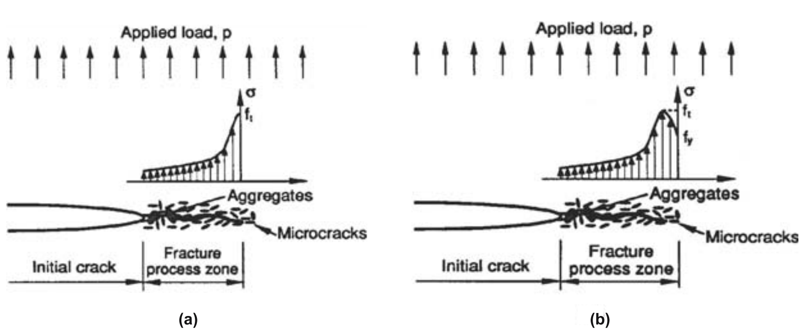


Figure 3. Concrete crack and: **(a)** not including effect of crack-tip micro-cracks, and **(b)** including effect of crack-tip micro-cracks.

2. Fracture mechanics of brittle materials: A critical review

Coulomb (1776) pioneered investigation of the fracture of stones in compression. Galilei (1638) investigated the influence of size in fracture of structures. The original concept of fracture energy was conceived by Griffith (1920). Griffith theory predicted that compressive strength of a material is eight times greater than its tensile strength. Later Irwin (1957) provided extension of Griffth theory to an arbitrary crack and proposed the criterion for crack growth. Further, Irwin showed using Westergaard's method, the stress field in the area of the crack tip is completely determined by the quantity K, called stress intensity factor (SIF). The first application of fracture mechanics to concrete was made by Kaplan (1961) using LEFM principles. Clintock & Walsh (1962) introduced the concept of friction between crack faces. Barenblatt (1959) and Dugdale (1960) made the first attempt at including the cohesive forces in the crack tip region within the limits of elasticity theory. Barenblatt (1959) assumed that cohesive forces acted in a small zone near the crack ends such that the faces closed smoothly. The distribution of these forces is generally unknown. For Dugdale (1960), the distribution of the closing forces is known and constant according to an elastic-perfectly plastic material. A major advance in concrete fracture was made by Hillerborg *et al* (1976). Hillerborg's model (1976) includes the tension softening process zone through a fictitious crack ahead of the pre-existing crack whose lips are acted upon by closing forces such that there is no stress concentration at the tip of this extended crack. Kesler *et al* (1972) showed that the classical LEFM of sharp cracks was inadequate for normal concrete structure and this conclusion was supported by the results of Walsh (1972, 1976) who tested geometrically similar notched beams of different sizes. Inspired by the softening and plastic models of FPZ initiated in the works of Barenblatt (1959) and Dugdale (1960) and the models for materials other than concrete was developed by Rice (1968), Smith (1974), Knauss (1973, 1974), Wnuk (1974), Palmer & Rice (1973) and Kfouri & Rice (1977). Bazant (1976) and Bazant & Cedolin (1979) used a smeared crack model to model cracking in concrete. In this model, the crack front is assumed to consist of a diffuse zone of micro-cracks and the stresses that close FPZ faces are represented through a stress–strain softening law. Hillerborg *et al* (1976) introduced the concept of a characteristic length, which is a unique material property. Carpinteri (1980) proposed a parameter's' as measure of concrete structural brittleness but later introduced the energy brittleness number. Extensive research work was carried out towards numerical modelling of fracture and size effect in plain concrete using lattice model (Hrennikoff 1941; Roelfstra *et al* 1985; Burt & Dougill 1977; Herrman *et al* 1989; Herrman 1991; Schlangen & Van 1991; 1992; Raghuprasad *et al* 1994; Ince *et al* 2003; Arslan *et al* 2002; Karihaloo *et al* 2003).

The concept of lattice model is discretization of the continuum by line elements such as bar and beam elements, which can transfer forces and moments. The advantage with lattice model is the heterogeneity of the material can be modelled/represented by assigning different strength and/or stiffness values to the individual lattice members. Another advantage with this model is that it is possible to identify micro cracking, crack branching, crack tortuosity and bridging.

Hillerborg (1983, 1985) improved the cohesive crack model and adapted to concrete. The finite element analysis showed that the cohesive crack model (also called the fictitious crack model) predicts, for the flexural failure of unnotched plain concrete beams, a deterministic size effect, different from the Weibull statistical size effect. This conclusion was strengthened and the model was further refined by Petersson (1981). Lange *et al* (1993) quantified the texture of fracture surfaces using image analysis techniques to compute a roughness parameter and fractal dimension. A positive correlation between fracture surface roughness and fracture toughness was demonstrated. Sundara Raja Iyengar *et al* (1996) applied the fictitious crack method to determine the load deflection diagrams of notched plain concrete beam under three point bending using various forms of strain softening in the stress-deformation relationship and indicated that there is a need to determine a more realistic relationship.

