Free vibration of semi-rigid connected Reddy–Bickford piles embedded in elastic soil

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MS received 14 February 2008; revised 28 April 2008

Abstract. The literature on free vibration analysis of Bernoulli–Euler and Timoshenko piles embedded in elastic soil is plenty, but that of Reddy–Bickford piles partially embedded in elastic soil with/without axial force effect is fewer. The soil that the pile partially embedded in is idealized by Winkler model and is assumed to be two-layered. The pile part above the soil is called the first region and the parts embedded in the soil are called the second and the third region, respectively. It is assumed that the behaviour of the material is linear-elastic, that axial force along the pile length to be constant and the upper end of the pile that is semi-rigid supported against rotation is modelled by an elastic spring. The governing differential equations of motion of the rectangular pile in free vibration are derived using Hamilton's principle and Winkler hypothesis. The terms are found directly from the solutions of the differential equations that describe the deformations of the cross-section according to the high-order theory. The models have six degrees of freedom at the two ends, one transverse displacement and two rotations, and the end forces are a shear force and two end moments. Natural frequencies of the pile are calculated using transfer matrix and the secant method for non-trivial solution of linear homogeneous system of equations obtained due to values of axial forces acting on the pile, total and embedded lengths of the pile, the linear-elastic rotational restraining stiffness at the upper end of the pile and to the boundary conditions of the pile. Two different boundary conditions are considered in the study. For the first boundary condition, the pile's end at the first region is semi-rigid connected and not restricted for horizontal displacement and the end at the third region is free and for the second boundary condition, the pile's end at the first region is semi-rigid connected and restricted for horizontal displacement and the end at the third region is fixed supported. The calculated natural frequencies of semi-rigid connected Reddy–Bickford pile embedded in elastic soil are given in tables and compared with results of Timoshenko pile model.

Keywords. Axial force effect; free vibration; Reddy–Bickford pile; semi-rigid connected; transfer matrix.

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Figure 1. Cross-section displacements in different beam theories (Wang *et al* 2000) (a) Bernoulli–Euler Beam Theory (BET); (b) Timoshenko Beam Theory (TBT); (c) Reddy–Bickford Beam Theory (RBT).

1. Introduction

The analysis of the piles embedded in elastic soil is similar to the analysis of beams on elastic foundations.

The analysis of beams has been performed over the years mostly using Bernoulli–Euler beam theory (BET). The classical Bernoulli–Euler beam is well studied for slender beams, where the transverse shear deformation can be safely disregarded. This theory is based on the assumption that plane sections of the cross-section remain plane and perpendicular to the beam axis. The cross-sectional displacements are shown in (figure 1a), and expressed as

$$u(x, z, t) = -z \cdot \frac{\partial w_0(x, t)}{\partial x} \quad (1)$$

$$w(x, z, t) = w_0(x, t), \quad (2)$$

where $w_0(x, t)$ is the lateral displacement of the beam neutral axis, z is the distance from the beam neutral axis.

For moderately thick beams Bernoulli–Euler beam theory can be modified in order to take into account the transverse shear effect in a simplified way. For example, the well-known Timoshenko beam theory (TBT) predicts a uniform shear distribution, so necessitating the use of a so-called shear factor (Cowper 1966, Gruttmann & Wagner 2001, Murthy 1970). The cross-sectional displacements of Timoshenko beam theory are shown in (figure 1b) and the

equations for Timoshenko beam theory which relaxes the restriction on the angle of shearing deformations are;

$$u(x, z, t) = z \cdot \phi(x, t) \quad (3)$$

$$w(x, z, t) = w_0(x, t), \quad (4)$$

where $\phi(x, t)$ represents the rotation of a normal to the axis of the beam. Han *et al* (1999) presented a comprehensive study of Bernoulli–Euler, Rayleigh, Shear and Timoshenko beam theories.

The real shear deformation distribution is not uniform along the depth of the beam, so that Timoshenko beam theory is not recommended for composite beams, where the accurate determination of the shear stresses is required. Especially, it was found that the Timoshenko shear deformation theory has some major numerical problems such as locking in the numerical analysis for composite materials. The other problem was the need to supply an artificially derived shear correction factor. Although some remedies were devised, as a result, several higher-order theories have emerged. These theories, with small variations, are due to several authors relax the restriction on the warping of the cross-section and allow variation in the longitudinal direction of the beam which is cubic (Bickford 1982, Heyliger & Reddy 1988, Levinson 1981, Wang *et al* 2000).

In this paper, Reddy–Bickford beam theory (RBT) is used, which seems a good compromise between accuracy and simplicity (Bickford 1982, Wang *et al* 2000). The cross-sectional displacements of Reddy–Bickford beam theory are shown in (figure 1c) and according to Reddy–Bickford beam theory, the displacements of the rectangular beam can be written as (Wang *et al* 2000, Reddy 2002, Reddy 2007)

$$u(x, z, t) = z \cdot \phi(x, t) - \alpha \cdot z^3 \cdot \left[\phi(x, t) + \frac{\partial w(x, t)}{\partial x} \right], \quad (5)$$

$$w(x, z, t) = w_0(x, t), \quad (6)$$

where $\alpha = \frac{4}{3 \cdot h^2}$; h is height of the beam.

Bernoulli–Euler beam theory does not consider the shear stress in the cross-section and the associated strains. Thus, the shear angle is taken as zero through the height of the cross-section. Timoshenko beam theory assumes constant shear stress and shear strain in the cross-section. On the top and bottom edges of the beam the free surface condition is thus violated. The use of a shear correction factor, in various forms including the effect of Poisson's ratio, does not correct this fault of the theory, but rather artificially adjusts the solutions to match the static or dynamic behaviour of the beam. Reddy–Bickford beam theory and the other high-order theories remedy this physical mismatch at the free edges by assuming variable shear strain and shear stress along the height of the cross-section. Then there is no need for the shear correction factor. The high-order theory is more exact and represents much better physics of the problem. It results in a sixth-order theory compared to the fourth-order of the other less-accurate theories. This yields a six-degree-of-freedom element with six end forces, a shear force, bending moment and a high-order moment, at the two ends of the beam element.

The determination of the natural frequencies is crucial in the dynamic analysis of piles that are partially embedded in the soil. Previously, it was widely assumed that the upper ends of piles used to support offshore structures, marine and harbor structures or bridges are fully rigidly connected. In reality, these connections are neither fully rigid nor flexible. They fall between fully rigid and flexible connections, depending on the cross-sectional and material

properties of the piles. A more reasonable way is to treat them as semi-rigid connections in the structural analysis.

