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Free vibration analysis of beams by using a third-order shear deformation theory

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Abstract. In this study, free vibration of beams with different boundary conditions is analysed within the framework of the third-order shear deformation theory. The boundary conditions of beams are satisfied using Lagrange multipliers. To apply the Lagrange's equations, trial functions denoting the deflections and the rotations of the cross-section of the beam are expressed in polynomial form. Using Lagrange's equations, the problem is reduced to the solution of a system of algebraic equations. The first six eigenvalues of the considered beams are calculated for different thickness-to-length ratios. The results are compared with the previous results based on Timoshenko and Euler–Bernoulli beam theories.

Keywords. Free vibrations of beams; the third-order shear deformation theory; Lagrange's equations; Lagrange multipliers.

1. Introduction

There are many studies on the theory and analysis of beam-type structures in the literature. The oldest and the well-known beam theory is the Euler–Bernoulli beam theory (or classical beam theory—CBT) which assumed that straight lines perpendicular to the mid-plane before bending remain straight and perpendicular to the mid-plane after bending. As a result of this assumption, transverse shear strain is neglected. Although this theory is useful for slender beams and plates, it does not give accurate solutions for thick beams and plates. The next theory is the Timoshenko beam theory (the first order shear deformation theory—FSDT) which assumed that straight lines perpendicular to the mid-plane before bending remain straight, but no longer remain perpendicular to the mid-plane after bending. In FSDT, the distribution of the transverse shear stress with respect to the thickness coordinate is assumed constant. Thus, a shear correction factor is required to compensate for the error because of this assumption in FSDT. The third-order shear deformation theory (TSDT) which assumed parabolic distribution of the transverse shear stress and strain with respect to the thickness coordinate was proposed for beams with rectangular cross-sections (Wang *et al* 2000). Also, zero transverse shear stress condition of the upper and lower fibres of the cross-section is satisfied without a shear correction factor in TSDT.

There are many studies related with the problem of free vibration of beams based on CBT and FSDT (Timoshenko & Young 1955; Hurty & Rubinstein 1967; Farghaly 1994; Banerjee 1998; Nallim & Grossi 1999; Kim & Kim 2001; Lee *et al* 2003; Auciello & Ercolano 2004; Zhou 2001; Lee & Schultz 2004; Şimşek 2005a, b; Kocatürk & Şimşek 2005a, b). The relationship between the bending solution of TSDT and those of CBT and FSDT was presented (Wang *et al* 2000). The exact stiffness matrix was derived from the solutions of differential equations according to TSDT for isotropic beams (Eisenberger 2003). Frequency equations and characteristic functions of homogeneous orthotropic beams having different boundary conditions were obtained, and the first six natural frequency parameter was tabulated for different values of stiffness ratios and values of thickness-to-length ratios (Soldatos & Sophocleous 2001). Static deflections of the laminated composite beams subjected to uniformly distributed load were studied using the classical, the first-order, the second-order and the third-order beam theories (Khdeir & Reddy 1997).

In the present study, free vibration of beams with different boundary conditions is analysed based on the third-order shear deformation theory (TSDT). Frequency equations of the beams are derived using Lagrange's equations. The boundary conditions of the beams are considered using Lagrange multipliers. The trial functions for the deflections and rotations of the cross-section of the beam are selected in polynomial form. The first six eigenvalues of the considered beams are calculated for different thickness-to-length ratios. The obtained results are compared with earlier results based on CBT and FSDT.

2. Theory and formulations

A straight uniform beam of length L, width b, depth h, having rectangular cross-section is shown in figure 1. A Cartesian coordinate system (x, y, z) is defined on the central axis of the beam, where the x axis is taken along the central axis, the y axis in the width direction and the z axis in the depth direction. Also, the origin of the coordinate system is chosen at the mid-point of the total length of the beam.