An analytical study of the size effect due to localization of distributed cracking was begun in 1976. Later, a simple formula for the size effect, which describes the size effect for quasi-brittle failures preceded by large stable crack growth and allows determination of material fracture parameters from maximum load tests, was derived by Bazant (1983, 1984). The crack band model proposed by Bazant (1982, 1983), provides an almost equivalent alternative to the cohesive crack model. Bazant size effect law (1984) gives a measure of the brittleness of concrete elements. This model was shown to be in good agreement with the basic fracture data and size effect data and has been found to be convenient for programming (Bazant 1984; Bazant & Schell 1993; Bazant & Kazemi 1990). Nallathambi *et al* (1985) conducted experiments to study the influence of pre-crack, aggregate and specimen sizes on the fracture of concrete. A simple formula based on the experimental data was proposed to account for all the three size dependent effects. Bazant & Sun (1987) improved the size effect formula for diagonal shear. The improvement was in two aspects, namely, the effect of maximum aggregate size distinct from the effect of the relative beam size and to cover the effect of stirr-ups on the shear capacity of concrete. Bazant (1996) discussed three methods of fracture testing in the perspective of the size effect including the merits and limitations.

Appa Rao & Raghu Prasad (2002a) investigated the fracture properties of high strength concrete. It was observed that concrete becomes brittle with increase in compressive strength. Further, it was noted that fracture energy increases as the maximum size of coarse aggregate and compressive strength of concrete increases. Appa Rao & Raghu Prasad (2002b) conducted experiments to investigate the bond strength of the interface between mortar and aggregate. It was observed that the bond strength of the interface in tension is significantly low, though the mortars exhibited higher strength. The bond strength of the interface in shear (mode III) significantly increases as the roughness and phase angle of the aggregate surface increase. Bazant & Yavari (2005) examined the theories on size effect, namely, energetic statistical scaling and fractal geometry. The advantages and disadvantages in modelling the structural size effect by fractals are discussed. Emphasis was made on design aspect and codal provisions considering the size effect. Ragu Prasad & Renuka Devi (2007) proposed a modified fictitious crack model for plain concrete beam with vertical tortuous crack and analysed the effect of tortuousity of the cracks on various fracture parameters. Carpinteri *et al* (2008) introduced finite fracture mechanics criterion and applied to structures with sharp V-notches. It was found that the predicted values are in good agreement with the experimental results.

When the structural components are subjected to repetitive live loads of high-stress amplitude, according to classical theory, applied loads result in in-plane tensile stresses at the bottom of the components. The stress-state in such structures is often simulated with three-point bending tests. Plain concrete subjected to flexural loading fails owing to crack propagation. Repeated loading results in a steady decrease in the stiffness of the structure, eventually leading to failure. It is of interest to characterize the material behaviour subjected to such loading and study the crack propagation and remaining life resulting from such loading. The current approaches used to evaluate fatigue performance are mainly empirical. Fatigue equations based on the well known S–N concept have been developed. Implementation of the conventional S–N approach requires time-consuming experimental data collection for a given design case followed by statistical analysis. The resulting information is not applicable to other design cases with different loading configurations or boundary conditions. A severe limitation of the S–N approach is the inherent empiricism. The approach does not use fundamental material parameters that can be determined for use in design or evaluation. Mechanistic approaches that utilize the concept of fracture mechanics to study crack propagation from fatigue loading have also been proposed. For example, Perdikaris & Calomino (1987) showed that compliance measurements provide a convenient method for estimating the traction-free crack length of fatigued concrete specimens. Since then, many experimental investigations on fatigue crack propagation in concrete have been reported (Baluch *et al* 1987; Ramsamooj 1994; Stuart 1982; Subramaniam *et al* 2000; Takashi *et al* 1999; Toumi & Turatsinze 1998; Slowik *et al* 1996; Bazant & Xu 1991).

The rate of fatigue crack growth in concrete exhibits an acceleration stage that follows an initial deceleration stage. In the deceleration stage the rate of crack growth decreases with increasing crack length, whereas in the acceleration stage there is a steady increase in crack growth rate up to failure. They (Baluch *et al* 1987; Ramsamooj 1994; Stuart 1982; Subramaniam *et al* 2000; Takashi *et al* 1999; Toumi & Turatsinze, 1998; Slowik *et al* 1996; Bazant & Xu 1991) have attempted to apply the fracture mechanics principles to describe the crack growth during the acceleration stage of fatigue crack growth in concrete. It has been observed that the Paris law coefficients are dependent on the material composition potentially explaining the large differences in the values of the Paris law coefficients. From literature, it has also been observed that the research work towards crack growth analysis and remaining life prediction of concrete structural components considering tension softening is limited.