The analysis of a partially supported pile is similar to that of a beam that is elastically supported. Previously, numerous researchers studied the behaviour of beams supported by elastic foundations (Hetenyi 1955). Doyle & Pavlovic (1982) solved the partial differential equation for free vibration of beams partially attached to elastic foundation using variable separating method and neglecting axial force and shear effects. Boroomand & Kaynia (1991) studied dynamic analysis of pile–soil–pile interaction for vertical piles in a homogeneous soil by using a Fourier expansion of variables. Aviles & Sanchez-Sesma (1983) studied the usefulness of a row of rigid piles as an isolating barrier for elastic waves. They formulated the problem as one of multiple scattering and diffraction. Aviles & Sanchez-Sesma (1988) presented a theoretical analysis to solve the problem of foundation isolation, using a row of elastic piles as an isolating barrier for elastic waves. Liao & Sangrey (1978) employed an acoustic model for the use of rows of piles as passive isolation barriers to reduce ground vibrations. West & Mafi (1984) solved the partial differential equation for free vibration of an elastic beam on elastic foundation that is subjected to axial force by using initial value method. Yokoyama (1991) studied the free vibration motion of Timoshenko beam on two-parameters elastic foundation. Esmailzadeh & Ohadi (2000) investigated vibration and stability analysis of non-uniform Timoshenko beams under axial and distributed tangential loads. Catal (2002) calculated natural frequencies and relative stiffness of the pile for non-trivial solution of linear homogeneous system of equations obtained due to the values of axial forces acting on the pile, the shape factors, and the boundary conditions of the pile with both ends free and both ends simply supported by using Timoshenko beam theory and transfer matrix. Further, he proceeded to determine the natural frequencies of Timoshenko piles partially embedded in the soil, but semi-rigidly connected at the upper ends, using the method of initial values (Catal 2006). Lin & Chang (2005) studied free vibration analysis of multi-span Timoshenko beam with an arbitrary number of flexible constraints by transfer matrix method. Demirdag & Catal (2007) investigated spectral analysis of semi-rigid supported single storey frames modelled as Timoshenko column with attached mass. Demirdag (2008) studied elastically-supported Timoshenko column with attached masses is under consideration to obtain its free vibration natural frequencies using two different algorithm; transfer matrix method and finite element method. Yesilce & Catal (2008) calculated normalized natural frequencies of Timoshenko pile due to the different values of axial force using transfer matrix and considering rotatory inertia.

2. The mathematical model and formulation

A rectangular pile partially embedded in the soil whose upper end is semi-rigid connected against rotation is presented in (figure 2). The pile part above the soil is called the first region and the parts embedded in the soil are called the second and the third region, respectively.

Using Hamilton's principle and Eqs. (5) and (6); the equations of motion can be written for each region as (Wang *et al* 2000, Eisenberger 2003):

$$\begin{aligned}
 & -\frac{68}{105} \cdot EI_x \cdot \frac{\partial^2 \phi_j(x_j, t)}{\partial x_j^2} + \frac{16}{105} \cdot EI_x \cdot \frac{\partial^3 w_j(x_j, t)}{\partial x_j^3} + \frac{8}{15} \cdot AG \\
 & \cdot \left[\phi_j(x_j, t) + \frac{\partial w_j(x_j, t)}{\partial x_j} \right] = 0 \quad (0 \leq x_j \leq L_j) \quad (j = 1, 2, 3)
 \end{aligned} \tag{7}$$

$$\begin{aligned}
& -m \cdot \frac{\partial^2 w_1(x_1, t)}{\partial t^2} + \frac{8}{15} \cdot AG \cdot \left[\frac{\partial \phi_1(x_1, t)}{\partial x_1} + \frac{\partial^2 w_1(x_1, t)}{\partial x_1^2} \right] + \frac{16}{105} \cdot EI_x \\
& \cdot \frac{\partial^3 \phi_1(x_1, t)}{\partial x_1^3} - \frac{1}{21} \cdot EI_x \cdot \frac{\partial^4 w_1(x_1, t)}{\partial x_1^4} - N \cdot \frac{\partial^2 w_1(x_1, t)}{\partial x_1^2} = 0 \quad (0 \leq x_1 \leq L_1)
\end{aligned} \quad (8)$$

$$\begin{aligned}
& -m \cdot \frac{\partial^2 w_k(x_k, t)}{\partial t^2} + \frac{8}{15} \cdot AG \cdot \left[\frac{\partial \phi_k(x_k, t)}{\partial x_k} + \frac{\partial^2 w_k(x_k, t)}{\partial x_k^2} \right] + \frac{16}{105} \cdot EI_x \cdot \frac{\partial^3 \phi_k(x_k, t)}{\partial x_k^3} \\
& - \frac{1}{21} \cdot EI_x \cdot \frac{\partial^4 w_k(x_k, t)}{\partial x_k^4} - C_{S(k-1)} \cdot w_k(x_k, t) - N \cdot \frac{\partial^2 w_k(x_k, t)}{\partial x_k^2} = 0 \quad (0 \leq x_k \leq L_k) \quad (k = 2, 3),
\end{aligned} \quad (9)$$

where $w_j(x_j, t)$ is displacement function for j^{th} region of the pile, $\phi_j(x_j, t)$ represents the rotation of a normal to the axis for j^{th} region of the pile, m is mass per unit length of the pile, L_1 is pile length above the soil, L_2 is pile length embedded in the second region, L_3 is pile length embedded in the third region, L is total length of the pile, N is the constant axial compressive force, A is the cross-section area, I_x is moment of inertia, E, G are Young's modulus and shear modulus of the pile, $C_{S1} = C_{R1} \cdot b$ and $C_{S2} = C_{R2} \cdot b$ in which C_{R1}, C_{R2}

Figure 2. Pile partially embedded in the elastic soil.

are the modulus of subgrade reaction for the second and the third regions, respectively and b is width of the pile, x_1 , x_2 and x_3 are pile positions for the first, the second and the third regions, t is time variable.

Assuming that the motion is harmonic we substitute for $w_j(z_j, t)$ and $\phi_j(r_j, t)$ the following:

$$w_j(z_j, t) = w_j(z_j) \cdot \sin(\omega \cdot t) \quad (10)$$

$$\phi_j(z_j, t) = \phi_j(z_j) \cdot \sin(\omega \cdot t) \quad (j = 1, 2, 3) \quad (11)$$

and obtain a system of two coupled ordinary equation for each region as:

$$\begin{aligned} & -\frac{68}{105} \cdot \frac{EI_x}{L^2} \cdot \frac{d^2\phi_j(z_j)}{dz_j^2} + \frac{16}{105} \cdot \frac{EI_x}{L^3} \cdot \frac{d^3w_j(z_j)}{dz_j^3} + \frac{8}{15} \cdot AG \\ & \cdot \left[\phi_j(z_j) + \frac{1}{L} \cdot \frac{dw_j(z_j)}{dz_j} \right] = 0 \quad (j = 1, 2, 3) \end{aligned} \quad (12)$$

$$\begin{aligned} & m \cdot \omega^2 \cdot w_1(z_1) + \frac{8}{15} \cdot \frac{AG}{L} \cdot \left[\frac{d\phi_1(z_1)}{dz_1} + \frac{1}{L} \cdot \frac{d^2w_1(z_1)}{dz_1^2} \right] + \frac{16}{105} \cdot \frac{EI_x}{L^3} \\ & \cdot \frac{d^3\phi_1(z_1)}{dz_1^3} - \frac{1}{21} \cdot \frac{EI_x}{L^4} \cdot \frac{d^4w_1(z_1)}{dz_1^4} - N_r \cdot \frac{\pi^2 \cdot EI_x}{L^4} \cdot \frac{d^2w_1(z_1)}{dz_1^2} = 0 \end{aligned} \quad (13)$$

$$\begin{aligned} & m \cdot \omega^2 \cdot w_k(z_k) + \frac{8}{15} \cdot \frac{AG}{L} \cdot \left[\frac{d\phi_k(z_k)}{dz_k} + \frac{1}{L} \cdot \frac{d^2w_k(z_k)}{dz_k^2} \right] + \frac{16}{105} \\ & \cdot \frac{EI_x}{L^3} \cdot \frac{d^3\phi_k(z_k)}{dz_k^3} - \frac{1}{21} \cdot \frac{EI_x}{L^4} \cdot \frac{d^4w_k(z_k)}{dz_k^4} - C_{S(k-1)} \cdot w_k(z_k) - N_r \\ & \cdot \frac{\pi^2 \cdot EI_x}{L^4} \cdot \frac{d^2w_k(z_k)}{dz_k^2} = 0 \quad (k = 2, 3), \end{aligned} \quad (14)$$

where $z = \frac{x}{L}$, dimensionless position parameter; $N_r = \frac{N \cdot L^2}{\pi^2 \cdot EI_x}$, non-dimensionalized multiplication factor for the axial compressive force and ω is natural frequency of the pile.