The third-order shear deformation theory (TSDT) is based on the following displacement fields (Wang *et al* 2000);

$$
\begin{aligned}
u_x(x, z, t) &= z\phi(x, t) - \alpha z^3[\phi(x, t) + w_{,x}(x, t)] \\
u_z(x, z, t) &= w(x, t),
\end{aligned} \tag{1}
$$

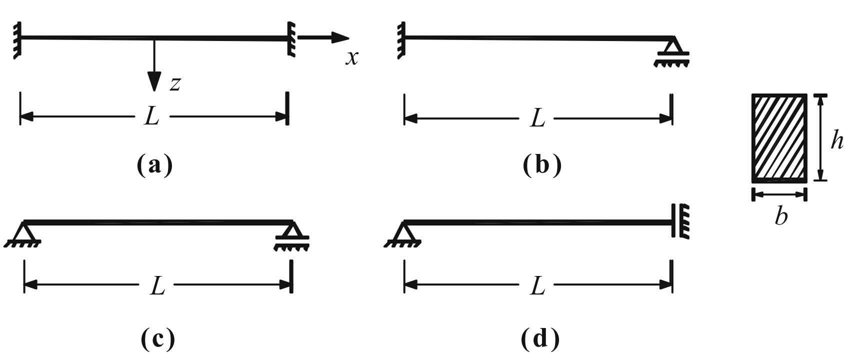

Figure 1. **(a)** Clamped-clamped, **(b)** clamped-pinned, **(c)** pinned-pinned, **(d)** pinned-guided straight uniform beams with rectangular cross-section.

where u_x and u_z are displacements in x and z directions at any material point in the (x, z) plane, $\alpha = 4/(3h^2)$, w is the transverse displacements, and ϕ represents the slope $\partial u_x/\partial z$ at $z = 0$ of the deformed line which was straight in the undeformed beam. In this case $\phi(x, t)$ and α together define the third-order nature of the deformed line. The symbol $()_{,x}$ indicates the derivative with respect to x. The strain-displacement relations are given by

$$\begin{aligned} \varepsilon_{xx} &= u_{x,x} = z\phi_{,x} - \alpha z^3(\phi_{,x} + w_{,xx}), \\ \gamma_{xz} &= u_{x,z} + u_{z,x} = (1 - 3\alpha z^2)(\phi + w_{,x}). \end{aligned} \tag{2}$$

The constitutive relations between stresses and strains for the linear elastic material become

$$\sigma_{xx} = E\varepsilon_{xx}; \quad \tau_{xz} = G\gamma_{xz}, \tag{3}$$

where σ_{xx} is the longitudinal normal stress, ε_{xx} the longitudinal normal strain, τ_{xz} the transverse shear stress, γ_{xz} the transverse shear strain, E the Young's modulus, and G the shear modulus. The strain energy of the beam in Cartesian coordinates is

$$V = \frac{1}{2}\int_{-L/2}^{L/2}\int_A (\sigma_{xx}\varepsilon_{xx} + \tau_{xz}\gamma_{xz})\mathrm{d}A\mathrm{d}x. \tag{4}$$

Using (2), (3) and (4), the strain energy of the beam at any instant can be expressed as:

$$\begin{aligned} V = \frac{1}{2}\int_{-L/2}^{L/2} &\{D_{xx}(\phi_{,x})^2 - 2\alpha F_{xx}\phi_{,x}(\phi_{,x} + w_{,xx}) \\ &+ \alpha^2 H_{xx}[(\phi_{,x})^2 + 2\phi_{,x}w_{,xx} + (w_{,xx})^2] \\ &+ (A_{xz} - 6\alpha D_{xz} + 9\alpha^2 F_{xz})[\phi^2 + 2\phi w_{,x} + (w_{,x})^2]\}\mathrm{d}x, \end{aligned} \tag{5}$$

where

$$(D_{xx}, F_{xx}, H_{xx}) = \int_A (z^2, z^4, z^6)E\mathrm{d}A, \quad (A_{xz}, D_{xz}, F_{xz}) = \int_A (1, z^2, z^4)G\mathrm{d}A. \tag{6}$$

It follows from (1) that the velocities of any point on the beam take the form;

$$v_x = \dot{u}_x(x, z, t) = z\dot{\phi} - \alpha z^3(\dot{\phi} + \dot{w}_{,x}), \quad v_z = \dot{u}_z(x, z, t) = \dot{w}. \tag{7}$$

The kinetic energy of the beam at any instant is

$$T = \frac{1}{2}\int_{-L/2}^{L/2}\int_A \rho(v_x^2 + v_z^2)\mathrm{d}A\mathrm{d}x, \tag{8}$$

where ρ is the mass of the beam per unit volume. By defining the following cross-sectional inertial coefficients

$$(I_A, I_D, I_F, I_H) = \int_A (1, z^2, z^4, z^6)\rho\mathrm{d}A \tag{9}$$

and after some algebraic manipulations, the kinetic energy of the beam at any instant is obtained in the following form;

$$T = \frac{1}{2}\int_{-L/2}^{L/2}\{I_A(\dot{w})^2 + I_D(\dot{\phi})^2 - 2\alpha I_F\dot{\phi}(\dot{\phi} + \dot{w}_{,x}) + \alpha^2 I_H[(\dot{\phi})^2 + 2\dot{\phi}\dot{w}_{,x} + (\dot{w}_{,x})^2]\}\mathrm{d}x. \tag{10}$$