This paper presents a critical review of literature on fracture analysis of concrete structural components. Review includes various fracture models, tension softening models, crack growth analysis, and remaining life prediction. The widely used fracture models which are based on fictitious crack approach and effective elastic crack approach have been explained. Various tension softening models such as linear, bi-linear, tri-linear etc. have been presented with appropriate expressions. Studies have been conducted on crack growth analysis and remaining life prediction using linear elastic fracture mechanics principles. Observations from remaining life prediction studies have been highlighted. Directions for further research in this area have been discussed.

3. Nonlinear fracture mechanics for mode I quasi-brittle material

It is known that the fracture behaviour of concrete is greatly influenced by FPZ. An effective quasi-brittle crack is shown in figure 4(a), where an initial crack and the associated FPZ are presented by a crack with length ‘a’ (Shah *et al* 1995). The toughening mechanisms in FPZ are modelled by a cohesive pressure acting on the crack surfaces as described by Jenq & Shah (1985). The cohesive pressure σ(w) is a monotonic decreasing function of crack opening displacement w. The value of $\sigma(w)$ is equal to material tensile strength, f_t for $w = 0$ at the crack ‘tip’ (the end of FPZ). This implies that micro-cracks ahead of the crack tip are not included in FPZ.

When a concrete structure with a quasi-brittle crack is subjected to loading, the applied load results in an energy release rate, G_q at the tip of the effective quasi-brittle crack, where the subscript q stands for quasi-brittle materials. The energy release rate G_q may be divided into two portions: (i) the energy rate consumed during material fracturing in creating two surfaces, G_{Ic}, which is equivalent to the material surface energy, and (ii) the energy rate to overcome the cohesive pressure $\sigma(w)$ in separating the surfaces, G_σ, where the subscript σ

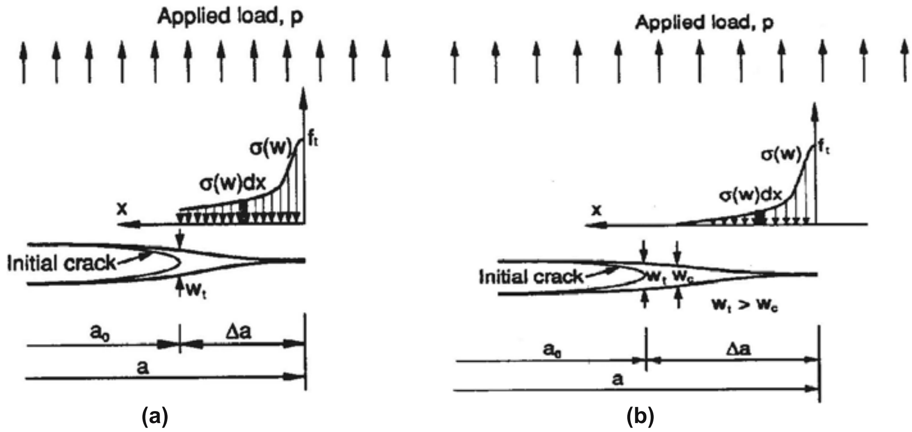


Figure 4. Modelling of quasi-brittle crack: **(a)** cohesive crack with crack surfaces in contact and **(b)** partially separated crack surfaces.

indicates this part of energy overcomes the cohesive pressure $\sigma(w)$ to open the crack. As a result, the energy release rate for a mode I quasi-brittle crack, G_q, can be expressed as

$$G_q = G_{Ic} + G_\sigma. \tag{1}$$

The value of G_{Ic}, can be evaluated based on LEFM and is called the critical energy release rate. Since G_σ is equal to the work done by the cohesive pressure over a unit length of the crack for a structure with a unit thickness, its value can be calculated using the following expression

$$G_\sigma = \frac{1}{\Delta a}\int_0^{\Delta a}\int_0^{w}\sigma(w)dxdw = \frac{1}{\Delta a}\int_0^{\Delta a}dx\int_0^{w}\sigma(w)dw = \int_0^{w}t\sigma(w)dw, \tag{2}$$

where $\sigma(w)$ is the normal cohesive pressure and w_t is the crack separation displacement at the initial crack tip, as shown in figure 4(a). Substituting (2) into (1) leads to

$$G_q = G_{Ic} + \int_0^{W_t}\sigma(w)dw. \tag{3}$$

It can be observed that when $w_t > w_c$ as shown in figure 4(b), the upper integral limit w_t in (3) should be replaced by w_c. Equation (3) indicates that for quasi-brittle fracturing, the energy release rate due to the applied load G_q is balanced by two fracture energy dissipation mechanisms. The Griffith–Irwin energy dissipation mechanism is represented by the fracture energy release rate G_{Ic}, whereas the Dugdale–Barenblatt energy dissipation mechanism is represented by the material traction term G_σ.