It is assumed that the solution is

$$w_j(z_j) = C_j \cdot e^{i \cdot s_j \cdot z_j} \quad (15)$$

$$\phi_j(z_j) = P_j \cdot e^{i \cdot s_j \cdot z_j} \quad (j = 1, 2, 3) \quad (16)$$

and substituting Eqs. (15) and (16) into Eqs. (12), (13) and (14) results in

$$\begin{aligned} & \left(\frac{8}{15} \cdot AG + \frac{68}{105} \cdot \frac{EI_x}{L^2} \cdot s_j^2 \right) \cdot P_j \\ & + \left(\frac{8}{15} \cdot \frac{AG}{L} \cdot s_j \cdot i - \frac{16}{105} \cdot \frac{EI_x}{L^3} \cdot s_j^3 \cdot i \right) \cdot C_j = 0 \quad (j = 1, 2, 3) \end{aligned} \quad (17)$$

$$\begin{aligned} & \left(\frac{8}{15} \cdot \frac{AG}{L} \cdot s_1 \cdot i - \frac{16}{105} \cdot \frac{EI_x}{L^3} \cdot s_1^3 \cdot i \right) \cdot P_1 \\ & + \left(m \cdot \omega^2 - \frac{8}{15} \cdot \frac{AG}{L^2} \cdot s_1^2 - \frac{1}{21} \cdot \frac{EI_x}{L^4} \cdot s_1^4 + N_r \cdot \frac{\pi^2 \cdot EI_x}{L^4} \cdot s_1^2 \right) \cdot C_1 = 0 \end{aligned} \quad (18)$$

$$\begin{aligned} & \left(\frac{8}{15} \cdot \frac{AG}{L} \cdot s_k \cdot i - \frac{16}{105} \cdot \frac{EI_x}{L^3} \cdot s_k^3 \cdot i \right) \cdot P_k \\ & + \left(m \cdot \omega^2 - \frac{8}{15} \cdot \frac{AG}{L^2} \cdot s_k^2 - \frac{1}{21} \cdot \frac{EI_x}{L^4} \cdot s_k^4 - C_{S(k-1)} + N_r \cdot \frac{\pi^2 \cdot EI_x}{L^4} \cdot s_k^2 \right) \\ & \cdot C_k = 0 \quad (k = 2, 3), \end{aligned} \quad (19)$$

where $i = \sqrt{-1}$.

Eqs. (17), (18) and (19) can be written in matrix form for the two unknowns P_j and C_j and the non-trivial solution will be obtained when the determinant of the coefficient matrix will be zero for each region, i.e.

$$\begin{aligned} & \left[-\frac{4}{525} \cdot \frac{(EI_x)^2}{L^6} \right] \cdot s_1^6 + \left[\frac{68}{105} \cdot \frac{N_r \cdot \pi^2 \cdot (EI_x)^2}{L^6} - \frac{8}{15} \cdot \frac{AG \cdot EI_x}{L^4} \right] \cdot s_1^4 \\ & + \left(\frac{68}{105} \cdot \frac{EI_x}{L^2} \cdot m \cdot \omega^2 + \frac{8}{15} \cdot \frac{N_r \cdot \pi^2 \cdot EI_x \cdot AG}{L^4} \right) \cdot s_1^2 \\ & + \frac{8}{15} \cdot AG \cdot m \cdot \omega^2 = 0 \end{aligned} \quad (20)$$

$$\begin{aligned} & \left[-\frac{4}{525} \cdot \frac{(EI_x)^2}{L^6} \right] \cdot s_k^6 + \left[\frac{68}{105} \cdot \frac{N_r \cdot \pi^2 \cdot (EI_x)^2}{L^6} - \frac{8}{15} \cdot \frac{AG \cdot EI_x}{L^4} \right] \cdot s_k^4 \\ & + \left(\frac{68}{105} \cdot \frac{EI_x}{L^2} \cdot (m \cdot \omega^2 - C_{S(k-1)}) + \frac{8}{15} \cdot \frac{N_r \cdot \pi^2 \cdot EI_x \cdot AG}{L^4} \right) \cdot s_k^2 \\ & + \frac{8}{15} \cdot AG \cdot (m \cdot \omega^2 - C_{S(k-1)}) = 0. \end{aligned} \quad (21)$$

Thus, we have a sixth-order equation with the unknowns for each region, resulting in six values and the general solution for each region can be written as:

$$\begin{aligned} w_j(z_j, t) = & [C_{j1} \cdot e^{i \cdot s_{j1} \cdot z_j} + C_{j2} \cdot e^{i \cdot s_{j2} \cdot z_j} + C_{j3} \cdot e^{i \cdot s_{j3} \cdot z_j} + C_{j4} \cdot e^{i \cdot s_{j4} \cdot z_j} \\ & + C_{j5} \cdot e^{i \cdot s_{j5} \cdot z_j} + C_{j6} \cdot e^{i \cdot s_{j6} \cdot z_j}] \cdot \sin(\omega \cdot t) \end{aligned} \quad (22)$$

$$\begin{aligned} \phi_j(z_j, t) = & [P_{j1} \cdot e^{i \cdot s_{j1} \cdot z_j} + P_{j2} \cdot e^{i \cdot s_{j2} \cdot z_j} + P_{j3} \cdot e^{i \cdot s_{j3} \cdot z_j} + P_{j4} \cdot e^{i \cdot s_{j4} \cdot z_j} \\ & + P_{j5} \cdot e^{i \cdot s_{j5} \cdot z_j} + P_{j6} \cdot e^{i \cdot s_{j6} \cdot z_j}] \cdot \sin(\omega \cdot t) \quad (j = 1, 2, 3). \end{aligned} \quad (23)$$

The thirty-six constants, $C_{j1}, \dots, C_{j6}$ and $P_{j1}, \dots, P_{j6}$, will be found from Eqs. (17), (18), (19) and boundary conditions.

For each region, the expression for bending rotation $w'_j(z_j, t)$ is given by

$$w'_j(z_j, t) = \frac{1}{L} \cdot \frac{dw_j(z_j)}{dz_j} \cdot \sin(\omega \cdot t) \quad (j = 1, 2, 3). \quad (24)$$

For each region, the shear force function $Q_j(z_j, t)$ can be obtained by using Eqs. (22) and (23) as:

$$\begin{aligned} Q_j(z_j, t) = & -\frac{8 \cdot AG}{15} \cdot \left(\phi_j(z_j) + \frac{1}{L} \cdot \frac{dw_j(z_j)}{dz_j} \right) \cdot \sin(\omega \cdot t) \\ & + N_r \cdot \frac{\pi^2 \cdot EI_x}{L^3} \cdot \frac{dw_j(z_j)}{dz_j} \cdot \sin(\omega \cdot t) \\ & + \frac{EI_x}{21 \cdot L^3} \cdot \frac{d^3 w_j(z_j)}{dz_j^3} \cdot \sin(\omega \cdot t) \\ & - \frac{16 \cdot EI_x}{105 \cdot L^2} \cdot \frac{d^2 \phi_j(z_j)}{dz_j^2} \cdot \sin(\omega \cdot t) \quad (j = 1, 2, 3). \end{aligned} \quad (25)$$

