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Free vibrations of a multi-span Timoshenko beam carrying multiple spring-mass systems

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Abstract. Structural elements supporting motors or engines are frequently seen in technological applications. The operation of a machine may introduce additional dynamic stresses on the beam. It is important, then, to know the natural frequencies of the coupled beam-mass system, in order to obtain a proper design of the structural elements. The literature regarding the free vibration analysis of Bernoulli–Euler single-span beams carrying a number of spring-mass system and Bernoulli–Euler multi-span beams carrying multiple spring-mass systems are plenty, but on Timoshenko multi-span beams carrying multiple spring-mass systems is fewer. This paper aims at determining the natural frequencies and mode shapes of a Timoshenko multi-span beam. The model allows to analyse the influence of the shear effect and spring-mass systems on the dynamic behaviour of the beams by using Timoshenko Beam Theory (TBT). The effects of attached spring-mass systems on the free vibration characteristics of the 1–4 span beams are studied. The natural frequencies of Timoshenko multi-span beam calculated by using secant method for non-trivial solution are compared with the natural frequencies of multi-span beam calculated by using Bernoulli–Euler Beam Theory (EBT) in literature; the mode shapes are presented in graphs.

Keywords. Free vibration; Timoshenko multi-span beam; spring-mass system; eigenvalue problem.

1. Introduction

Extensive research has been carried out with regard to the vibration analysis of beams carrying concentrated masses at arbitrary positions and additional complexities.

Introducing the mass by the Dirac delta function, Chen (1963) solved analytically the problem of a simply supported beam carrying a concentrated mass. Chang (2000) solved a

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simply supported Rayleigh beam carrying a rigidly attached centered mass. Dowell (1979) studied general properties of beams carrying springs and concentrated masses. Gürgöze (1984, 1985) used the normal mode summation technique to determine the fundamental frequency of the cantilever beams carrying masses and springs. Gürgöze (1996) obtained the frequency equation of a clamped-free Bernoulli–Euler beam with attached tip mass and a spring-mass system by using the Lagrange multipliers method. Gürgöze (1998) presented two alternative formulations for the frequency equation of a clamped-free Bernoulli–Euler beam to which several spring-mass systems. Cha (2001) obtained the natural frequencies of a continuous structure with spring-mass attachments by using the classical assumed-modes method in conjunction with Lagrange's equations. Wu (2002) obtained the natural frequencies and mode shapes of the beams carrying any number of two-degrees of freedom spring-mass systems by using finite element method (FEM). Wu & Chen (2001) used the numerical assembly technique for free vibration analysis of a Timoshenko beam carrying multiple spring-mass systems. Lin & Tsai (2007) determined the natural frequencies and mode shapes of Bernoulli–Euler multi-span beam carrying multiple spring-mass systems. Liu *et al* (1998) formulated the frequency equation for beams carrying intermediate concentrated masses by using the Laplace Transformation Technique. Wu & Chou (1999) obtained the exact solution of the natural frequency values and mode shapes for a beam carrying any number of spring masses. Naguleswaran (2002, 2003) obtained the natural frequency values of the beams on up to five resilient supports including ends and carrying several particles by using EBT and obtained a fourth-order determinant equated to zero. Other studies on the vibration analysis of beams carrying masses are presented in references (Kukla & Posiadala 1994, Su & Banerjee 2005). In structural engineering, this problem is similar to free vibration of an elastic beam on elastic foundation and a pile embedded in soil. Doyle & Pavlovic (1982) solved the partial differential equation for free vibration of beams partially attached to elastic foundation using separation of variable method and neglecting shear effect. Catal (2002, 2006a) calculated natural circular frequencies and relative stiffness of the pile due to the values of axial forces acting on the pile and the shape factors. Yesilce & Catal (2008) calculated normalized natural frequencies of the pile using carry-over matrix and considering rotatory inertia. Catal (2006b) calculated frequency factors of a beam on elastic soil using differential transform method.

In this paper, free vibration of a multi-span Timoshenko beam carrying multiple spring-mass systems is studied. Exact solution is obtained for natural frequencies and mode shapes. 1–4 span beams carrying different number of spring-mass systems are used for the numerical analysis as part of the paper. Natural frequencies of Timoshenko multi-span beams are calculated by using secant method and the mode shapes are presented in graphs. The frequency values are compared with the ones of the same beams calculated by using Bernoulli–Euler Beam Theory (EBT) in literature.

2. The mathematical model and formulation

A uniform Timoshenko beam supported by $\bar{T}$ pins included at the two ends of beam and carrying S spring-mass systems is presented in figure 1. From figure 1, the total number of stations is $N = S + \bar{T}$ (Lin & Tsai 2007). The kinds of coordinates which are used in this study are given below:

$x_{v'}$ are the position coordinates for the stations, $(1 \leq v' \leq N)$,
x_p^* are the position coordinates of the spring-mass systems, $(1 \leq p \leq S)$,
$\bar{x}_r$ are the position coordinates of the pinned supports, $(1 \leq r \leq \bar{T})$.

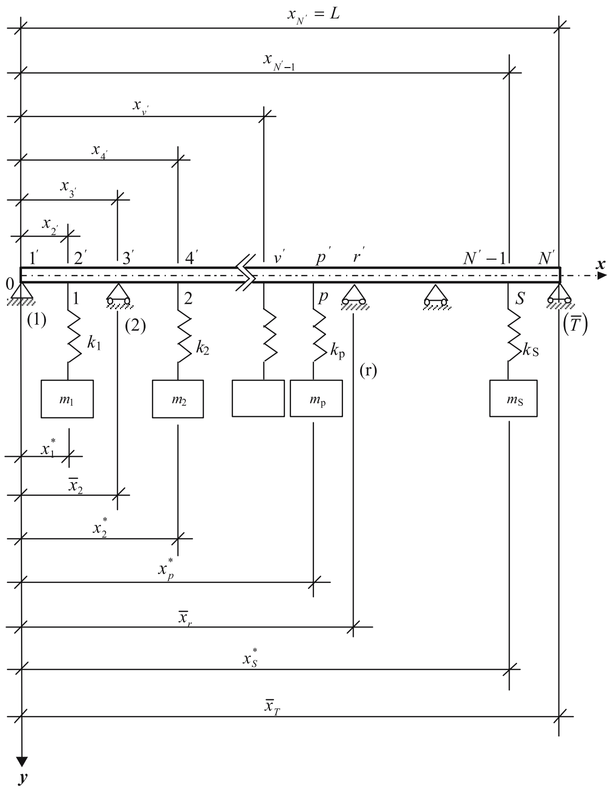

Figure 1. A uniform Timoshenko beam supported by T pins and carrying S spring-mass systems.