3.1 *Concrete fracture models*

Based on different energy dissipation mechanisms used, non-linear fracture mechanics models for quasi-brittle materials can be classified as a fictitious crack approach and an equivalent-elastic crack approach (or an effective-elastic crack approach). Fracture mechanics models using only the Dugdale–Barenblatt energy dissipation mechanism are usually referred to as the fictitious crack approach, whereas fracture mechanics models using only the Griffth–Irwin energy dissipation mechanism are usually referred to as the effective-elastic crack approach or equivalent elastic crack approach. Brief description of various models based on fictitious crack approach as well as effective-elastic crack approach is described in table 1.

4. Tension softening models

It is known that the cohesive crack model requires a unique $\sigma(w)$ curve to quantify the value of energy dissipation. The choice of the $\sigma(w)$ function influences the prediction of the structural response significantly, and the local fracture behaviour, for example the crack opening displacement, is particularly sensitive to the shape of $\sigma(w)$. Many different shapes $\sigma(w)$ curves, including linear, bilinear, trilinear, exponential, and power functions, have been previously used. Some of the widely used $\sigma(w)$ curves are listed in table 2.

The CEB–FIP Model Code (1990) also recommended a bilinear curve for $\sigma(w)$. However, the value of w_c depends on maximum aggregate size d_a. The value of σ_1 was assigned to be equal to $0{\cdot}15 f_t$ and the value of w_1 in units of millimeters is given by

$$w_1 = \frac{G_F - 22 w_c (G_F/k_d)^{0\cdot95}}{150(G_F/k_d)^{0\cdot95}}, \tag{4}$$

where the coefficient k_d also depends on the maximum aggregate size d_a. In the absence of test data, CEB–FIP Model Code also specifies empirical relations for the fracture parameters of the fictitious crack model in terms of the mean compressive strength f_c:

$$f_t = 0{\cdot}3(f_c + 8)^{2/3}, G_F = k_d f_c^{0\cdot7}, E = 10^4 f_c^{1/3}, \tag{5}$$

in which f_c, f_t, and E are in megapascals and G_F is in newtons per meter.

Since the assessment of the fracture behavior of a concrete structure is influenced by using different $\sigma(w)$ functions, reasonable and accurate determinations of the $\sigma(w)$ curve and the corresponding parameters become crucial for the cohesive crack approach. Experimental determination of $\sigma(w)$ directly from tension tests has been suggested by Gopalaratnam and shah (1985), but this is difficult and the results may vary with specimen size and shape. Li *et al* (1987) have proposed a J-integral-based method for obtaining the entire $\sigma(w)$ curve. Miller *et al* (1991) have computed the $\sigma(w)$ curve from parabolic crack profiles observed in fracture tests through laser holography interferometry. Yon *et al* (1997) and Du *et al* (1990) have applied Moire interferometry to deduce crack profiles that were used to determine the stress-opening relation in the process zone.

From the critical review of models, it can be observed that the models are subjective in view of the following reasons:

- Some of the models are complex and do not readily lend themselves to mathematical manipulation such as differentiation and integration.
- Number of models are divided into two separate expressions hence adding to the complexity of the model.
- Some of the models require parameters obtained through curve fitting methods

There is scope and need to develop more generalised models to represent realistic closing pressure distribution.

Table 1. Concrete fracture models.

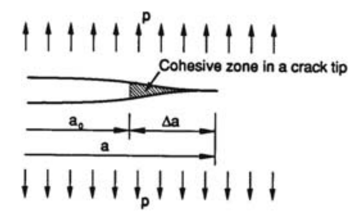
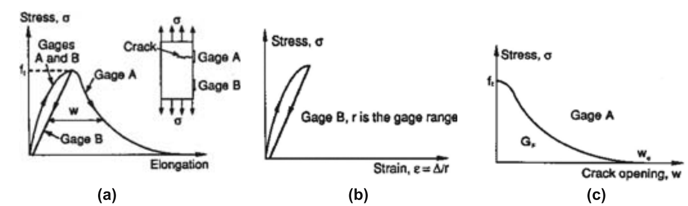
Sl. No	Model	Description	Figures
1	Fictitious crack approach	• Assumes that energy to create the new surface is small compared to that required to separate them • All energy produced by the applied load is completely balanced by the cohesive pressure (figure 5). $$Gq = \int_0^{W_t} \sigma(w)dw.$$	 **Figure 5.** Mode I crack for fictitious crack approach.
1·1	Fictitious crack model by Hillerborg *et al* (1976)	• First proposed a fictitious crack model for fracture of concrete • Area under entire softening stress elongation curve (figure 6) is given by $$G_F = \int_0^{wc} \sigma(w)dw,$$ where, w_c: critical crack opening displacement • Characteristic length, $l_{ch} = \frac{EG_F}{f_t^2}$ Where E = modulus of elasticity and G_F = fracture energy	 **Figure 6.** **(a)** Complete tensile stress-elongation curve, **(b)** Stress strain curve for uncracked section, **(c)** Stress-elongation curve for cracked section.