Similarly, the bending moment function $M_j(z_j, t)$ can be obtained by using Eqs. (22) and (23) as:

$$\begin{aligned} M_j(z_j, t) = & \left(-\frac{EI_x}{21 \cdot L^2} \cdot \frac{d^2 w_j(z_j)}{dz_j^2} - N_r \cdot \frac{\pi^2 \cdot EI_x}{L^2} \cdot w_j(z_j) \right. \\ & \left. + \frac{16 \cdot EI_x}{105 \cdot L} \cdot \frac{d\phi_j(z_j)}{dz_j} \right) \cdot \sin(\omega \cdot t). \end{aligned} \quad (26)$$

For each region, the higher-order moment function $M_{hj}(z_j, t)$ can be obtained as:

$$M_{hj}(z_j, t) = \left(\frac{16 \cdot EI_x}{105 \cdot L^2} \cdot \frac{d^2 w_j(z_j)}{dz_j^2} - \frac{68 \cdot EI_x}{105 \cdot L} \cdot \frac{d\phi_j(z_j)}{dz_j} \right) \cdot \sin(\omega \cdot t) \quad (j = 1, 2, 3). \quad (27)$$

3. Obtaining the transfer matrices of the pile

The position for each region is written due to the values of the transverse displacement $w_j(z_j, t)$, bending rotation $w'_j(z_j, t)$, rotation of normal $\phi_j(z_j, t)$, shear force $Q_j(z_j, t)$, bending moment $M_h(z_j, t)$ and higher-order moment function $M_{hj}(z_j, t)$ at the locations of z_j and t for Reddy–Bickford pile, as ($j = 1, 2, 3$):

$$\langle S_j(z_j, t) \rangle^T = \langle w_j(z_j) \quad w'_j(z_j) \quad \phi_j(z_j) \quad M_j(z_j) \quad M_{hj}(z_j) \quad Q_j(z_j) \rangle^T \cdot \sin(\omega \cdot t), \quad (28)$$

where $\{S_j(z_j, t)\}$ shows the position vector for each region.

All terms of the position vector in Eq. (28), is written reducing the $\sin(\omega t)$ terms as:

$$\{S_1(z_1)\} = \begin{Bmatrix} w_1(z_1) \\ w'_1(z_1) \\ \phi_1(z_1) \\ M_1(z_1) \\ M_{h_1}(z_1) \\ Q_1(z_1) \end{Bmatrix} = \begin{bmatrix} A_{11}(z_1) & A_{12}(z_1) & A_{13}(z_1) & A_{14}(z_1) & A_{15}(z_1) & A_{16}(z_1) \\ A_{21}(z_1) & A_{22}(z_1) & A_{23}(z_1) & A_{24}(z_1) & A_{25}(z_1) & A_{26}(z_1) \\ A_{31}(z_1) & A_{32}(z_1) & A_{33}(z_1) & A_{34}(z_1) & A_{35}(z_1) & A_{36}(z_1) \\ A_{41}(z_1) & A_{42}(z_1) & A_{43}(z_1) & A_{44}(z_1) & A_{45}(z_1) & A_{46}(z_1) \\ A_{51}(z_1) & A_{52}(z_1) & A_{53}(z_1) & A_{54}(z_1) & A_{55}(z_1) & A_{56}(z_1) \\ A_{61}(z_1) & A_{62}(z_1) & A_{63}(z_1) & A_{64}(z_1) & A_{65}(z_1) & A_{66}(z_1) \end{bmatrix} \begin{Bmatrix} C_{11} \\ C_{12} \\ C_{13} \\ C_{14} \\ C_{15} \\ C_{16} \end{Bmatrix} \quad (29)$$

$$\{S_2(z_2)\} = \begin{Bmatrix} w_2(z_2) \\ w'_2(z_2) \\ \phi_2(z_2) \\ M_2(z_2) \\ M_{h_2}(z_2) \\ Q_2(z_2) \end{Bmatrix} = \begin{bmatrix} B_{11}(z_2) & B_{12}(z_2) & B_{13}(z_2) & B_{14}(z_2) & B_{15}(z_2) & B_{16}(z_2) \\ B_{21}(z_2) & B_{22}(z_2) & B_{23}(z_2) & B_{24}(z_2) & B_{25}(z_2) & B_{26}(z_2) \\ B_{31}(z_2) & B_{32}(z_2) & B_{33}(z_2) & B_{34}(z_2) & B_{35}(z_2) & B_{36}(z_2) \\ B_{41}(z_2) & B_{42}(z_2) & B_{43}(z_2) & B_{44}(z_2) & B_{45}(z_2) & B_{46}(z_2) \\ B_{51}(z_2) & B_{52}(z_2) & B_{53}(z_2) & B_{54}(z_2) & B_{55}(z_2) & B_{56}(z_2) \\ B_{61}(z_2) & B_{62}(z_2) & B_{63}(z_2) & B_{64}(z_2) & B_{65}(z_2) & B_{66}(z_2) \end{bmatrix} \begin{Bmatrix} C_{21} \\ C_{22} \\ C_{23} \\ C_{24} \\ C_{25} \\ C_{26} \end{Bmatrix} \quad (30)$$

$$\{S_3(z_3)\} = \begin{Bmatrix} w_3(z_3) \\ w'_3(z_3) \\ \phi_3(z_3) \\ M_3(z_3) \\ M_{h_3}(z_3) \\ Q_3(z_3) \end{Bmatrix} = \begin{bmatrix} \bar{C}_{11}(z_3) & \bar{C}_{12}(z_3) & \bar{C}_{13}(z_3) & \bar{C}_{14}(z_3) & \bar{C}_{15}(z_3) & \bar{C}_{16}(z_3) \\ \bar{C}_{21}(z_3) & \bar{C}_{22}(z_3) & \bar{C}_{23}(z_3) & \bar{C}_{24}(z_3) & \bar{C}_{25}(z_3) & \bar{C}_{26}(z_3) \\ \bar{C}_{31}(z_3) & \bar{C}_{32}(z_3) & \bar{C}_{33}(z_3) & \bar{C}_{34}(z_3) & \bar{C}_{35}(z_3) & \bar{C}_{36}(z_3) \\ \bar{C}_{41}(z_3) & \bar{C}_{42}(z_3) & \bar{C}_{43}(z_3) & \bar{C}_{44}(z_3) & \bar{C}_{45}(z_3) & \bar{C}_{46}(z_3) \\ \bar{C}_{51}(z_3) & \bar{C}_{52}(z_3) & \bar{C}_{53}(z_3) & \bar{C}_{54}(z_3) & \bar{C}_{55}(z_3) & \bar{C}_{56}(z_3) \\ \bar{C}_{61}(z_3) & \bar{C}_{62}(z_3) & \bar{C}_{63}(z_3) & \bar{C}_{64}(z_3) & \bar{C}_{65}(z_3) & \bar{C}_{66}(z_3) \end{bmatrix} \begin{Bmatrix} C_{31} \\ C_{32} \\ C_{33} \\ C_{34} \\ C_{35} \\ C_{36} \end{Bmatrix} \quad (31)$$