In order to apply Lagrange's equations, the trial functions $w(x,t)$ and $\phi(x,t)$ are approximated by space-dependent polynomial terms $x^0, x^1, x^2, \ldots, x^{N-1}$ and time-dependent generalized coordinates $a_n(t)$ and $b_n(t)$. Therefore, by using Lagrange's equations, by assuming the displacement $w(x,t)$ and the rotation of cross-sections $\phi(x,t)$ to be representable by a series of admissible functions and adjusting the coefficients in the series to satisfy Lagrange's equations, approximate solutions are found for the displacement and rotation functions. Thus;

$$w(x,t) = \sum_{n=1}^{N} a_n(t)x^{n-1},$$
$$\phi(x,t) = \sum_{n=1}^{N} b_n(t)x^{n-1}. \tag{11}$$

The constraint conditions of the supports are satisfied using the Lagrange multipliers. It should be noted at this stage that while both CBT and FSDT have two boundary conditions at each support of the beam, TSDT has three at each support. Essential and natural boundary conditions for TSDT theory are given below (Wang *et al* 2000):

The essential (kinematic or geometric) boundary conditions:

$$w,\ w_{,x},\ \phi. \tag{12}$$

The natural (dynamic) boundary conditions:

$$\widehat{V}_x = \alpha\frac{\mathrm{d}P_{xx}}{\mathrm{d}x} + \widehat{Q}_x,\ \alpha P_{xx},\ \widehat{M}_{xx}, \tag{13}$$

where $\widehat{V}_x$ is the effective shear force, and the quantities $\widehat{M}_{xx}$, P_{xx}, $\widehat{Q}_x$ are defined as follows (Wang *et al* 2000):

$$\widehat{M}_{xx} = M_{xx} - \alpha P_{xx},\ \widehat{Q}_{xx} = Q_x - 3\alpha R_x, \tag{14}$$

where

$$M_{xx} = \int_A z\sigma_{xx}\mathrm{d}A,\quad P_{xx} = \int_A z^3\sigma_{xx}\mathrm{d}A,\quad Q_x = \int_A \tau_{xz}\mathrm{d}A,\quad R_x = \int_A z^2\tau_{xz}\mathrm{d}A. \tag{15}$$

It is known that some expressions satisfying essential (geometric) boundary conditions are chosen for $w(x,t)$, $\phi(x,t)$, and by using the Lagrange's equations, the natural boundary conditions are also satisfied. Therefore, by choosing the appropriate boundary conditions given by (12) and (13), the constraint conditions of the beams are given as follows:

(i) For the clamped-clamped beam

$$w(x_A, t) = 0, w_{,x}(x_A, t) = 0, \phi(x_A, t) = 0, w(x_B, t) = 0,$$
$$w_{,x}(x_B, t) = 0, \phi(x_B, t) = 0. \tag{16}$$

(ii) For the clamped-pinned beam

$$w(x_A, t) = 0, w_{,x}(x_A, t) = 0, \phi(x_A, t) = 0, w(x_B, t) = 0. \tag{17}$$

(iii) For the pinned-guided beam

$$w(x_A, t) = 0, w_{,x}(x_B, t) = 0, \phi(x_B, t) = 0. \tag{18}$$

(iv) For the pinned-pinned beam

$$w(x_A, t) = 0, w(x_B, t) = 0. \tag{19}$$

x_A and x_B denote the location of left and right supports of the beam respectively. By introducing the Lagrange multipliers formulation, the Lagrangian functional of the problem is obtained as follows:

$$J = T - V + G_S, \tag{20}$$

where for the clamped-clamped beam

$$G_S = \theta_1 \cdot w(x_A, t) + \beta_1 \cdot w_{,x}(x_A, t) + \delta_1 \cdot \phi(x_A, t) + \theta_2 \cdot w(x_B, t)$$
$$+ \beta_2 \cdot w_{,x}(x_B, t) + \delta_2 \cdot \phi(x_B, t); \tag{21}$$

for the clamped-pinned beam

$$G_S = \theta_1 \cdot w(x_A, t) + \beta_1 \cdot w_{,x}(x_A, t) + \delta_1 \cdot \phi(x_A, t) + \theta_2 \cdot w(x_B, t); \tag{22}$$