Table 1. Continued.

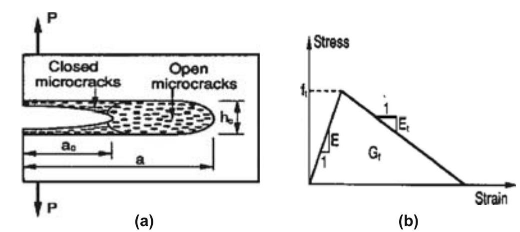
Sl. No	Model	Description	Figures
1·2	Crack band model by Bazant & Oh (1983)	• Modelled FPZ by a band of uniformly and continuously distributed micro-cracks (figure 7) • The energy consumed due to the crack advance per unit area of the crack band, G_f $$G_f = h_c\left(1 + \frac{E}{E_t}\right)\frac{f_t^2}{2E}$$	 **Figure 7.** Crack band model **(a)** Microcrack band fraction, **(b)** Stress–strain curve.
2	Effective-elastic crack approach	• Due to Griffith–Irwin energy dissipation mechanism • Models FPZ by using an equivalent traction-free elastic crack • energy release rate $G_q = G_{Ic}$	
2·1	Two-parameter Fracture Model by Jenq & Shah (1985)	• Based on elastic fracture response of structures • The value of CMOD at peak load, w_c is given by (figure 8) $CMOD_c = CMOD_c^e + CMOD_c^p$ $CMOD_c^e$ = elastic component of CMODc $CMOD_c^p$ = inelastic components of CMODc	

Table 1. Continued.

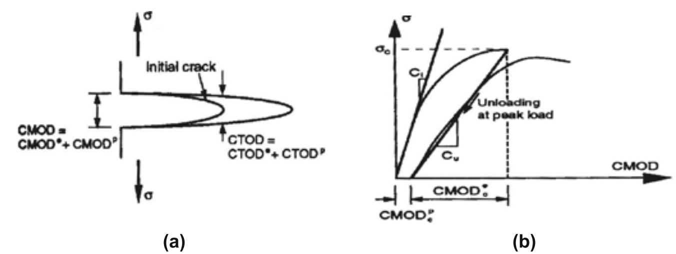
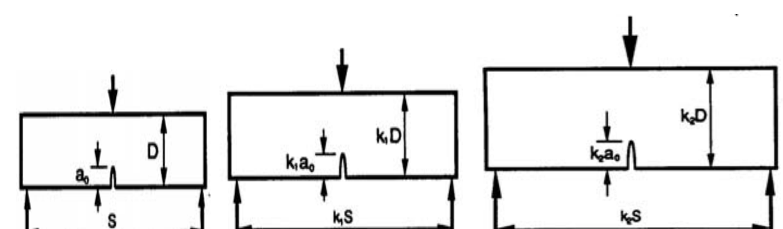
Sl. No	Model	Description	Figures
		• The value of $CMOD_c^e$ is given by $$CMOD_c^e = \frac{4\sigma_c a_c}{E} g_2\left(\frac{a_c}{b}\right)$$ • Critical crack tip opening displacement is given by $$CTOD_c^e = CMOD_c^e g_3\left(\frac{a_c}{b}, \frac{a_0}{a_c}\right)$$ • At critical fracture load the following conditions should be satisfied $$K_I = K_{IC}^S, \quad CTOD = CTODc$$	 **Figure 8.** **(a)** Elastic and fracture responses and **(b)** Loading and unloading procedure.
2·2	Size Effect Model by Bazant & Kazemi (1990) and Bazant (1986, 1989)	• The nominal stress σ_{NC} for geometrically similar structures (figure 9) at failure is given by $$\sigma_{Nc} = \frac{c_n P_c}{tD}$$ where p_c: critical fracture load, c_n: coefficient representing different types of structures	 **Figure 9.** Series of geometrically similar structures.

Table 1. Continued.

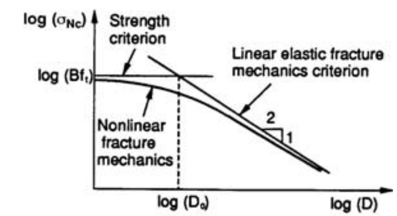
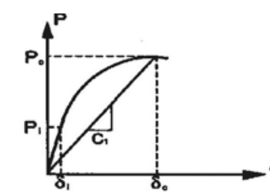
Sl. No	Model	Description	Figures
		• Nominal strength based on dimensional analysis and similitude arguments is given by $$\sigma_{Nc} = c_n \left[\frac{EG_f}{g'(a_0/D)c_f + g(a_0/D)D} \right]^{1/2}.$$ Figure 10 shows the nominal strength for a series of geometrically similar structures based on dimensional analysis	 **Figure 10.** Size effect on nominal failure stress.
2·3	Effective Crack Model by Karihaloo & Nallathambi (1989)	• Similar to two parameter fracture model • Secant compliance at the maximum load is used to determine the effective elastic crack length (figure 11)	 **Figure 11.** Load vs. displacement curve.
2·4	Finite Element Analysis (a) Discrete crack approach (Bazant 1976; Ngo & Scordelis 1967; Ingraffea 1977; Ingraffea & Saouma 1984; Swenson 1986; Wawrzynek & Ingraffea 1987)	• Crack path is assumed a priori and a mesh is arranged so that the path coincides with boundaries between elements • There are three basic issues – determination of location and initiation of crack – determining how the crack extends – determination of direction of crack extension	

Table 1. Continued.