Eqs. (29), (30) and (31) can be written in closed form, as $[(j = 1, \dots, 6), (m = 1, \dots, 6), (n = 1, \dots, 6)]$:

$$\{S_1(z_1)\} = [A_{jm}(z_1)] \cdot \{C_{1n}\} \quad (32)$$

$$\{S_2(z_2)\} = [B_{jm}(z_2)] \cdot \{C_{2n}\} \quad (33)$$

$$\{S_3(z_3)\} = [\bar{C}_{jm}(z_3)] \cdot \{C_{3n}\}. \quad (34)$$

The coefficient vectors $\{C_{1n}\}$, $\{C_{2n}\}$, $\{C_{3n}\}$ are obtained from Eqs. (32), (33) and (34) for $z_1 = 0$, $z_2 = 0$ and $z_3 = 0$ with the condition that the matrices $[A_{jm}(z_1)]$, $[B_{jm}(z_2)]$ and $[\bar{C}_{jm}(z_3)]$ are not singular. Substituting these coefficient vectors into the Eqs. (32), (33) and (34) respectively, gives,

$$\{S_1(z_1)\} = [F_1(z_1)] \cdot \{S_1(0)\} \quad (35)$$

$$\{S_2(z_2)\} = [F_2(z_2)] \cdot \{S_2(0)\} \quad (36)$$

$$\{S_3(z_3)\} = [F_3(z_3)] \cdot \{S_3(0)\} \quad (37)$$

where;

$$[F_1(z_1)] = [A_{jm}(z_1)] \cdot [A_{jm}(0)]^{-1} \quad (38)$$

$$[F_2(z_2)] = [B_{jm}(z_2)] \cdot [B_{jm}(0)]^{-1} \quad (39)$$

$$[F_3(z_3)] = [\bar{C}_{jm}(z_3)] \cdot [\bar{C}_{jm}(0)]^{-1} \quad (40)$$

are the transfer matrices for the first, the second and the third regions of the pile, respectively.

The transfer matrices for the three regions can be combined to yield one global transfer matrix using the characteristics of transfer matrix. Thus, the position vector of the pile end at the third region can be related to the position vector of the pile end at the first region as

$$\{S_1(z_1 = 1)\} = \left[F_1 \left(z_1 = \frac{L_1}{L} \right) \right] \cdot \left[F_2 \left(z_2 = \frac{L_2}{L} \right) \right] \cdot \left[F_3 \left(z_3 = \frac{L_3}{L} \right) \right] \cdot \{S_3(z_3 = 0)\} \quad (41)$$

$$\{S_1(z_1 = 1)\} = [F(z = 1)] \cdot \{S_3(z_3 = 0)\}, \quad (42)$$

where $[F(z = 1)]$ shows the global transfer matrix that transfers the values of the position vector of the pile end at the third region to the values of the position vector of the pile end at the first region and can be written as:

$$[F(z = 1)] = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} \\ F_{21} & F_{22} & F_{23} & F_{24} & F_{25} & F_{26} \\ F_{31} & F_{32} & F_{33} & F_{34} & F_{35} & F_{36} \\ F_{41} & F_{42} & F_{43} & F_{44} & F_{45} & F_{46} \\ F_{51} & F_{52} & F_{53} & F_{54} & F_{55} & F_{56} \\ F_{61} & F_{62} & F_{63} & F_{64} & F_{65} & F_{66} \end{bmatrix}, \quad (43)$$

where the term F_{jm} shows the terms of the global transfer matrix $[F(z = 1)]$.

4. Obtaining natural frequencies for semi-rigid connected pile

The behaviour of the pile end that is semi-rigid supported against rotation at the first region is modelled by an elastic spring. The rotational spring rigidities are related with fixity factor that is defined as below (Monforton & Wu 1963):

$$f = \frac{1}{1 + \frac{3 \cdot EI_x}{C_\theta \cdot L}}, \quad (44)$$

where C_θ is the rotational restraining stiffness at the upper end of the pile in the first region.

Bending moment function at semi-rigid connected end is written as a linear function of rotational restraining stiffness and bending rotation as (Wang *et al* 2000):

$$M_1 \left(z_1 = \frac{L_1}{L} \right) = -C_\theta \cdot \phi_1 \left(z_1 = \frac{L_1}{L} \right), \quad (45)$$

Figure 3. (a) Pile whose end at the first region is semi-rigid connected and not restricted for horizontal displacement and the end at the third region is free (The first model). (b) Pile whose end at the first region is semi-rigid connected and restricted for horizontal displacement and the end at the third region is fixed supported (The second model).

where $M_1(z_1 = \frac{L_1}{L})$ and $\phi_1(z_1 = \frac{L_1}{L})$ are the bending moment and the rotation of normal for the first region.

Similarly, high-order moment function at semi-rigid connected end is written as a linear function of rotational restraining stiffness and rotation of normal as (Wang *et al* 2000):

$$M_{h1}\left(z_1 = \frac{L_1}{L}\right) = 0, \quad (46)$$

where $M_{h1}(z_1 = \frac{L_1}{L})$ is the high-order moment for the first region.

Using Eq. (42), the position vectors at free and the fixed supported ends are transferred to the position vectors at the semi-rigid connected end by Eqs. (47) and (48), respectively. For the first model, the pile's end at the first region is semi-rigid connected and not restricted for horizontal displacement and the end at the third region is free as in figure 3a. For the second model, the pile's end at the first region is semi-rigid connected and restricted for horizontal displacement and the end at the third region is fixed supported as in figure 3b.

$$\begin{Bmatrix} w_1(z_1 = \frac{L_1}{L}) \\ w'_1(z_1 = \frac{L_1}{L}) \\ \phi_1(z_1 = \frac{L_1}{L}) \\ C_\theta \cdot w'_1(z_1 = \frac{L_1}{L}) \\ C_\theta \cdot \phi_1(z_1 = \frac{L_1}{L}) \\ 0 \end{Bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} \\ F_{21} & F_{22} & F_{23} & F_{24} & F_{25} & F_{26} \\ F_{31} & F_{32} & F_{33} & F_{34} & F_{35} & F_{36} \\ F_{41} & F_{42} & F_{43} & F_{44} & F_{45} & F_{46} \\ F_{51} & F_{52} & F_{53} & F_{54} & F_{55} & F_{56} \\ F_{61} & F_{62} & F_{63} & F_{64} & F_{65} & F_{66} \end{bmatrix} \cdot \begin{Bmatrix} w_3(z_3 = 0) \\ w'_3(z_3 = 0) \\ \phi_3(z_3 = 0) \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (47)$$

$$\begin{Bmatrix} 0 \\ w'_1(z_1 = \frac{L_1}{L}) \\ \phi_1(z_1 = \frac{L_1}{L}) \\ C_\theta \cdot w'_1(z_1 = \frac{L_1}{L}) \\ C_\theta \cdot \phi_1(z_1 = \frac{L_1}{L}) \\ Q_1(z_1 = \frac{L_1}{L}) \end{Bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} \\ F_{21} & F_{22} & F_{23} & F_{24} & F_{25} & F_{26} \\ F_{31} & F_{32} & F_{33} & F_{34} & F_{35} & F_{36} \\ F_{41} & F_{42} & F_{43} & F_{44} & F_{45} & F_{46} \\ F_{51} & F_{52} & F_{53} & F_{54} & F_{55} & F_{56} \\ F_{61} & F_{62} & F_{63} & F_{64} & F_{65} & F_{66} \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M_3(z_3 = 0) \\ M_{h3}(z_3 = 0) \\ Q_3(z_3 = 0) \end{Bmatrix} \quad (48)$$

Eqs. (47) and (48) can be written in matrix form as:

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} & -1 & 0 & 0 \\ F_{21} & F_{22} & F_{23} & 0 & -1 & 0 \\ F_{31} & F_{32} & F_{33} & 0 & 0 & -1 \\ F_{41} & F_{42} & F_{43} & 0 & -C_\theta & 0 \\ F_{51} & F_{52} & F_{53} & 0 & 0 & -C_\theta \\ F_{61} & F_{62} & F_{63} & 0 & 0 & 0 \end{bmatrix} \cdot \begin{Bmatrix} w_3(z_3 = 0) \\ w'_3(z_3 = 0) \\ \phi_3(z_3 = 0) \\ w_1(z_1 = \frac{L_1}{L}) \\ w'_1(z_1 = \frac{L_1}{L}) \\ \phi_1(z_1 = \frac{L_1}{L}) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (49)$$

$$\begin{bmatrix} F_{14} & F_{15} & F_{16} & 0 & 0 & 0 \\ F_{24} & F_{25} & F_{26} & -1 & 0 & 0 \\ F_{34} & F_{35} & F_{36} & 0 & -1 & 0 \\ F_{44} & F_{45} & F_{46} & -C_\theta & 0 & 0 \\ F_{54} & F_{55} & F_{56} & 0 & -C_\theta & 0 \\ F_{64} & F_{65} & F_{66} & 0 & 0 & -1 \end{bmatrix} \cdot \begin{Bmatrix} M_3(z_3 = 0) \\ M_{h3}(z_3 = 0) \\ Q_3(z_3 = 0) \\ w'_1(z_1 = \frac{L_1}{L}) \\ \phi_1(z_1 = \frac{L_1}{L}) \\ Q_1(z_1 = \frac{L_1}{L}) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (50)$$

For non-trivial solutions of this problem, following relations are written by using Eqs. (49) and (50), as:

$$\begin{vmatrix} F_{11} & F_{12} & F_{13} & -1 & 0 & 0 \\ F_{21} & F_{22} & F_{23} & 0 & -1 & 0 \\ F_{31} & F_{32} & F_{33} & 0 & 0 & -1 \\ F_{41} & F_{42} & F_{43} & 0 & -C_\theta & 0 \\ F_{51} & F_{52} & F_{53} & 0 & 0 & -C_\theta \\ F_{61} & F_{62} & F_{63} & 0 & 0 & 0 \end{vmatrix} = 0 \quad (51)$$

$$\begin{vmatrix} F_{14} & F_{15} & F_{16} & 0 & 0 & 0 \\ F_{24} & F_{25} & F_{26} & -1 & 0 & 0 \\ F_{34} & F_{35} & F_{36} & 0 & -1 & 0 \\ F_{44} & F_{45} & F_{46} & -C_\theta & 0 & 0 \\ F_{54} & F_{55} & F_{56} & 0 & -C_\theta & 0 \\ F_{64} & F_{65} & F_{66} & 0 & 0 & -1 \end{vmatrix} = 0. \quad (52)$$

5. Numerical analysis and discussions

In this paper, for numerical analysis, two models that are partially embedded in Winkler soil and whose ends above the soil are semi-rigid connected by an elastic spring having the rotational spring rigidities of C_θ , as one being not restricted for horizontal displacement and the other being restricted for horizontal displacement, respectively, are considered. For two examples, natural frequencies of the pile, ω_i ($i = 1, 2, 3$) are calculated by using computer programs prepared by authors. Natural frequencies are found by determining values for which the determinant of the coefficient matrix is equal to zero. There are various methods for calculating the roots of the frequency equation. One commonly used and simple technique is the secant method in which a linear interpolation is employed. The eigenvalues, the natural frequencies, are determined by a trial and error method based on interpolation and the bisection approach. One such procedure consists of evaluating the determinant for a range of frequency values, ω_i . When there is a change of sign between successive evaluations, there must be a root lying in this interval. The iterative computations are determined when the value of the determinant changed sign due to a change of 10^{-4} in the value of ω_i .

For each example, $C_{S1} = 15000 \text{ kN/m}^3$ and $C_{S2} = 60000 \text{ kN/m}^3$. The length of the pile is taken as 15 m and 30 m.

The all numerical results of this paper are obtained based on uniform, rectangular Reddy–Bickford and Timoshenko piles with the following data:

$$h = 0.50 \text{ m}; \quad b = 0.30 \text{ m}; \quad EI_x = 6.5625 \times 10^3 \text{ kNm}^2; \\ AG = 121500 \text{ kN}; \quad m = 0.50 \text{ kN.sec}^2/\text{m};$$

for the axial force effect $N_r = 0.25$ and 1.00 , fixity factors are taken as $f = 0.25$ and $f = 0.75$.

The all numerical results are given for the following three models: Timoshenko model with two values for shear correction factor k and Reddy–Bickford model. Many values for the shear correction factor k were suggested, but in this paper, the original values suggested by Timoshenko $k = \frac{5}{6}$ and $k = \frac{14}{17}$ are used (Gruttmann & Wagner 2001).

Natural frequencies of the pile are obtained from the solution of Eqs. (51) and (52) according to the boundary conditions, by using the computer program for $N_r = 0.25$ and $N_r = 1.0$ with the values of L , L_1/L , L_2/L and L_3/L taken from table 1.

For $L = 15 \text{ m}$, the frequency values obtained for the first three modes of Reddy–Bickford and Timoshenko piles whose end at the first region is semi-rigid connected and not restricted for horizontal displacement and the end at the third region is free are presented in table 2; for $L = 30 \text{ m}$, the frequency values obtained for the first three modes of the same piles are presented in table 3 being compared with the frequency values obtained for $N_r = 0.25$ and 1.00 , $f = 0.25$ and 0.75 .

Table 1. Values of L_1 , L_2 and L_3 with respect to values of L , L_1/L , L_2/L and L_3/L .

$L_1/L = 0.50 \quad L_2/L = 0.30 \quad L_3/L = 0.20$				$L_1/L = 0.50 \quad L_2/L = 0.20 \quad L_3/L = 0.30$			
$L(m)$	$L_1(m)$	$L_2(m)$	$L_3(m)$	$L(m)$	$L_1(m)$	$L_2(m)$	$L_3(m)$
15	7.5	4.5	3.0	15	7.5	3.0	4.5
30	15.0	9.0	6.0	30	15.0	6.0	9.0

Table 2. The first three natural frequencies of Reddy–Bickford and Timoshenko piles whose end at the first region is semi-rigid connected and not restricted for horizontal displacement and the end at the third region is free, $L = 15$ m.

$L = 15\text{ m}$																		
$f = 0.25$																		
ω_i (rad/sec)	$L_1/L = 0.50$			$L_2/L = 0.30$			$L_3/L = 0.20$			$L_1/L = 0.50$			$L_2/L = 0.20$			$L_3/L = 0.30$		
	$N_r = 0.25$			$N_r = 1.00$			$N_r = 1.00$			$N_r = 0.25$			$N_r = 1.00$			$N_r = 1.00$		
	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	
	RBT			RBT			RBT			RBT			RBT			RBT		
ω_1	6.1274	6.0826	6.0823	6.2982	6.1985	6.1981	6.1340	6.0891	6.0888	6.3051	6.2054	6.2050	6.3051	6.2054	6.2050	6.3051	6.2054	6.2050
ω_2	34.0097	33.8781	33.8733	32.8500	32.6954	32.6906	34.0377	33.9054	33.9006	32.8792	32.7240	32.7191	32.8792	32.7240	32.7191	32.8792	32.7240	32.7191
ω_3	88.8914	88.9223	88.9028	87.5489	87.6241	87.6048	88.9436	88.9725	88.9529	87.6051	87.6781	87.6587	87.6051	87.6781	87.6587	87.6051	87.6781	87.6587
$f = 0.75$																		
ω_i (rad/sec)	$L_1/L = 0.50$			$L_2/L = 0.30$			$L_3/L = 0.20$			$L_1/L = 0.50$			$L_2/L = 0.20$			$L_3/L = 0.30$		
	$N_r = 0.25$			$N_r = 1.00$			$N_r = 1.00$			$N_r = 0.25$			$N_r = 1.00$			$N_r = 1.00$		
	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	
	RBT			RBT			RBT			RBT			RBT			RBT		
ω_1	7.6571	7.3756	7.3751	7.0486	6.6565	6.6560	7.6652	7.3834	7.3829	7.0574	6.6650	6.6645	7.0574	6.6650	6.6645	7.0574	6.6650	6.6645
ω_2	40.4182	39.1142	39.1080	39.2134	37.7777	37.7716	40.4502	39.1444	39.1381	39.2468	37.8093	37.8031	39.2468	37.8093	37.8031	39.2468	37.8093	37.8031
ω_3	96.9907	95.2717	95.2492	95.6900	93.9080	93.8857	97.0410	95.3205	95.2979	95.7448	93.9610	93.9386	95.7448	93.9610	93.9386	95.7448	93.9610	93.9386

Table 3. The first three natural frequencies of Reddy–Bickford and Timoshenko piles whose end at the first region is semi-rigid connected and not restricted for horizontal displacement and the end at the third region is free, $L = 30$ m.