From figure 1, the symbols of $1', 2', \ldots, v', \ldots, N'-1, N'$ above the x-axis refer to the numbering of stations. The symbols of $1, 2, \ldots, p, \ldots, S$ below the x-axis refer to the numbering of spring-mass systems. The symbols of (1), (2), ..., (r), ..., $\bar{T}$ below the x-axis refer to the numbering of pinned supports.

The equation of motion for a Timoshenko beam can be written as (Catal 2002, Benaroya 2004, Catal 2006a, Yesilce & Catal 2008):

$$\frac{\partial^4 y(x,t)}{\partial x^4} - \frac{m \cdot k}{AG} \cdot \frac{\partial^4 y(x,t)}{\partial x^2 \cdot \partial t^2} + \frac{m}{EI_x} \cdot \frac{\partial^2 y(x,t)}{\partial t^2} = 0 \qquad (0 \le x \le L), \tag{1}$$

where $y(x,t)$ is elastic curve function of a Timoshenko beam; m is mass per unit length of the beam; x, is the beam position; t is time variable; L is length of the beam, A is the cross-section area; I_x is moment of inertia; k is the shape factor due to cross-section geometry of the beam; E, G are Young's modulus and shear modulus of the beam, respectively.

The p^{th} spring-mass system's equation of motion can be written as (Lin & Tsai 2007):

$$m_p \cdot \ddot{z}_p + k_p \cdot (z_p - y_p) = 0, \tag{2}$$

where m_p is the mass of the p^{th} spring-mass system, k_p is the spring constant of the p^{th} spring-mass system, $\ddot{z}_p$ and z_p are the acceleration and displacement of the mass of the p^{th} spring-mass system relative to its static equilibrium position, respectively, y_p is the value of elastic curve function of the beam at the attaching point of the p^{th} spring-mass system.

The solution of differential equation (1) can be obtained by the method of separation of variables (Doyle & Pavlovic 1982, Catal 2002, Catal 2006a).

The solution of (1) and (2) are obtained as:

$$y(z,t) = \phi(z)\cdot\sin(\omega\cdot t) \qquad (0 \le z \le 1) \tag{3a}$$

$$z_p = Z_p \cdot \sin(\omega\cdot t) \tag{3b}$$

in which

$$\phi(z) = [C_1\cdot\cosh(D_1\cdot z) + C_2\cdot\sinh(D_1\cdot z) + C_3\cdot\cos(D_2\cdot z) + C_4\cdot\sin(D_2\cdot z)];\quad D_1 = (\Delta_1)^{0\cdot 5};\quad D_2 = (\Delta_2)^{0\cdot 5};$$

$$\Delta_1 = -\frac{\beta}{2} + (\gamma)^{0,5};\quad \Delta_2 = -\frac{\beta}{2} - (\gamma)^{0,5};\quad \beta = \frac{m\cdot\omega^2\cdot k\cdot L^2}{AG};$$

$$\gamma = \left(\frac{\beta}{2}\right)^2 + \lambda^4;\quad \lambda = \sqrt[4]{\frac{m\cdot\omega^2\cdot L^4}{EI_x}}\ \text{(frequency factor)};\quad z = \frac{x}{L};$$

$\omega =$ natural circular frequency; $C_1, \ldots, C_4 =$ constants of integration; $L =$ total length of the beam.

The bending moment function of Timoshenko beam with respect to z is given below (Benaroya 2004):

$$M(z,t) = \left\{-\frac{EI_x}{L^2}\cdot\frac{\partial^2\phi(z)}{\partial z^2} - m\cdot\omega^2\cdot\frac{k\cdot EI_x}{AG}\cdot\phi(z)\right\}\sin(\omega\cdot t). \tag{4}$$

Differentiating (4) with respect to z gives the shear force function of the beam, as:

$$T(z,t) = \left\{-\frac{EI_x}{L^3}\cdot\frac{\partial^3\phi(z)}{\partial z^3} - m\cdot\omega^2\cdot\frac{k\cdot EI_x}{AG}\cdot\frac{1}{L}\cdot\frac{\partial\phi(z)}{\partial z}\right\}\sin(\omega\cdot t). \tag{5}$$

Slope function for a Timoshenko beam is given with respect to z in terms of total displacements as follows (Geradin & Rixen 1998)

$$\theta(z,t) = \left\{\frac{k}{AG}\cdot\left\{\frac{EI_x}{L^3}\cdot\frac{\partial^3\phi(z)}{\partial z^3} + \left[m\cdot\omega^2\cdot\frac{k\cdot EI_x}{AG} + \frac{AG}{k}\right]\cdot\frac{1}{L}\cdot\frac{\partial\phi(z)}{\partial z}\right\}\right\}\sin(\omega\cdot t). \tag{6}$$

3. Determination of natural frequencies and mode shapes

The position is written due to the values of the displacement, slope, bending moment and shear force functions at the locations of z and t for Timoshenko beam, as:

$$\langle S(z,t)\rangle^T = \langle\phi(z)\quad\theta(z)\quad M(z)\quad T(z)\rangle\cdot\sin(\omega\cdot t), \tag{7}$$

where $\langle S(z,t)\rangle$ shows the state vector.