Sl. No	Model	Description	Figures
		• Advantages – Interface elements are a natural way of describing the crack – Interface elements are economical – Mesh regeneration associated with crack propagation and direction change, can be accomplished automatically A graphical representation of the geometric discontinuity can be obtained automatically	• Disadvantages – compute intensive – Difficult to handle multi-cracks
	(b) Smeared Crack Approach (Rashid 1968; Pudhaphongsiriporn 1978; Bazant & Cedolin 1979; ACI Committee 446 1995; Sitaram 1993)	• Based on the concept of replacing the crack by a continuous medium with altered physical properties • Concept is similar to crack band model • Advantages – computational convenience – remeshing is not required – parallel cracks can be modelled easily – crack tortuosity can be modelled	• Disadvantages – Spurious mesh sensitivity – convergence of the solution for decreasing mesh size cannot be checked.

Table 2. Different types of closing pressure for FPZ.

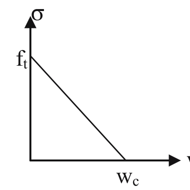
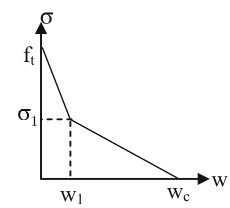
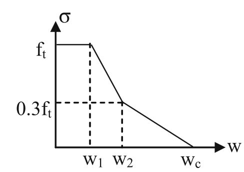
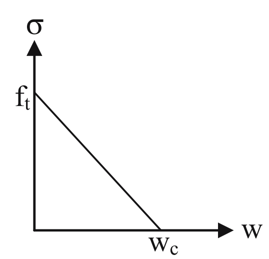
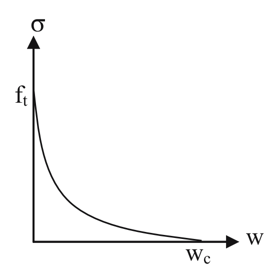
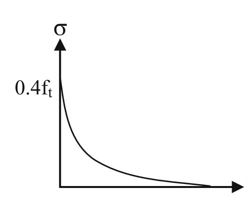
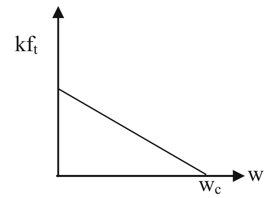
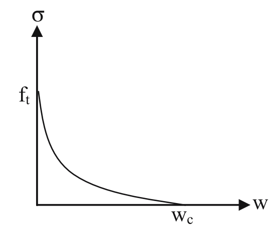
Type	Expression	Shape
Linear curve (Hillerborg *et al* 1976)	$\sigma = f_t(1 - w/w_c)$	
Bilinear curve (Roelfstra & Wittmann 1986)	$\sigma = \begin{cases} f_t - (f_t - \sigma_1)w/w_1 & \text{for } w \le w_1 \\ \sigma_1 - \sigma_1(w - w_1)/(w_c - w_1) & \text{for } w_1 > w \end{cases}$	
Trilinear curve (Liaw *et al* 1990)	$\sigma = \begin{cases} f_t & \text{for } w \le w_1 \\ f_t - 0{\cdot}7f_t(w - w_1)(w_2 - w_1) & \text{for } w_1 < w \le w_2 \\ 0{\cdot}3f_t(w_c - w)/(w_c - w_2) & \text{for } w_2 < w \le w_c \end{cases}$	
Exponential curve (Footer *et al* 1986)	$\sigma = f_t\left(1 - \frac{w}{w_c}\right)^n$ where n is a fitting parameter	
(Reinhardt 1985)	$\sigma = f_t\left\{1 - \left(\frac{w}{w_c}\right)^n\right\}$ where $0 < n < 1$ is a fitting parameter	
Gopalaratnam & Shah (1985) similar relationship was also suggested by Cedolin *et al* (1987)	$\sigma = f_t \exp(kw^{\lambda})$ where k and λ are material parameters $k = -0{\cdot}06163$ and $\lambda = 1{\cdot}01$ for concrete with f_c' values of 33–47 MPa.	
Power curve (Du *et al* 1990)	$\sigma = 0{\cdot}4f_t(1 - w/w_c)^{1{\cdot}5}$	

Table 2. Continued.