for the pinned-guided beam

$$G_S = \theta_1 \cdot w(x_A, t) + \beta_2 \cdot w_{,x}(x_B, t) + \delta_2 \cdot \phi(x_B, t); \tag{23}$$

for the pinned-pinned beam

$$G_S = \theta_1 \cdot w(x_A, t) + \theta_2 \cdot w(x_B, t). \tag{24}$$

In equations (21) to (24), θ_i, β_i and δ_i are the Lagrange multipliers. The Lagrange's equations are given as follows:

$$\frac{\partial J}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial J}{\partial \dot{q}_k} = 0 \quad k = 1, 2, \ldots, 2N + M \tag{25}$$

where the overdot stands for the partial derivative with respect to time, M is the number of the Lagrange multipliers, and

$$q_k = a_n \quad k = 1, 2, \ldots, N$$
$$q_k = b_{n-N} \quad k = N + 1, \ldots, 2N \tag{26}$$

for the clamped-clamped beam

$$q_{2N+1} = \theta_1, q_{2N+2} = \theta_2, q_{2N+3} = \beta_1, q_{2N+4} = \beta_2, q_{2N+5} = \delta_1, q_{2N+6} = \delta_2; \tag{27}$$

for the clamped-pinned beam

$$q_{2N+1} = \theta_1, q_{2N+2} = \theta_2, q_{2N+3} = \beta_1, q_{2N+4} = \delta_1, q_{2N+5} = 0, q_{2N+6} = 0, \tag{28}$$

for the pinned-guided beam

$$q_{2N+1} = \theta_1, q_{2N+2} = \beta_2, q_{2N+3} = \delta_2, q_{2N+4} = 0, q_{2N+5} = 0, q_{2N+6} = 0, \tag{29}$$

for the pinned-pinned beam

$$q_{2N+1} = \theta_1, q_{2N+2} = \theta_2, q_{2N+3} = 0, q_{2N+4} = 0, q_{2N+5} = 0, q_{2N+6} = 0. \tag{30}$$

For free vibration of the beam, the time-dependent generalized displacement coordinates can be expressed as follows:

$$\begin{aligned} a_n(t) &= \bar{a}_n \exp^{i\omega t}, \\ b_n(t) &= \bar{b}_n \exp^{i\omega t}, \end{aligned} \tag{31}$$

where ω is the natural frequency of the beam. Dimensionless amplitudes of the displacement and normal rotation of a cross-section of the beam can be expressed as follows:

$$\begin{aligned} \bar{w}(x) &= \sum_{n=1}^{N} \bar{a}_n x^{n-1}, \\ \bar{\phi}(x) &= \sum_{n=1}^{N} \bar{b}_n x^{n-1}. \end{aligned} \tag{32}$$

Introducing the following non-dimensional parameters

$$\widehat{x} = \frac{x}{L}, \widehat{w} = \frac{w}{L}, \widehat{\phi} = \phi, \quad \lambda^2 = \frac{\rho A \omega^2 L^4}{EI}, \quad \mu = \frac{h^2}{L^2}, \quad \kappa = \frac{L^2}{h^2(1+\nu)} \tag{33}$$

and using (25), the following simultaneous sets of linear algebraic equations (frequency equation) are obtained which can be expressed in the following matrix form

$$\tilde{K}\tilde{q} - \lambda^2 \tilde{M}\tilde{q} = 0 \tag{34}$$

The elements of stiffness matrix $\tilde{K}$ and the mass matrix $\tilde{M}$ are given in the Appendix. The eigenvalues (characteristic values) λ are found from the condition that the determinant of the system of equations given by (34) must vanish.

3. Numerical results

The first six dimensionless frequency parameters (eigenvalues) of the beams with clamped-clamped (CC), clamped-pinned (CP), pinned-pinned (PP), pinned-guided (PG) boundary

Table 1. Convergence study of the first six dimensionless frequency parameters λ_i of the pinned-pinned (PP) beam for $h/L = 0{\cdot}1$ according to TSDT.