The boundary conditions for the left-end support of the beam are written as:

$$\phi_{1'}(z = 0) = 0 \tag{8a}$$

$$M_{1'}(z = 0) = 0. \tag{8b}$$

From (3a) and (4), the boundary conditions for the left-end support can be written in matrix notation as:

$$[B_{1'}] \cdot \{C_{1'}\} = \{0\} \tag{9a}$$

$$\begin{matrix} 1 & 2 & 3 & 4 & \\ \end{matrix} \left[\begin{matrix} 1 & 0 & 1 & 0 \\ K_1 & 0 & -K_2 & 0 \end{matrix}\right] \begin{matrix} 1 \\ 2 \end{matrix} \cdot \begin{Bmatrix} C_{1',1} \\ C_{1',2} \\ C_{1',3} \\ C_{1',4} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{9b}$$

where

$$K_1 = -\frac{EI_x}{L^2} \cdot D_1^2 - \frac{m \cdot \omega^2 \cdot k \cdot EI_x}{AG};\ K_2 = -\frac{EI_x}{L^2} \cdot D_2^2 + \frac{m \cdot \omega^2 \cdot k \cdot EI_x}{AG}.$$

The boundary conditions for the p^{th} intermediate spring-mass system are written by using the continuity of deformations and slopes, and the equilibrium of bending moments and shear forces as (the station numbering corresponding to the p^{th} intermediate spring-mass system is represented by p'):

$$\phi^L_{p'}(z_{p'}) = \phi^R_{p'}(z_{p'}) \tag{10a}$$

$$\theta^L_{p'}(z_{p'}) = \theta^R_{p'}(z_{p'}) \tag{10b}$$

$$M^L_{p'}(z_{p'}) = M^R_{p'}(z_{p'}) \tag{10c}$$

$$T^L_{p'}(z_{p'}) - m_p \cdot \omega^2 \cdot Z_p = T^R_{p'}(z_{p'}), \tag{10d}$$

where L and R refer to the left side and right side of the p^{th} intermediate spring-mass system, respectively.

Substituting (3a) and (3b) into (2) gives:

$$\phi_{p'} + (\alpha_p^2 - 1) \cdot Z_p = 0, \tag{11}$$

where $\alpha_p = \frac{\omega}{\omega_p}$; $\omega_p = \sqrt{\frac{k_p}{m_p}}$.

From (3a), (4), (5) and (6), the boundary conditions for the p^{th} intermediate spring-mass system can be written in matrix notation as:

$$[B_{p'}] \cdot \{C_{p'}\} = \{0\}, \tag{12}$$

where

$$\{C'_p\}^T = \{C_{p'-1,1} \quad C_{p'-1,2} \quad C_{p'-1,3} \quad C_{p'-1,4} \quad C_{p',1} \quad C_{p',2} \quad C_{p',3} \quad C_{p',4} \quad Z_p\} \tag{13a}$$

$$[B_{p'}] = \begin{array}{c} \begin{array}{cccccc} 4p'-3 & 4p'-2 & 4p'-1 & 4p' & 4p'+1 & 4p'+2 \end{array} \\ \left[\begin{array}{cccccc} ch_1 & sh_1 & cs_2 & sn_2 & -ch_1 & -sh_1 \\ K_3 \cdot sh_1 & K_3 \cdot ch_1 & K_4 \cdot sn_2 & -K_4 \cdot cs_2 & -K_3 \cdot sh_1 & -K_3 \cdot ch_1 \\ K_1 \cdot ch_1 & K_1 \cdot sh_1 & -K_2 \cdot cs_2 & -K_2 \cdot sn_2 & -K_1 \cdot ch_1 & -K_1 \cdot sh_1 \\ K_5 \cdot sh_1 & K_5 \cdot ch_1 & K_6 \cdot sn_2 & -K_6 \cdot cs_2 & -K_5 \cdot sh_1 & -K_5 \cdot ch_1 \\ 0 & 0 & 0 & 0 & ch_1 & sh_1 \end{array}\right. \end{array}$$

$$\begin{array}{c} \begin{array}{ccc} 4p'+3 & 4p'+4 & 4p'+5 \end{array} \\ \left.\begin{array}{ccc} -cs_2 & -sn_2 & 0 \\ -K_4 \cdot sn_2 & K_4 \cdot cs_2 & 0 \\ K_2 \cdot cs_2 & K_2 \cdot sn_2 & 0 \\ -K_6 \cdot sn_2 & K_6 \cdot cs_2 & -m_p \cdot \omega^2 \\ cs_2 & sn_2 & \alpha_p^2 - 1 \end{array}\right] \begin{array}{c} 4p'-1 \\ 4p' \\ 4p'+1 \\ 4p'+2 \\ 4p'+3 \end{array} \end{array}, \tag{13b}$$

where

$$ch_1 = \cosh(D_1 \cdot z_{p'}); \quad ch_2 = \cosh(D_2 \cdot z_{p'}); \quad sh_1 = \sinh(D_1 \cdot z_{p'});$$

$$sh_2 = \sinh(D_2 \cdot z_{p'}); \quad cs_1 = \cos(D_1 \cdot z_{p'}); \quad cs_2 = \cos(D_2 \cdot z_{p'});$$

$$sn_1 = \sin(D_1 \cdot z_{p'}); \quad sn_2 = \sin(D_2 \cdot z_{p'});$$

$$K_3 = \frac{k \cdot EI_x}{AG \cdot L^3} \cdot D_1^3 + \left(\frac{m \cdot \omega^2 \cdot k^2 \cdot EI_x}{(AG)^2} + 1\right) \cdot \frac{D_1}{L};$$

$$K_4 = \frac{k \cdot EI_x}{AG \cdot L^3} \cdot D_2^3 - \left(\frac{m \cdot \omega^2 \cdot k^2 \cdot EI_x}{(AG)^2} + 1\right) \cdot \frac{D_2}{L}$$

$$K_5 = -\frac{EI_x}{L^3} \cdot D_1^3 - \frac{m \cdot \omega^2 \cdot k \cdot EI_x}{AG} \cdot \frac{D_1}{L};$$

$$K_6 = -\frac{EI_x}{L^3} \cdot D_2^3 + \frac{m \cdot \omega^2 \cdot k \cdot EI_x}{AG} \cdot \frac{D_2}{L}.$$