Type	Expression	Shape
Bilinear curve with $w_1 = 0$ (Figueiras & Owen 1984)	$\sigma = kf_t(1 - w/w_c)$ where $k = a$ constant	
Power curve (Hordijk 1991)	$\sigma = f_c \left\{ \left[1 + \left(a_1 \frac{w}{w_c} \right)^3 \right] \exp\left(-a_2 \frac{w}{wc} \right) - \frac{w}{w_c} \left(1 + a_1^3 \right) \exp(-a_2) \right\}$ where a_1 and a_2 are fitting parameters	

5. Numerical studies

Numerical studies have been conducted on remaining life prediction of concrete structural components. Section 5·1 presents the details of remaining life prediction studies using LEFM principles.

5.1 *Remaining life prediction using LEFM*

Crack growth studies and remaining life prediction has been carried out for concrete three-point bending specimens under constant amplitude loading. The details of the studies are presented below.

5.1a *Problem 1:* This problem was studied by Toumi & Turatsinze (1998) for three-point bending concrete specimen (figure 12).

Length (S) = 320 mm
Depth (b) = 80 mm
Thickness (t) = 50 mm
Initial crack length (a_o) = 2 to 4 mm
Compressive strength = 57 MPa
Tensile strength = 4·2 MPa
Fracture toughness = 0·63 MPa$\sqrt{}$m
Crack growth equation = Paris
Min. load = 198·72 N

The bending tensile stress (f_b) can be calculated by using the formula given below

$$f_b = 3Pl/2tb^2. \tag{6}$$

Table 3. Predicted remaining life values.

S. No.	Max. Stress (MPa)	Crack growth constants: C (μm/cycle)	Crack growth constants: m	Remaining life (Cycles): Present study	Remaining life (Cycles): Toumi & Turatsinze 1998 (Exptl.)	% diff.
1	1·125	6·45	4·18	28689	32222	10·96
2	1·05	0·33	2·31	57251	63611	9·99
3	0·975	0·26	2·25	62603	69444	9·85
4	0·9	2·04	2·6	16188	18333	11·71

Crack growth analysis and remaining life prediction has been carried out for various loadings using corresponding crack growth constants. Remaining life has been predicted for the different loading cases using LEFM Principles. Geometric factor has been calculated by using the expression given below (Tada *et al* 1985).

$$g_1\left(\frac{a}{b}\right) = \frac{1{\cdot}0 - 2{\cdot}5a/b + 4{\cdot}49(a/b)^2 - 3{\cdot}98(a/b)^3 + 1{\cdot}33(a/b)^4}{(1-a/b)^{3/2}}. \tag{7}$$

Table 3 shows the predicted remaining life values for the above cases along with the experimental values presented by Toumi & Turatsinze (1998). From table 3, it can be observed that there is about 12% difference between the predicted and experimental observations. The difference in the values can be attributed to not considering the tension softening effect in the analysis. Figure 13 shows the variation of predicted remaining life with crack length for various loading cases.

5.1b *Problem 2:* Another example problem has been chosen for crack growth study and remaining life prediction. This problem was studied earlier by Baluch *et al* (1987).

Length of supported span (s) = 1360 mm
Thickness (t) = 51 mm
Depth (b) = 152 mm Fracture toughness = $1{\cdot}16 \times 10^6$ N/m$^{3/2}$

Other input details are shown in table 4. Table 4 shows the predicted remaining life for different loading cases. From table 4, it can be observed that there is about 11% difference between the predicted value and the corresponding experimental observation.

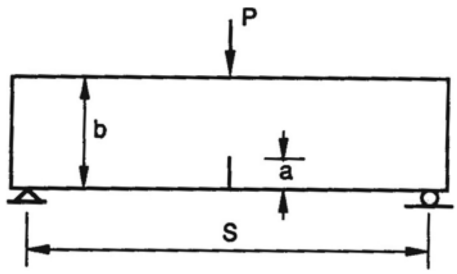


Figure 12. Three point bending problem.