N	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
6	3·115916	6·102613	9·829280	13·85760	-	-
8	3·115696	6·090875	8·873625	11·52865	16·07145	-
10	3·115696	6·090825	8·841825	11·35213	13·84615	16·36586
12	3·115696	6·090825	8·841488	11·34637	13·63057	15·75051
14	3·115696	6·090825	8·841488	11·34631	13·62094	15·69566
16	3·115696	6·090825	8·841488	11·34631	13·62079	15·69385
18	3·115696	6·090825	8·841462	11·34630	13·62074	15·69364

conditions are given in tables 2 to 5 for the different thickness-to-length ratios. The frequencies obtained are compared with the previously published results of CBT (Hurty & Rubinstein 1967) and FSDT (Lee & Schultz 2004; Kocatürk & Şimşek 2005a). Convergence study of the beam with pinned-pinned boundary conditions is carried out for $h/L = 0{\cdot}1$ and the results are given in table 1. In all the following calculations, Poisson's ratio is taken as $\nu = 0{\cdot}3$ and thickness-to-length ratios range from $h/L = 0{\cdot}002$ to $0{\cdot}2$.

It is observed from table 1 that the natural frequencies decrease as the number of polynomial terms increases. It means that the convergence to the exact value is from above, i.e. by increasing the number of the polynomial terms, the exact value can be approached from above. It should be remembered that energy methods always overestimate the fundamental frequency, so with more refined analyses, the exact value can be approached from above. From here on, the number of the polynomial terms N is taken as 16 in all of the numerical investigations.

It is known that for simplifying some problems such as beam problems and plate problems, some restrictions are made and some unknown functions are expressed by other unknown

Table 2. The first six dimensionless frequency parameters λ_i of the clamped-clamped (CC) beam for different h/L values.

Method	h/L	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
CBT[1]		4·7300	7·8532	10·9956	14·1372	17·2788	20·4204
FSDT[2]	0·002	4·7299	7·8529	10·9949	14·1358	17·2765	20·4166
TSDT[3]		4·7299	7·8529	10·9949	14·1359	17·2766	20·4170
FSDT	0·005	4·7296	7·8516	10·9916	14·1293	17·2650	20·3983
TSDT		4·7296	7·8516	10·9917	14·1294	17·2652	20·3989
FSDT	0·01	4·7283	7·8468	10·9799	14·1061	17·2244	20·3336
TSDT		4·7284	7·8469	10·9801	14·1064	17·2249	20·3350
FSDT	0·02	4·7234	7·8281	10·9339	14·0154	17·0675	20·0866
TSDT		4·7235	7·8283	10·9345	14·0167	17·0696	20·0911
FSDT	0·05	4·6898	7·7035	10·6399	13·4611	16·1586	18·7316
TSDT		4·6902	7·7052	10·6447	13·4703	16·1754	18·7573
FSDT	0·1	4·5795	7·3312	9·8559	12·1453	14·2323	16·1478
TSDT		4·5820	7·3407	9·8810	12·1861	14·3018	16·2373
FSDT	0·2	4·2419	6·4179	8·2852	9·9036	11·3486	12·6357
TSDT		4·2563	6·4642	8·3758	10·0364	11·5314	12·8563

([1]Hurty & Rubinstein 1967; [2]Kocatürk & Şimşek 2005a; [3]present study)

Table 3. The first six dimensionless frequency parameters λ_i of the clamped-pinned (CP) beam for different h/L values.

Method	h/L	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
CBT[1]		3·9269	7·0685	10·2101	13·3517	16·4933	19·6349
FSDT[2]	0·002	3·9265	7·0684	10·2097	13·3508	16·4916	19·6319
TSDT[3]		3·9265	7·0684	10·2097	13·3508	16·4916	19·6322
FSDT	0·005	3·9264	7·0676	10·2074	13·3458	16·4825	19·6169
TSDT		3·9264	7·0676	10·2074	13·3459	16·4826	19·6173
FSDT	0·01	3·9258	7·0646	10·1992	13·3283	16·4504	19·5638
TSDT		3·9258	7·0647	10·1992	13·3284	16·4506	19·5646
FSDT	0·02	3·9234	7·0530	10·1668	13·2595	16·3256	19·3601
TSDT		3·9234	7·0531	10·1671	13·2600	16·3266	19·3624
FSDT	0·05	3·9071	6·9747	9·9562	12·8306	15·5852	18·2150
TSDT		3·9072	6·9754	9·9582	12·8349	15·5932	18·2290
FSDT	0·1	3·8517	6·7305	9·3658	11·7583	13·9329	15·9194
TSDT		3·8525	6·7346	9·3769	11·7802	13·9692	15·9742
FSDT	0·2	3·6656	6·0726	8·0743	9·7860	11·2866	12·6191
TSDT		3·6708	6·0947	8·1219	9·8636	11·3979	12·7717

([1]Hurty & Rubinstein 1967; [2]Kocatürk & Şimşek 2005a; [3]present study)

functions. This situation results in decreasing the freedom of the considered problem. As is known, the frequencies become greater when the considered element becomes more rigid. Therefore, by decreasing the freedom of the element, the frequencies become greater from the exact frequencies. In CBT, the plane cross sections remain plane and perpendicular to the elastic curve after bending. In this case, rotations of the cross sections of the beam are expressed in terms of displacements. The first derivative of the elastic curve of the beam with

Table 4. The first six dimensionless frequency parameters λ_i of the pinned-pinned (PP) beam for different h/L values.