The boundary conditions for the r^{th} support are written by using continuity of deformations, slopes and equilibrium of bending moments, as (the station numbering corresponding to the r^{th} intermediate support is represented by r'):

$$\phi_{r'}^L(z_{r'}) = \phi_{r'}^R(z_{r'}) = 0 \tag{14a}$$

$$\theta_{r'}^L(z_{r'}) = \theta_{r'}^R(z_{r'}) \tag{14b}$$

$$M_{r'}^L(z_{r'}) = M_{r'}^R(z_{r'}). \tag{14c}$$

From (3a), (4), and (5), the boundary conditions for the r^{th} intermediate support can be written in matrix notation as:

$$[B_{r'}] \cdot \{C_{r'}\} = \{0\}, \tag{15}$$

where

$$\{C'_r\}^T = \{C_{r'-1,1} \quad C_{r'-1,2} \quad C_{r'-1,3} \quad C_{r'-1,4} \quad C_{r',1} \quad C_{r',2} \quad C_{r',3} \quad C_{r',4}\} \tag{16a}$$

$$[B'_r] = \begin{array}{c} \begin{array}{ccccc} 4r'-3 & 4r'-2 & 4r'-1 & 4r' & 4r'+1 \end{array} \\ \left[\begin{array}{ccccc} chr_1 & shr_1 & csr_2 & snr_2 & 0 \\ 0 & 0 & 0 & 0 & chr_1 \\ K_3 \cdot shr_1 & K_3 \cdot chr_1 & K_4 \cdot snr_2 & -K_4 \cdot csr_2 & -K_3 \cdot shr_1 \\ K_1 \cdot chr_1 & K_1 \cdot shr_1 & -K_2 \cdot csr_2 & -K_2 \cdot snr_2 & -K_1 \cdot chr_1 \end{array}\right. \end{array}$$

$$\begin{array}{c} \begin{array}{ccc} 4r'+2 & 4r'+3 & 4r'+4 \end{array} \\ \left.\begin{array}{ccc} 0 & 0 & 0 \\ shr_1 & csr_2 & snr_2 \\ -K_3 \cdot chr_1 & -K_4 \cdot snr_2 & K_4 \cdot csr_2 \\ -K_1 \cdot shr_1 & K_2 \cdot csr_2 & K_2 \cdot snr_2 \end{array}\right] \begin{array}{c} 4r'-1 \\ 4r' \\ 4r'+1 \\ 4r'+2 \end{array} \end{array}, \tag{16b}$$

where

$$chr_1 = \cosh(D_1 \cdot z_{r'}); \quad chr_2 = \cosh(D_2 \cdot z_{r'}); \quad shr_1 = \sinh(D_1 \cdot z_{r'});$$

$$shr_2 = \sinh(D_2 \cdot z_{r'}); \quad csr_1 = \cos(D_1 \cdot z_{r'}); \quad csr_2 = \cos(D_2 \cdot z_{r'});$$

$$snr_1 = \sin(D_1 \cdot z_{r'}); \quad snr_2 = \sin(D_2 \cdot z_{r'}).$$

The boundary conditions for the right-end support of the beam are written as:

$$\phi_{N'}(z = 1) = 0 \tag{17a}$$

$$M_{N'}(z = 1) = 0. \tag{17b}$$

From (3a) and (4), the boundary conditions for the right-end support can be written in matrix notation as:

$$[B_{N'}] \cdot \{C_{N'}\} = \{0\}, \tag{18}$$

where

$$\{C'_N\}^T = \{C_{N',1} \quad C_{N',2} \quad C_{N',3} \quad C_{N',4}\} \tag{19a}$$

$$[B'_N] = \begin{array}{c} \begin{array}{cccc} 4N_{i'}+1 & 4N_{i'}+2 & 4N_{i'}+3 & 4N_{i'}+4 \end{array} \\ \left[\begin{array}{cccc} \cosh(D_1) & \sinh(D_1) & \cos(D_2) & \sin(D_2) \\ K_1 \cdot \cosh(D_1) & K_1 \cdot \sinh(D_1) & -K_2 \cdot \cos(D_2) & -K_2 \cdot \sin(D_2) \end{array}\right] \begin{array}{c} q \\ q-1 \end{array} \end{array} \tag{19b}$$

where N'_i is the total number of intermediate stations and is given by:

$$N'_i = S + \bar{T} - 2. \tag{20}$$

The total number of equations for the integration constants is obtained as:

$$q = 2 + 4 \cdot (\bar{T} - 2) + 5 \cdot S + 2. \tag{21}$$

From (21), it can be seen that; the left-end support of the beam has two equations, each intermediate support of the beam has four equations, each intermediate spring-mass system of the beam has five equations and the right-end support of the beam has two equations.

In this paper, by using (9b), (13b), (16b) and (19b), the coefficient matrices for left-end support, each intermediate spring-mass system, each intermediate pinned support and right-end support of a Timoshenko beam are derived, respectively. In the next step, the numerical assembly technique is used to establish the overall coefficient matrix for the whole vibrating system as given in (22). In the last step, for non-trivial solution, equating the last overall coefficient matrix to zero one determines the natural frequencies of the vibrating system as given in (23); and substituting of the last integration constants into the related eigenfunctions one determines the associated mode shapes.

$$[B] \cdot \{C\} = \{0\} \tag{22}$$

$$|B| = 0. \tag{23}$$

4. Numerical analysis and discussions

For numerical analysis, five models are considered. For these five models, natural frequencies of the beam, $\omega_i \, (i = 1, 2, 3, \ldots)$ are calculated by using a computer program prepared by the authors. In this program, the secant method is used in which determinant values are evaluated for a range of (ω_i) values. The (ω_i) value causing a sign change between the successive determinant values is a root of frequency equation and means a frequency for the system.