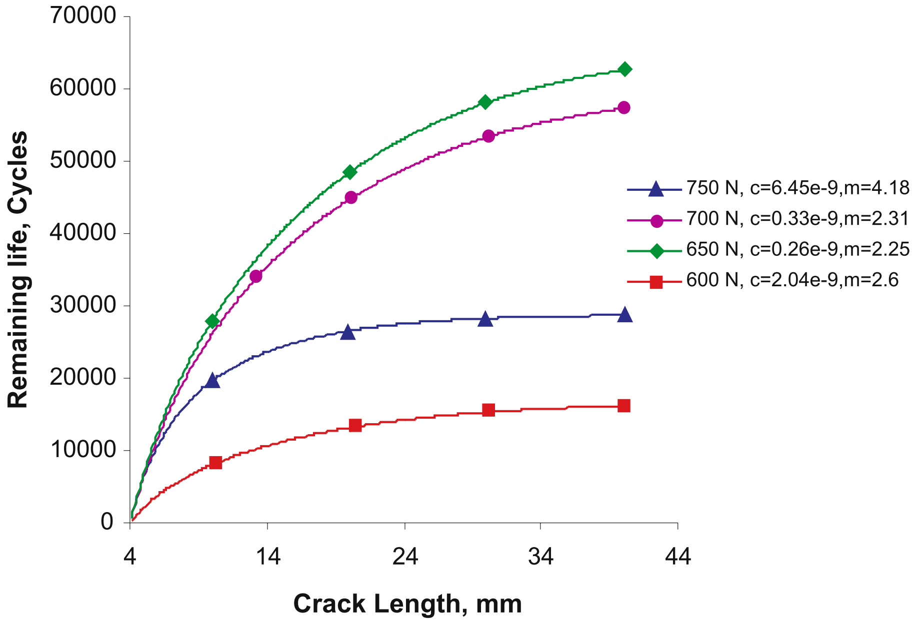


Figure 13. Crack length vs. remaining life.

6. Summary and concluding remarks

A critical review of literature on fracture analysis of concrete has been carried out. Review includes various fracture models, tension softening models and methodologies for crack growth analysis and remaining life prediction. It has been observed that fracture process zone (FPZ) plays an important role in the fracture analysis of quasi-brittle materials like concrete. The widely used fracture models are based on (i) fictitious crack approach and (ii) effective elastic crack approach have been explained. Fictitious crack approach model uses the Dugdale–Barenblatt energy dissipation mechanism which assumes that all energy produced

Table 4. Predicted remaining life values.

Sl. No.	Max. stress (MPa)	Stress ratio	Initial crack length, mm	Crack growth constants C	Crack growth constants m	Remaining life (Cycles)
1		0·1		7·71e–25	3·12	38078*
2	0·5194	0·2	75	5·78e–24	3·12	33176
3		0·3		1·72e–24	3·15	25436
4		0·1		7·71e–25	3·12	24536
5	0·692	0·2	75	5·78e–24	3·12	21987
6		0·3		1·72e–24	3·15	14789
7		0·1		7·71e–25	3·12	25123
8	0·4328	0·2	85	5·78e–24	3·12	22453
9		0·3		1·72e–24	3·15	17936

*-Experimental value 44000

by the applied load is completely balanced by the cohesive pressure. The fictitious crack is assumed to initiate and propagate when the principal tensile stress reaches the tensile strength of the material. Effective elastic crack approach uses the Griffith–Irwin energy dissipation mechanism. The effective elastic crack approach models FPZ by using an equivalent traction-free elastic crack.

Various tension softening models such as linear, bi-linear, tri-linear, etc. have been presented with appropriate expressions. From the critical review of models, it has been observed that the models are subjective in view of the following reasons:

- Some of the models are complex and do not readily lend themselves to mathematical manipulation such as differentiation and integration
- Number of models are divided into two separate expressions hence adding to the complexity of the model
- Some of the models require parameters obtained through curve fitting methods.

There is scope and need to develop more generalised models to represent closing pressure distribution.

Studies have also been conducted on crack growth analysis and remaining life prediction using LEFM principles. From the studies, it has been observed that there is about 12% difference between predicted and experimental observations. The difference in the values can be attributed to not considering the tension softening effect in the analysis. For reliable remaining life prediction, effect of tension softening should be considered appropriately in the analysis by employing NLFM principles.

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List of symbols

f_t	Tensile strength of the material
f_y	Yield strength of the material
$\sigma(w)$	Cohesive pressure
w	Crack opening displacement
w_c	Critical crack opening displacement
G_q	Energy release rate
G_{Ic}	Critical energy release rate
G_σ	Work done by the cohesive pressure
w_t	Crack separation displacement
E	Modulus of elasticity
G_F	Fracture energy
$CMOD_c$	Critical crack mouth opening displacement
$CMOD_c^e$	Elastic component of $CMOD_c$
$CMOD_c^p$	In elastic component of $CMOD_c$
σ_c	Critical stress
a_c	Critical crack length

$g_2\left(\frac{a_c}{b}\right)$	Geometry factor for CMOD
$CTOD_c^e$	Critical crack tip opening displacement
$g_3\left(\frac{a_c}{b}, \frac{a_0}{a_c}\right)$	Geometry factor for CTOD
K_I	Stress intensity factor (SIF) for mode I
K_{IC}^S	Critical SIF for mode I
σ_{NC}	Nominal stress
P_c	Critical fracture load
c_n	Coefficient representing different types of structures
D	Depth of the beam
f_c	Compressive strength of concrete
f_b	Tensile bending stress
l	Length of the beam
t	thickness of the beam
b	width of the beam
$g_1\left(\frac{a}{b}\right)$	Geometry factor using LEFM
a	Crack length
C, m	Paris crack growth constants

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