Method	h/L	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
CBT[1]		3·1415	6·2831	9·4247	12·5664	15·7080	18·8496
FSDT[2]	0·002	3·1415	6·2831	9·4244	12·5656	15·7066	18·8471
TSDT[3]		3·1415	6·2831	9·4244	12·5656	15·7066	18·8472
FSDT	0·005	3·1415	6·2826	9·4229	12·5621	15·6996	18·8351
TSDT		3·1415	6·2826	9·4229	12·5621	15·6996	18·8352
FSDT	0·01	3·1413	6·2810	9·4176	12·5494	15·6749	18·7925
TSDT		3·1413	6·2810	9·4176	12·5494	15·6749	18·7926
FSDT	0·02	3·1405	6·2747	9·3962	12·4993	15·5784	18·6280
TSDT		3·1405	6·2747	9·3963	12·4994	15·5784	18·6283
FSDT	0·05	3·1349	6·2313	9·2553	12·1812	14·9926	17·6802
TSDT		3·1349	6·2313	9·2554	12·1816	14·9935	17·6829
FSDT	0·1	3·1156	6·0906	8·8404	11·3430	13·6131	15·6769
TSDT		3·1156	6·0908	8·8414	11·3463	13·6207	15·6938
FSDT	0·2	3·0453	5·6715	7·8394	9·6569	11·2219	12·5971
TSDT		3·0454	5·6731	7·8469	9·6769	11·2625	12·6723

([1]Hurty & Rubinstein 1967; [2]Kocatürk & Şimşek 2005a; [3]present study)

Table 5. The first six dimensionless frequency parameters λ_i of the pinned-guided (PG) beam for different h/L values.

Method	h/L	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
CBT[1]		1·5708	4·7123	7·8539	10·9955	14·1371	17·2787
FSDT[4]	0·002	1·5708	4·7123	7·8538	10·9951	14·1362	17·2770
TSDT[3]		1·5708	4·7123	7·8538	10·9951	14·1362	17·2770
FSDT	0·005	1·5708	4·7121	7·8529	10·9927	14·1311	17·2677
TSDT		1·5708	4·7121	7·8529	10·9927	14·1311	17·2677
FSDT	0·01	1·5707	4·7114	7·8498	10·9842	14·1130	17·2348
TSDT		1·5707	4·7114	7·8498	10·9842	14·1131	17·2349
FSDT	0·02	1·5706	4·7088	7·8374	10·9505	14·0423	17·1073
TSDT		1·5706	4·7088	7·8375	10·9505	14·0423	17·1073
FSDT	0·05	1·5699	4·6902	7·7542	10·7319	13·6020	16·3524
TSDT		1·5699	4·6903	7·7542	10·7320	13·6025	16·3537
FSDT	0·1	1·5674	4·6276	7·4963	10·1223	12·5056	14·6697
TSDT		1·5675	4·6277	7·4967	10·1241	12·5106	14·6805
FSDT	0·2	1·5578	4·4202	6·8065	8·7852	10·4663	11·9320
TSDT		1·5578	4·4207	6·8103	8·7979	10·4953	11·9861

([4]Lee & Schultz 2004)