The first four numerical results of this paper are obtained based on a uniform, circular Timoshenko beam with the following data as in (Lin & Tsai 2007):

Diameter $d = 0{\cdot}05$ m; $EI_x = 6{\cdot}34761 \times 10^4$ Nm^2; $m = 15{\cdot}3875$ kg/m; $L = 1{\cdot}0$ m; total mass $m_b = m \cdot L = 15{\cdot}3875$ kg; reference spring constant $k_b = EI_x/L^3 = 6{\cdot}34761$ N/m.

As in (Lin & Tsai 2007), two non-dimensional $\hat{m}_p$ and $\hat{k}_p$, are defined by $\hat{m}_p = m_p/m_b$ and $\hat{k}_p = k_p/k_b$. Different from (Lin & Tsai 2007), $k = 4/3$ and $AG = 1{\cdot}562489231 \times 10^8$ N for the shear effect.

In the fifth numerical example of this paper, a different circular beam is considered to give an example from structural engineering. The characteristics of this beam are presented as:

Diameter $d = 0{\cdot}20$ m; $A = 3{\cdot}14 \times 10^{-2}$ m^2; $m = 245{\cdot}04$ kg/m; $EI_x = 1{\cdot}649336 \times 10^7$ Nm^2; $AG = 2{\cdot}54 \times 10^9$ N; $k = 4/3$; $L = 4{\cdot}0$ m; total mass $m_b = m \cdot L = 980{\cdot}16$ kg; reference spring constant $k_b = EI_x/L^3 = 25770877{\cdot}23$ N/m.

4.1 *Free vibration analysis of the uniform single-span pinned-pinned Timoshenko beam carrying five intermediate spring-mass systems*

In the first numerical example, the uniform single-span pinned-pinned Timoshenko beam carrying one, three and five intermediate spring-mass systems, respectively are considered. For this numerical example, the natural frequency of the five spring-mass systems (see figure 2) with respect to the static beam defined in § 4 are given in table 1. The frequency values obtained for the first five modes are presented in table 2 by comparing with the frequency

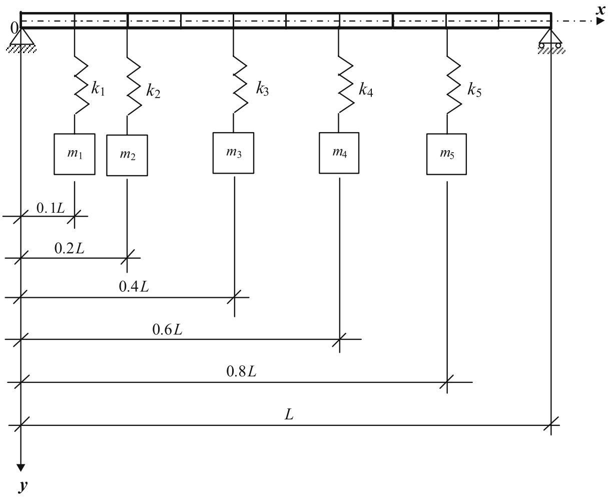

Figure 2. A single-span pinned-pinned Timoshenko beam carrying five intermediate spring-mass systems, $p = 1$ to 5.

values obtained for the same models for a Bernoulli–Euler beam in (Lin & Tsai 2007), for three cases, which are presented below:

1^{st} *case:* In this case, there is one spring-mass system and $z_p^* = x_p^*/L = 0{\cdot}75$, $p = 1$.

2^{nd} *case:* In this case, there are three spring-mass systems and $z_p^* = 0{\cdot}1$, $0{\cdot}4$ and $0{\cdot}8$, respectively, $p = 1, 3, 5$.

Table 1. Natural frequencies of the five spring-mass systems with respect to the bare beam.

Numbering, p	1	2	3	4	5
$\hat{m}_p = m_p/m_b$	0·2	0·3	0·5	0·65	1·0
$\hat{k}_p = k_p/k_b$	3·0	3·5	4·5	5·0	6·0
$\omega_p = \sqrt{k_p/m_p}$	248·7521	219·3787	192·6825	178·1351	157·3246

Table 2. The first five natural frequencies of the uniform single-span pinned-pinned Timoshenko beam and Bernoulli–Euler beam in (Lin & Tsai 2007).

		Natural frequencies, ω_i (rad/sec)				
Cases	Methods	ω_1	ω_2	ω_3	ω_4	ω_5
1	TBT	243·8150	643·5531	2513·9003	5573·7283	9734·6103
	EBT	243·8577	645·2028	2540·5298	5706·1880	10142·4002
2	TBT	152·6903	185·0458	247·8140	676·0293	2522·1120
	EBT	152·7339	185·0949	247·8313	677·5959	2548·6572
3	TBT	150·8958	169·4255	187·8848	217·1026	247·9713
	EBT	150·9571	169·4728	187·9146	217·1278	247·9867

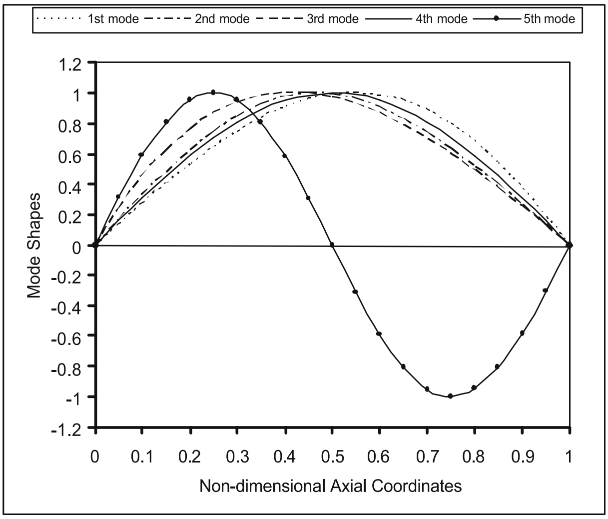

Figure 3. The first five mode shapes for the third case.

3^{rd} *case:* In this case, there are five spring-mass systems and $z_p^* = 0{\cdot}1, 0{\cdot}2, 0{\cdot}4, 0{\cdot}6$ and $0{\cdot}8$, respectively, $p = 1, 2, 3, 4, 5$.