$L = 30\text{ m}$																		
$f = 0.25$																		
ω_i (rad/sec)	$L_1/L = 0.50$			$L_2/L = 0.30$			$L_3/L = 0.20$			$L_1/L = 0.50$			$L_2/L = 0.20$			$L_3/L = 0.30$		
	$N_r = 0.25$			$N_r = 1.00$			$N_r = 1.00$			$N_r = 0.25$			$N_r = 1.00$			$N_r = 1.00$		
	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	RBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	RBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	RBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	
	RBT																	
ω_1	1.7489	1.7438	1.7438	1.7885	1.7762	1.7761	1.7490	1.7439	1.7439	1.7439	1.7886	1.7763	1.7763	1.7886	1.7763	1.7763	1.7762	
ω_2	9.9464	9.9296	9.9290	9.6504	9.6295	9.6290	9.9465	9.9297	9.9291	9.9291	9.6505	9.6296	9.6296	9.6505	9.6296	9.6296	9.6290	
ω_3	26.9903	26.9513	26.9484	26.6369	26.6002	26.5973	26.9904	26.9514	26.9485	26.9485	26.6370	26.6003	26.6003	26.6370	26.6003	26.6003	26.5974	
$f = 0.75$																		
ω_i (rad/sec)	$L_1/L = 0.50$			$L_2/L = 0.30$			$L_3/L = 0.20$			$L_1/L = 0.50$			$L_2/L = 0.20$			$L_3/L = 0.30$		
	$N_r = 0.25$			$N_r = 1.00$			$N_r = 1.00$			$N_r = 0.25$			$N_r = 1.00$			$N_r = 1.00$		
	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	RBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	RBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	RBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	
	RBT																	
ω_1	2.1671	2.1284	2.1284	2.0088	1.9551	1.9551	2.1672	2.1285	2.1285	2.1285	2.0089	1.9552	1.9552	2.0089	1.9552	1.9552	1.9552	
ω_2	11.6509	11.4670	11.4663	11.3302	11.1273	11.1265	11.6510	11.4671	11.4664	11.4664	11.3303	11.1274	11.1266	11.3303	11.1274	11.1266	11.1266	
ω_3	29.2498	28.9423	28.9390	28.8924	28.5723	28.5690	29.2499	28.9424	28.9391	28.9391	28.8925	28.5724	28.5691	28.8925	28.5724	28.5691	28.5691	

For $L = 15$ m, the frequency values obtained for the first three modes of Reddy–Bickford and Timoshenko piles whose end at the first region is semi-rigid connected and restricted for horizontal displacement and the end at the third region is fixed supported are presented in table 4; for $L = 30$ m, the frequency values obtained for the first three modes of the same piles are presented in table 5 being compared with the frequency values obtained for $N_r = 0.25$ and 1.00 , $f = 0.25$ and 0.75 .

The first two natural frequency values of Reddy–Bickford pile are higher than the first two natural frequency values of Timoshenko pile for both models and for all values of N_r , L and f . For both models, the third natural frequency values of Reddy–Bickford pile are lower for $f = 0.25$ and $L = 15$ m; are higher for $f = 0.75$ and $L = 30$ m, than the third natural frequency values of Timoshenko pile. The third natural frequency values of Reddy–Bickford pile are higher than the third natural frequency values of Timoshenko pile for both models, $L = 30$ m and all values of f . The differences between Reddy–Bickford beam theory and Timoshenko beam theory are more prominent for the higher frequencies.

For all boundary conditions, the differences between natural frequency values of Timoshenko pile models with $k = \frac{5}{6}$ and $k = \frac{14}{17}$ are small.

For $f = 0.25$ and for the condition of the other variables (L , L_1/L , L_2/L and L_3/L ratios) are constant, as the axial compressive force acting to Reddy–Bickford pile and Timoshenko pile whose end at the first region is semi-rigid connected and not restricted for horizontal displacement and the end at the third region is free, is increased, the first natural frequency values of Reddy–Bickford pile and Timoshenko pile are increased but the second and the third natural frequency values are decreased. The natural frequency values of Reddy–Bickford pile and Timoshenko pile are decreased as the axial force is increased for $f = 0.25$ in the second model and for $f = 0.75$ at all boundary conditions. These results indicate that, the boundary conditions (especially the end at the third region) of Reddy–Bickford pile and Timoshenko pile are important for the effect of axial force.

As the total length of the pile is increased, a decrease is observed in natural frequency values for both beam theories; for all boundary conditions and the condition of the other variables (f , N_r , L_1/L , L_2/L and L_3/L ratios) are being constant. This result indicates that the increase in the length of the pile leads to a reduction in natural frequency values for both beam theories.

A decrease is observed in natural frequency values of the first three modes of the pile for the condition of total pile length, N_r and L_1/L ratio being constant and of L_2/L ratio being greater than L_3/L ratio, for both support conditions and both beam theories. This decrease is more prominent for the short piles.

Natural frequencies values are different for two combinations of boundary conditions. For the condition of all variables are constant, the first three natural frequency values of the pile whose end at the first region is semi-rigid connected and restricted for horizontal displacement and the end at the third region is fixed supported are higher than the first three natural frequency values of the pile whose end at the first region is semi-rigid connected and not restricted for horizontal displacement and the end at the third region is free. This result indicates that the types of supporting affect the natural frequency values of the pile.

6. Summary and conclusion

In this study, starting from the governing differential equations of motion in free vibration, transfer matrices are developed by using Reddy–Bickford beam theory and the iterative-

Table 4. The first three natural frequencies of Reddy–Bickford and Timoshenko piles whose end at the first region is semi-rigid connected and restricted for horizontal displacement and the end at the third region is fixed supported, $L = 15$ m.