respect to the coordinate along the axis of the beam gives the rotation function. In FSDT, plane sections remain plane but not necessarily perpendicular to the elastic curve after bending. In TSDT, plane sections are not plane and are not perpendicular to the elastic curve in general cases. In these theories, TSDT satisfies the free surface stress conditions. At the upper and lower surfaces of the beam, the shear stresses are zero in the third-order shear deformation theory. It can be deduced from these explanations that in these three categories of beam theories, frequencies of TSDT should be lower than those of others. However, it is interesting to note that this is not so because it can be deduced from tables 2 to 5 that the frequencies of TSDT remain between the frequencies of CBT and FSDT. This situation can be explained as follows. The displacement field u_x for the first-order shear deformation theory (FSDT) is $u_x = z\phi^{\mathrm{FSDT}}$ (Wang *et al* 2000), and for the third-order shear deformation theory (TSDT) $u_x = z\phi^{\mathrm{TSDT}} - \alpha z^3(\phi^{\mathrm{TSDT}} + w_{,x})$ in equation (1). It is expected that the rotations ϕ^{FSDT} and ϕ^{TSDT} must be very close to each other. In this case, because of the negative terms in the displacement equation of TSDT, the displacements of TSDT become generally smaller than the displacements of FSDT. As a result of this situation, the strains of TSDT become generally smaller than the strains of FSDT. Therefore, the more flexible FSDT scheme results in greater displacements and smaller frequencies compared to the TSDT scheme. A similar situation is encountered in the frequency tables given for the classical plate, the first-order plate and the third-order plate theories (Reddy 1984).

The three solutions are close to each other for small values of h/L (i.e. $h/L = 0{\cdot}002$ and 0·005) as seen from tables 2 to 5. The results of TSDT are a little greater than those of FSDT.

It should be remembered that the eigenvalues obtained using the first-order or higher-order beam theories are lower than the corresponding eigenvalues obtained by the classical beam theory. The FSDT and TSDT results are very close to each other for the considered parameters. It can be seen from tables 2 to 5 that the results obtained using FSDT are fairly accurate; the difference between the results of FSDT and TSDT increases with increasing mode number.

Although the TSDT results are more accurate than the CBT and FSDT results for beams with rectangular cross-sections, higher-order beam theories cannot be used for beams with cross-sections other than rectangular shape.

It can be deduced from tables 2 to 5 that the difference between the eigenvalues of CBT and the other two theories increases for increasing mode numbers. This means that the effect of shear deformations increases for increasing mode numbers.

Tables 2 to 5 also show that the difference between the eigenvalues of CBT and the other two theories increases for increasing thickness-to-length ratios h/L. The effect of shear deformations increases for increasing values of h/L.

4. Conclusions

The free vibrations of the beams have been investigated for different thickness-to-length ratios according to TSDT. The eigenvalues of the beams obtained with various boundary conditions are compared with the previously available results of CBT and FSDT. Using Lagrange's equations with the trial functions in the polynomial form and satisfying the constraint conditions by the use of Lagrange multipliers is a nice way for studying the free vibration characteristics of the beams.

Numerical calculations have been carried out to clarify the effects of the thickness-to-length ratio on the eigenvalues of the beams. It is observed from the investigations that the CBT, FSDT and TSDT results are close to each other for small values of h/L. However, as the thickness-to-length ratio becomes greater, the results of the classical beam theory significantly differ from others. This situation is also observed as the mode numbers increase. It is interesting to note that the frequencies of TSDT are slightly greater than that of FSDT. Although it is not investigated here, it is expected that the results of the third-order shear deformation theory give the closest frequency values to the exact frequency values in the considered three beam theories as it is proved for plates (Reddy 1984). The results obtained are accurate and are expected to be useful to other researchers for comparison.

Appendix

$$\tilde{K}_{(m)(n)} = \frac{16}{5}\kappa \int_{-1/2}^{1/2} (x^{m-1})'(x^{n-1})'\mathrm{d}x + \frac{1}{21}\int_{-1/2}^{1/2} (x^{m-1})''(x^{n-1})''\mathrm{d}x$$

$$m, n = 1, 2, \ldots, N$$

$$\tilde{K}_{(m)(N+n)} = \frac{16}{5}\kappa \int_{-1/2}^{1/2} (x^{m-1})(x^{n-1})'\mathrm{d}x - \frac{16}{105}\int_{-1/2}^{1/2} (x^{m-1})'(x^{n-1})''\mathrm{d}x$$

$$m, n = 1, 2, \ldots, N$$

$$\tilde{K}_{(N+m)(n)} = \frac{16}{5}\kappa \int_{-1/2}^{1/2} (x^{m-1})'(x^{n-1})\mathrm{d}x - \frac{16}{105}\int_{-1/2}^{1/2} (x^{m-1})''(x^{n-1})'\mathrm{d}x$$