For the third case, the first five mode shapes are shown in figure 3. Figure 3 indicates that the first four mode shapes are similar, but the fifth mode shape is different from the first four mode shapes.

4.2 *Free vibration analysis of the uniform two-span Timoshenko beam carrying one intermediate spring-mass system*

In the second numerical example (figure 4), the uniform two-span Timoshenko beam carrying one spring-mass system is considered. In this numerical example, for the intermediate support, $\bar{z}_1 = 0{\cdot}4$; for the intermediate spring-mass system, $z_1^* = 0{\cdot}75$, $\hat{m}_p = m_p/m_b = 0{\cdot}2$ and $\hat{k}_p = k_p/k_b = 3{\cdot}0$. The frequency values obtained for the first five modes are presented in table 3 by comparing with the frequency values obtained for the same model for a

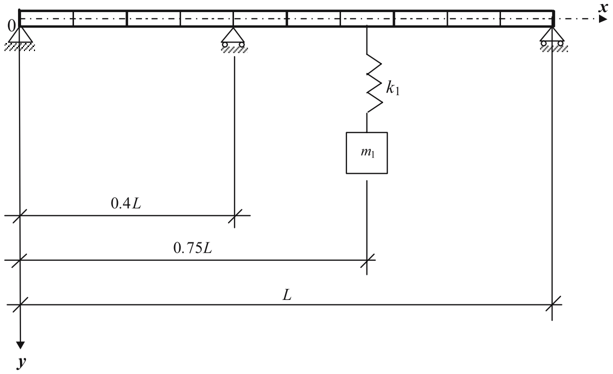

Figure 4. A two-span Timoshenko beam carrying one intermediate spring-mass system.

Table 3. The first five natural frequencies of the uniform two-span Timoshenko beam and Bernoulli–Euler beam in (Lin & Tsai 2007).

		Natural frequencies, ω_i (rad/sec)				
p	Methods	ω_1	ω_2	ω_3	ω_4	ω_5
1	TBT	247·6031	2131·6091	4804·8941	7860·1450	14884·4672
	EBT	247·6358	2156·8778	4938·1451	8220·1624	15847·8928
0	TBT	2122·2831	4804·2786	7859·9203	14884·0514	17462·8254
	EBT	2147·6735	4937·5110	8219·9705	15847·5054	19288·5388

Bernoulli–Euler beam in (Lin & Tsai 2007) and mode shapes for the model with one spring-mass system of Timoshenko beam are presented in figure 5.

The first two mode shapes have similar forms in figure 5, this is because the frequency values for the 2^{nd}, 3^{rd}, 4^{th} and 5^{th} modes of the beam with one attachment are close to the ones for the 1^{st}, 2^{nd}, 3^{rd} and 4^{th} modes of the beam with no attachments.

4.3 *Free vibration analysis of the uniform three-span Timoshenko beam carrying three intermediate spring-mass systems*

In the third numerical example (figure 6), the uniform three-span Timoshenko beam carrying three spring-mass systems is considered. In this numerical example, for the first intermediate support, $\bar{z}_1 = 0{\cdot}3$; for the second intermediate support, $\bar{z}_2 = 0{\cdot}7$; for the first intermediate spring-mass system, $z_1^* = 0{\cdot}1$, $\hat{m}_1 = 0{\cdot}2$ and $\hat{k}_1 = 3{\cdot}0$; for the second intermediate spring-mass system, $z_2^* = 0{\cdot}4$, $\hat{m}_2 = 0{\cdot}3$ and $\hat{k}_2 = 3{\cdot}5$; for the third intermediate spring-mass system, $z_3^* = 0{\cdot}8$, $\hat{m}_3 = 0{\cdot}5$ and $\hat{k}_3 = 4{\cdot}5$. The frequency values obtained for the first five modes are presented in table 4 by comparing with the frequency values obtained for the same

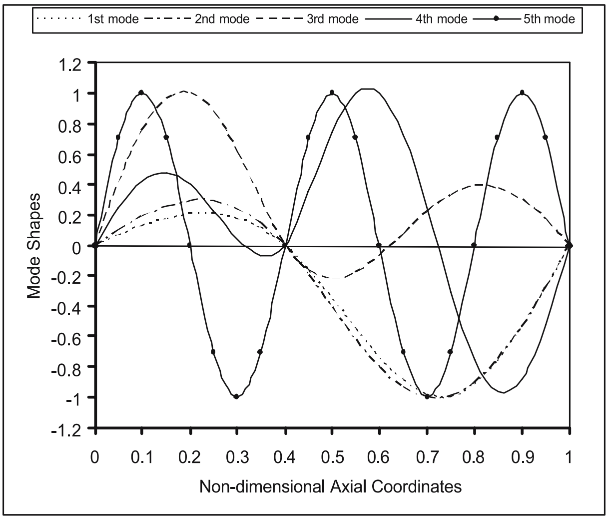

Figure 5. The first five mode shapes for the model with one spring-mass system of Timoshenko beam.

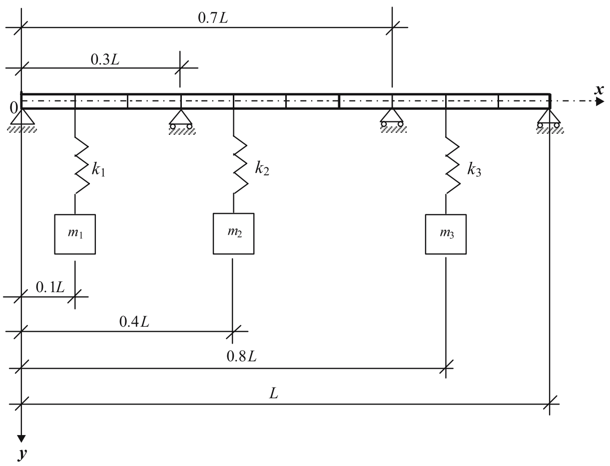

Figure 6. A three-span Timoshenko beam carrying three intermediate spring-mass systems.

model for a Bernoulli–Euler beam in (Lin & Tsai 2007) and mode shapes for the model with three spring-mass systems of Timoshenko beam are presented in figure 7.