$L = 15\text{ m}$																		
$f = 0.25$																		
ω_i (rad/sec)	$L_1/L = 0.50$			$L_2/L = 0.30$			$L_3/L = 0.20$			$L_1/L = 0.50$			$L_2/L = 0.20$			$L_3/L = 0.30$		
	$N_r = 0.25$			$N_r = 1.00$			$N_r = 1.00$			$N_r = 0.25$			$N_r = 0.25$			$N_r = 1.00$		
	RBT	TBT	TBT	RBT	TBT	TBT	RBT	TBT	TBT	RBT	TBT	TBT	RBT	TBT	TBT	RBT	TBT	TBT
	($k = 5/6$)	($k = 14/17$)	($k = 14/17$)	($k = 5/6$)	($k = 14/17$)	($k = 14/17$)	($k = 5/6$)	($k = 14/17$)	($k = 14/17$)	($k = 5/6$)	($k = 14/17$)	($k = 14/17$)	($k = 5/6$)	($k = 14/17$)	($k = 14/17$)	($k = 5/6$)	($k = 14/17$)	($k = 14/17$)
ω_1	23.5027	23.4191	23.4159	22.0924	22.0306	22.0275	23.5239	23.4399	23.4366	22.1163	22.0524	22.0492	23.5239	23.4399	23.4366	22.1163	22.0524	22.0492
ω_2	72.2261	72.1754	72.1585	70.8519	70.6625	70.6458	72.2753	72.2225	72.2055	70.9039	70.7125	70.6957	72.2753	72.2225	72.2055	70.9039	70.7125	70.6957
ω_3	139.2844	139.6384	139.5974	137.8130	138.2810	138.2401	139.3032	139.6627	139.6217	137.8316	138.3044	138.2635	139.3032	139.6627	139.6217	137.8316	138.3044	138.2635
$f = 0.75$																		
ω_i (rad/sec)	$L_1/L = 0.50$			$L_2/L = 0.30$			$L_3/L = 0.20$			$L_1/L = 0.50$			$L_2/L = 0.20$			$L_3/L = 0.30$		
	$N_r = 0.25$			$N_r = 1.00$			$N_r = 1.00$			$N_r = 0.25$			$N_r = 0.25$			$N_r = 1.00$		
	RBT	TBT	TBT	RBT	TBT	TBT	RBT	TBT	TBT	RBT	TBT	TBT	RBT	TBT	TBT	RBT	TBT	TBT
	($k = 5/6$)	($k = 14/17$)	($k = 14/17$)	($k = 5/6$)	($k = 14/17$)	($k = 14/17$)	($k = 5/6$)	($k = 14/17$)	($k = 14/17$)	($k = 5/6$)	($k = 14/17$)	($k = 14/17$)	($k = 5/6$)	($k = 14/17$)	($k = 14/17$)	($k = 5/6$)	($k = 14/17$)	($k = 14/17$)
ω_1	28.3752	27.1491	27.1446	27.2392	25.9921	25.9877	28.3993	27.1718	27.1673	27.2642	26.0158	26.0114	28.3993	27.1718	27.1673	27.2642	26.0158	26.0114
ω_2	78.1441	76.3510	76.3310	76.6952	74.9345	74.9148	78.1939	76.3984	76.3783	76.7481	74.9848	74.9650	78.1939	76.3984	76.3783	76.7481	74.9848	74.9650
ω_3	144.8557	143.3242	143.2794	143.4875	142.0368	141.9921	144.8906	143.3586	143.3137	143.5151	142.0662	142.0215	144.8906	143.3586	143.3137	143.5151	142.0662	142.0215

Table 5. The first three natural frequencies of Reddy–Bickford and Timoshenko piles whose end at the first region is semi-rigid connected and restricted for horizontal displacement and the end at the third region is fixed supported, $L = 30$ m.

$L = 30\text{ m}$																		
$f = 0.25$																		
ω_i (rad/sec)	$L_1/L = 0.50$			$L_2/L = 0.30$			$L_3/L = 0.20$			$L_1/L = 0.50$			$L_2/L = 0.20$			$L_3/L = 0.30$		
	$N_r = 0.25$			$N_r = 1.00$			$N_r = 1.00$			$N_r = 0.25$			$N_r = 1.00$			$N_r = 1.00$		
	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	
	RBT			RBT			RBT			RBT			RBT			RBT		
ω_1	6.8482	6.8402	6.8398	6.4948	6.4886	6.4882	6.8483	6.8403	6.8399	6.4949	6.4886	6.4883	6.8403	6.8399	6.4949	6.4886	6.4883	
ω_2	21.7283	21.6986	21.6963	21.3219	21.2975	21.2952	21.7284	21.6987	21.6964	21.3220	21.2976	21.7283	21.6987	21.6964	21.3220	21.2976	21.2953	
ω_3	44.6694	44.6216	44.6140	44.2449	44.2075	44.2001	44.6695	44.6217	44.6141	44.2450	44.2076	44.6694	44.6217	44.6141	44.2450	44.2076	44.2002	
$f = 0.75$																		
ω_i (rad/sec)	$L_1/L = 0.50$			$L_2/L = 0.30$			$L_3/L = 0.20$			$L_1/L = 0.50$			$L_2/L = 0.20$			$L_3/L = 0.30$		
	$N_r = 0.25$			$N_r = 1.00$			$N_r = 1.00$			$N_r = 0.25$			$N_r = 1.00$			$N_r = 1.00$		
	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	TBT	TBT ($k = 5/6$)	TBT ($k = 14/17$)	
	RBT			RBT			RBT			RBT			RBT			RBT		
ω_1	8.0657	7.9103	7.9098	7.7722	7.6137	7.6132	8.0658	7.9104	7.9099	7.7723	7.6138	8.0657	7.9104	7.9099	7.7723	7.6138	7.6133	
ω_2	23.2304	22.9722	22.9696	22.8494	22.5935	22.5909	23.2305	22.9723	22.9697	22.8495	22.5936	23.2304	22.9722	22.9697	22.8495	22.5936	22.5910	
ω_3	46.3291	45.9953	45.9871	45.9199	45.5945	45.5864	46.3292	45.9954	45.9872	45.9200	45.5946	46.3291	45.9953	45.9872	45.9200	45.5946	45.5865	

based computer programs are developed for solution of linear-homogeneous frequency equation set relating to free vibration of different supported two piles partially embedded in elastic soil. Variation in free vibration natural frequencies for the first three modes of the pile is investigated for $N_r = 0.25$ and $N_r = 1.0$, $f = 0.25$ and $f = 0.75$, due to supporting conditions of pile ends and different lengths of the pile. Natural frequency values obtained from Reddy–Bickford beam theory are compared with the results of Timoshenko beam theory. As shown, the differences between Reddy–Bickford beam theory and Timoshenko beam theory become more prominent for free vibration of piles embedded in elastic soil. So that, to be on safer side it is recommended to use the higher-order theory.

Notation

The following symbols are used in this paper.

A	cross-section area of the pile
b	width of the pile
C_{R1}	modulus of subgrade reaction for the second region
C_{R2}	modulus of subgrade reaction for the third region
C_θ	rotational restraining stiffness at the upper end of the pile in the first region
E	Young's modulus
f	fixity factor
$[F(z = 1)]$	global transfer matrix
F_{jm}	terms of the global transfer matrix
G	shear modulus
h	height of the pile
I_x	moment of inertia
k	shape factor due to cross-section geometry
L	total length of the pile
L_1	pile length above the soil
L_2	pile length embedded in the second region
L_3	pile length embedded in the third region
m	mass per unit length of the pile
$M_h(z_j, t)$	bending moment function for j^{th} region
$M_{hj}(z_j, t)$	higher-order moment function for j^{th} region
N	axial compressive force
N_r	non-dimensionalized multiplication factor for the axial compressive force
$Q_j(z_j, t)$	shear force function for j^{th} region
$\{S_j(z_j, t)\}$	position vector for j^{th} region
z	dimensionless position parameter
t	time variable
$w_j(z_j, t)$	transverse displacement function for j^{th} region
$w_0(x, t)$	lateral displacement of the beam neutral axis
$w'_j(z_j, t)$	bending rotation function for j^{th} region
x_j	position for j^{th} region
$\phi_j(z_j, t)$	rotation of a normal to the axis of the pile for j^{th} region
ω	natural frequency

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