$$m, n = 1, 2, \ldots, N$$

$$
\begin{aligned}
\tilde{K}_{(N+m)(N+n)} &= \frac{16}{5}\kappa \int_{-1/2}^{1/2} (x^{m-1})(x^{n-1})\mathrm{d}x + \frac{68}{105}\int_{-1/2}^{1/2} (x^{m-1})'(x^{n-1})'\mathrm{d}x \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad m,n = 1,2,\ldots,N \\
\tilde{M}_{(m)(n)} &= \int_{-1/2}^{1/2} (x^{m-1})(x^{n-1})\mathrm{d}x + \frac{\mu}{252}\int_{-1/2}^{1/2} (x^{m-1})'(x^{n-1})'\mathrm{d}x \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad m,n = 1,2,\ldots,N \\
\tilde{M}_{(m)(N+n)} &= \left(\frac{1}{252} - \frac{1}{60}\right)\mu \int_{-1/2}^{1/2} (x^{m-1})(x^{n-1})'\mathrm{d}x \qquad m,n = 1,2,\ldots,N \\
\tilde{M}_{(N+m)(n)} &= \left(\frac{1}{252} - \frac{1}{60}\right)\mu \int_{-1/2}^{1/2} (x^{m-1})'(x^{n-1})\mathrm{d}x \qquad m,n = 1,2,\ldots,N \\
\tilde{M}_{(N+m)(N+n)} &= \left(\frac{1}{12} + \frac{1}{252} - \frac{1}{30}\right)\mu \int_{-1/2}^{1/2} (x^{m-1})(x^{n-1})\mathrm{d}x \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad m,n = 1,2,\ldots,N
\end{aligned}
\tag{35}
$$

The elements of matrices $\tilde{K}$ and $\tilde{M}$ are obtained from the boundary conditions (Lagrange multipliers) are not given here.

List of symbols

a_n	time-dependent generalized coordinate of the displacements of the cross-section;
$\bar{a}_n$	amplitude of time-dependent generalized coordinate a_n;
b_n	time-dependent generalized coordinate of the rotations of the cross-sections;
$\bar{b}_n$	amplitude of time-dependent generalized coordinate b_n;
b	width of the cross-section;
h	depth of the cross-section;
q_k	generalized coordinates;
t	time;
u_x	displacements in x direction at any material point;
u_z	displacements in z direction at any material point;
v_x	velocity of any point on the beam in x direction;
v_z	velocity of any point on the beam in z direction;
w	displacements of the beam;
$\widehat{w}$	dimensionless displacements of the beam;

$\bar{w}$	dimensionless amplitudes of the displacements;
x	x coordinate;
$\widehat{x}$	dimensionless x coordinate;
y	y coordinate;
z	z coordinate;
A_{xz}	a cross-sectional stiffness coefficients;
D_{xx}	a cross-sectional stiffness coefficients;
D_{xz}	a cross-sectional stiffness coefficients;
E	elastic modulus of the beam;
F_{xx}	a cross-sectional stiffness coefficients;
F_{xz}	a cross-sectional stiffness coefficients;
G	shear modulus of the beam;
G_S	Lagrange multipliers formulation;
H_{xx}	a cross-sectional stiffness coefficients;
I_A	a cross-sectional inertial coefficients;
I_D	a cross-sectional inertial coefficients;
I_F	a cross-sectional inertial coefficients;
I_H	a cross-sectional inertial coefficients;
$\tilde{K}$	stiffness matrix of the beam;
L	length of the beam;
M	number of the Lagrange multipliers;
$\tilde{M}$	mass matrix of the beam;
M_{xx}	bending moment;
N	number of the polynomial terms;
P_{xx}	a higher order stress resultants;
Q_x	shear force;
R_x	a higher order stress resultants;
T	kinetic energy of the beam;
V	strain energy of the beam;
$\widehat{V}_x$	effective shear force;
α	a parameter in the displacement fields (equal to $4/3h^2$);
β_1, β_2	Lagrange multipliers;
δ_1, δ_2	Lagrange multipliers;
ε_{xx}	longitudinal normal strain;
ϕ	rotations of the cross-sections;
$\widehat{\phi}$	dimensionless rotations of the cross-sections;
$\bar{\phi}$	dimensionless amplitudes of the rotations of the cross-sections;
γ_{xz}	transverse shear strain;
κ	a dimensionless quantity;
$\lambda_1, \lambda_2, \ldots$	dimensionless frequency parameter of the beam;
μ	a dimensionless quantity;
ν	Poisson's ratio;
θ_1, θ_2	Lagrange multipliers;
ρ	mass of the beam per unit volume;
σ_{xx}	longitudinal normal stress;
τ_{xz}	transverse shear stress;
ω	natural frequency of the beam.

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