From table 4 one can see that the 4th and 5th mode frequency values are very close to the 1st and 2nd modes values obtained for the model with no attachment.

4.4 *Free vibration analysis of the uniform four-span Timoshenko beam carrying three intermediate spring-mass systems*

In the fourth numerical example (figure 8), the uniform four-span Timoshenko beam carrying three spring-mass systems is considered. In this numerical example, for the first intermediate support, $\bar{z}_1 = 0{\cdot}3$; for the second intermediate support, $\bar{z}_2 = 0{\cdot}5$; for the third intermediate support $\bar{z}_3 = 0{\cdot}7$; for the first, the second and the third intermediate spring-mass systems, locations and non-dimensional parameters are taken as which are given in § 4·3. The frequency values obtained for the first five modes are presented in table 5 by comparing with the frequency values obtained for the same model for a Bernoulli–Euler beam in (Lin & Tsai 2007) and mode shapes for the model with three spring-mass systems of four-span Timoshenko beam are shown in figure 9.

It is seen from table 5 that, as in example 4·3, the 4th and 5th mode frequency values are close to the 1st and 2nd modes frequency values obtained for the model with no attachment.

Table 4. The first five natural frequencies of the uniform three-span Timoshenko beam and Bernoulli–Euler beam in (Lin & Tsai 2007).

		Natural frequencies, ω_i (rad/sec)				
p	Methods	ω_1	ω_2	ω_3	ω_4	ω_5
3	TBT	192·5355	219·2253	248·6066	5114·0063	8208·9722
	EBT	192·5522	219·2418	248·6207	5252·5167	8595·0801
0	TBT	5109·5903	8204·3652	9567·9158	17836·4578	26723·0012
	EBT	5248·1555	8590·6934	10305·8151	19750·4270	30163·2250

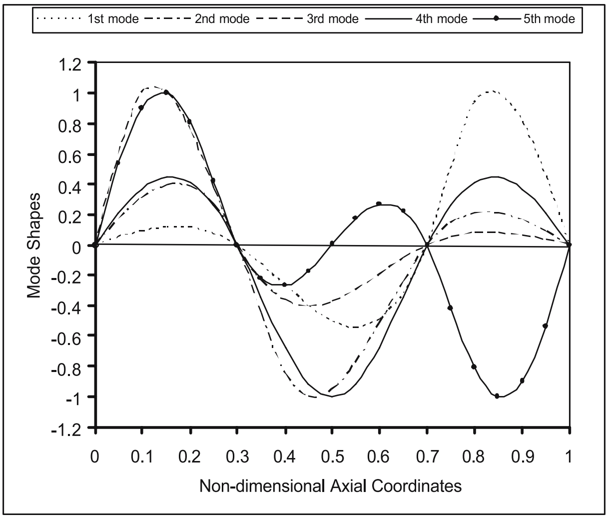

Figure 7. The first five mode shapes for the model with three spring-mass systems of Timoshenko beam.

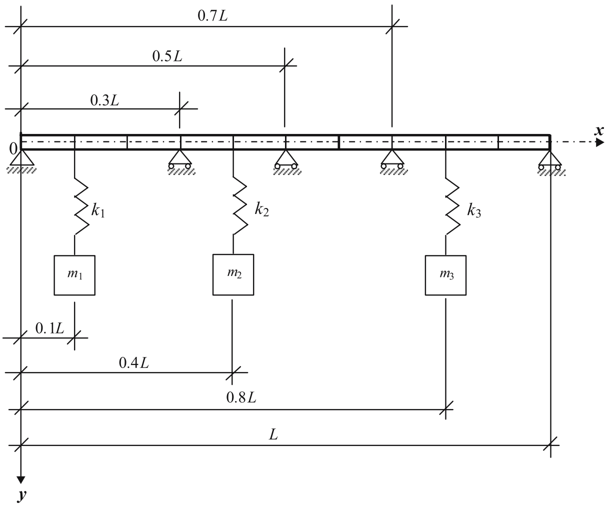

Figure 8. A four-span Timoshenko beam carrying three intermediate spring-mass systems.

Table 5. The first five natural frequencies of the uniform four-span Timoshenko beam and Bernoulli–Euler beam in (Lin & Tsai 2007).

		Natural frequencies, ω_i (rad/sec)				
p	Methods	ω_1	ω_2	ω_3	ω_4	ω_5
3	TBT	192·5558	219·3310	248·6177	8208·9721	8489·6965
	EBT	192·5744	219·3417	248·6328	8595·0801	9016·0676
0	TBT	8204·3652	8485·2542	17836·4578	23008·5397	28062·3113
	EBT	8590·6942	9011·9071	19750·4674	26562·3933	32882·0445

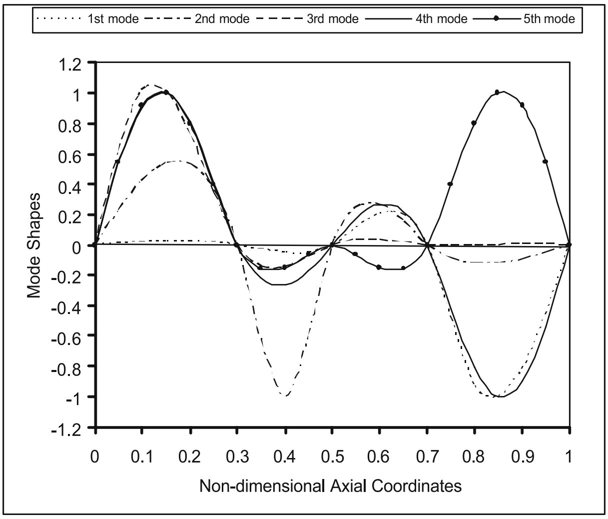

Figure 9. The first five mode shapes for the model with three spring-mass systems of four-span Timoshenko beam.

4.5 *Free vibration analysis of the circular four-span Timoshenko beam carrying three intermediate spring-mass systems,* $d = 0{\cdot}20\,m$, $L = 4{\cdot}0\,m$

In the fifth numerical example (figure 8), the circular four-span Timoshenko beam carrying three spring-mass systems is considered. In this numerical example; for the first, the second and the third intermediate supports, locations are taken which are given in § 4·4 and for the first, the second and the third intermediate spring-mass systems, locations and non-dimensional parameters are taken as in § 4·4. The frequency values obtained for the first five modes are presented in table 6 by comparing with the frequency values obtained by authors for the same model for a Bernoulli–Euler beam and mode shapes for the model with three spring-mass systems of four-span Timoshenko beam are given in figure 10.

It is seen from table 6 that, as in examples 4·3 and 4·4, the 4th and 5th mode frequency values are close to the 1st and 2nd modes frequency values obtained for the model with no attachment. It is seen from figure 10 that; especially, the 2nd and 3rd mode shapes are different from the mode shapes in example 4·3. when $0{\cdot}7 < z_p < 1{\cdot}0$. This result indicates that the behaviour of the beam comes into prominence especially in the 2nd and 3rd modes if the length and cross-section area of the beam are increased.

Table 6. The first five natural frequencies of the uniform four-span Timoshenko beam and Bernoulli–Euler beam; $d = 0{\cdot}20$ m, $L = 4{\cdot}0$ m.

		Natural frequencies, ω_i (rad/sec)				
p	Methods	ω_1	ω_2	ω_3	ω_4	ω_5
3	TBT	48·6129	55·3727	62·7664	1036·3325	1071·8015
	EBT	48·6177	55·3754	62·7702	1084·9644	1138·1057
0	TBT	1035·7498	1071·2396	2251·9977	2905·2692	3545·0012
	EBT	1084·4095	1137·5794	2493·0064	3352·0186	3931·0215

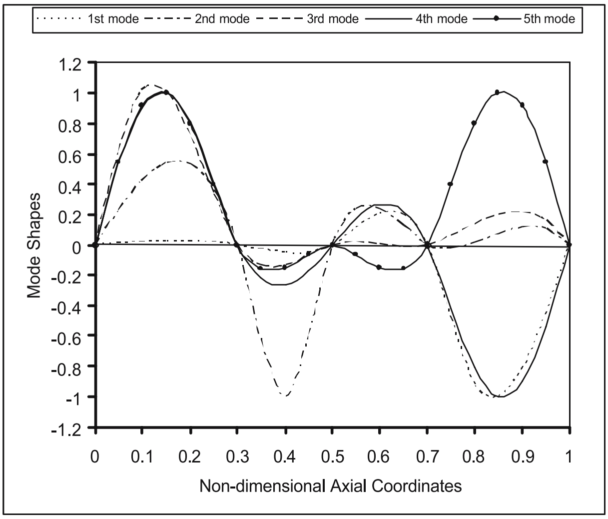

Figure 10. The first five mode shapes for the model with three spring-mass systems of four-span Timoshenko beam; $d = 0{\cdot}20$ m, $L = 4{\cdot}0$ m.

5. Conclusion

In this study, frequency values and mode shapes for free vibration of the multi-span Timoshenko beam with multiple spring-mass systems are obtained for different number of spans and spring-masses with different locations. In the first four numerical examples, the frequency values are presented in tables with the values obtained before for Bernoulli–Euler beam in (Lin & Tsai 2007). In the fifth numerical example, the frequency values are determined for Timoshenko and Bernoulli–Euler beams. The frequency values obtained for the Timoshenko beam in this study are a little less than the values obtained for the Bernoulli–Euler beam in the reference study and in the fifth numerical example. This result indicates that shear effects lead to reduction in natural frequency values.

It can be seen from the tables that the frequency values show a decrease as a spring-mass system is attached to the bare beam; this decrease show a continuity as the number of spring-mass attachments is increased.

It is also seen from the tables and the mode shapes that one can use the relation $\omega_{p+i} \approx \omega_{bi}$ $(i = 1, 2, 3, \ldots)$ for the Timoshenko beams with spring-mass attachments that is given in (Lin & Tsai 2007) for the Bernoulli–Euler beams with spring-mass attachments where p is the number of spring-mass attachments, i is the number of modes considered and ω_b is the frequency of the beam carrying no spring-mass system. Therefore, the first $(p + i)$ mode shapes of the beam carrying p spring-mass attachments have similar forms since the $(p+i)^{\text{th}}$ mode frequency value of the beam with attachments is very close to the 1^{st} mode frequency value of the beam with no attachment.

6. Notation

A	cross-section area of the beam
$[B]$	coefficient matrix
$\{C\}$	unknown coefficients vector
E	Young's modulus
G	shear modulus

I_x	moment of inertia
k	shape factor due to cross-section geometry
k_b	reference spring constant
k_p	spring constant of the p^{th} spring-mass system
L	length of the beam
N_i'	total number of intermediate stations
N'	total number of stations
m	mass per unit length of the beam
$M(z,t)$	bending moment function
m_b	total mass of the beam
m_p	mass of the p^{th} spring-mass system
n	number of nodes
q	total number of equations
S	number of spring-mass systems
$\langle S(z,t)\rangle$	state vector
$\bar{T}$	number of pins
t	time variable
$T(z,t)$	shear force function
x	state variable
$x_{v'}$	state coordinates for the stations
x_p^*	state coordinates of the spring-mass systems
$\bar{x}_r$	state coordinates of the pinned supports
$y(x,t)$	elastic curve function
y_p	value of elastic curve function of the beam at the attaching point of the p^{th} spring-mass system
z	dimensionless state parameter
$\ddot{z}_p$	acceleration of the mass of the p^{th} spring-mass system
z_p	displacement of the mass of the p^{th} spring-mass system
$\theta(z,t)$	slope function
λ	frequency factor
ω	natural frequency

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