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Stochastic response of nonlinear system in probability domain

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Abstract. A stochastic averaging procedure for obtaining the probability density function (PDF) of the response for a strongly nonlinear single-degree-of-freedom system, subjected to both multiplicative and additive random excitations is presented. The procedure uses random Van Der Pol transformation, Ito's equation of limiting diffusion process and stochastic averaging technique as outlined by Zhu and others. However, the equations are rederived in generalized form and arranged in such a way that the procedure lends itself to a numerical computational scheme using FFT. The main objective of the modification is to consider highly irregular nonlinear functions which cannot be integrated in closed form and also to solve problems where analytical expressions for probability density function cannot be obtained. The procedure is applied to obtain the PDF of the response of Duffing oscillator subjected to additive and multiplicative random excitations represented by rational power spectral density functions (PSDFs). The results are verified by digital simulation. It is shown that the procedure provides results which compare very well with those obtained from simulation analysis not only for wide-band excitations but also for very narrow-band excitations, which are weak (when normalized with respect to mass of the system).

Keywords. Stochastic average procedure; nonlinear single-DOF system; probability density function.

1. Introduction

Stochastic response analysis of nonlinear systems has been extensively studied in the frequency, time and probability domains. In the frequency domain, the stochastic linearization technique is generally used for obtaining the moments of the response, in particular, the mean square response of the system (Proppe 2003; Proppe *et al* 2003; Fang *et al* 1995). Iterative frequency domain approach using the Newton–Raphson technique has also been used to obtain the nonlinear response of the system (Jain & Datta 1987). The difficulty with the frequency domain approach is that it does not work well for strongly nonlinear systems under

This paper is dedicated to Prof R N Iyengar of the Indian Institute of Science on the occasion of his formal retirement.

resonating conditions. Time domain analysis essentially relies on simulation procedures and converts the stochastic response analysis to the response analysis for a specified (simulated) time history of excitation. Owing to the nonlinearity, the iterative procedure is adopted at each time step for finding the response in time domain using time-marching schemes. Alternatively, stochastic numerical integration schemes have been developed to obtain the nonlinear response (Ray & Dash 2005). The main disadvantage of time domain analysis is that it takes much computational time for obtaining the steady state response, although system nonlinearities are better handled in the time domain in comparison to that in the frequency domain. Further, convergence of the steady state solution is highly dependent on the choice of the initial condition. Nonlinear response analysis in the probability domain essentially uses Ito's differential equation and the corresponding FPK equation. For nonlinear systems, use of FPK equation has been particularly favoured by many investigators for Gaussian white noise excitation (Haung *et al* 2002; Blankenship & Papanicolaou 1978). The stochastic averaging technique has been extensively used for obtaining the response of strongly nonlinear system using the FPK equation.

The stochastic averaging method for strongly nonlinear systems, excited by Gaussian white noise, was initially proposed by Landau & Stratnovich (1962) and Khasminiskii (1964), and later modified by Zhu (1983), and Zhu & Lin (1991). The method has been extended further for non-white wide-band random excitation (Cai 1995; Dimentberg *et al* 1995; Cai *et al* 1999; Krenk & Roberts 1999). Also, the method has been applied to obtain the response of quasi-Hamiltonian MDOF system with Gaussian white-noise excitation (Zhu & Yang 1997; Zhu *et al* 1997; Bellizzi & Bouc 1999). Use of stochastic averaging methods for obtaining the response of strongly nonlinear systems under non-white random excitation is not widely reported. Zhu *et al* (2001) obtained the stationary response stability and bifurcation of strongly nonlinear systems using the stochastic averaging technique for non-white, but not-so-narrow-band excitation. The procedure is developed for systems where closed form solutions for drift and diffusion coefficients and the probability density function can be obtained involving long integration.

In the present paper, a generalized procedure using the stochastic averaging technique is presented for determining the response of strongly nonlinear SDOF systems for cases where closed form solutions are not possible. The procedure is based on the same method as that proposed by Zhu *et al* (2001), but different steps of the formulation are rederived and arranged in a form that allows an efficient numerical scheme to be developed using FFT. The scheme of operation is explained using an illustrative problem. A Van Der Pol duffing oscillator is taken as the problem to find the response of the system to non-white excitation varying from very narrow to wide-band excitations. The results obtained from the method are validated by simulation results.

2. Theory

Consider free vibration of a nonlinear SDOF system without damping. The equation of motion is

$$\ddot{x} + f(x) = 0. \tag{1}$$

The total energy E of the system is given by

$$E = \frac{1}{2}\dot{x}^2 + V(x), \tag{2}$$

in which $V(x)$ is the potential energy

$$V(x) = \int_0^x f(z)\mathrm{d}z. \tag{3}$$

For the system to be stable, $f(x)$ and $V(x)$ should be such that (1) has periodic solutions surrounding the origin in the z domain on the phase plane $(x, \dot{x})$ and the origin is an equilibrium point. The periodic solution of (1) can be conveniently written in the form,

$$x(t) = a\cos\varphi(t) + b, \tag{4a}$$

$$\varphi(t) = \psi(t) + \theta, \tag{4b}$$

in which a, b and θ are constants. a and b are related such that

$$V(a + b) = V(-a + b) = E. \tag{4c}$$

Differentiating (4a) w.r.t. t,

$$\dot{x}(t) = -a\frac{\mathrm{d}\psi}{\mathrm{d}t}\sin\varphi(t). \tag{5a}$$

Substitution of (5a) and (4a) in (2) yields,

$$\frac{1}{2}a^2\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2\sin^2\varphi(t) + V(a\cos\varphi + b) = V(a + b), \tag{5b}$$

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right) = \frac{1}{a\sin\varphi}(2[V(a + b) - V(a\cos\varphi + b)])^{1/2} = \beta(a, \varphi). \tag{5c}$$

Thus, $\dot{x}$ may be written as,

$$\dot{x}(t) = -a\beta(a, \varphi)\sin\varphi(t), \tag{5d}$$

$\cos\varphi(t)$ and $\sin\varphi(t)$ are called generalised harmonic functions with instantaneous frequency of oscillation $\beta(a, \varphi)$, as evident from (5d). In order to obtain the averaged frequency, the function $\beta(a, \varphi)$ is expanded in the form

$$\beta(a, \varphi) = c_0(a) + \sum_{n=1}^{\infty} c_n(a)\cos n\varphi, \tag{6}$$

with

$$c_0(a) = \frac{1}{\pi}\int_0^{2\pi} \beta(a, \varphi)\mathrm{d}\varphi, \tag{7}$$

$$c_n(a) = \frac{1}{2\pi}\int_0^{2\pi} \beta(a, \varphi)\cos n\varphi\mathrm{d}\varphi. \tag{8}$$

The averaged frequency is clearly

$$\omega(a) = c_0(a). \tag{9}$$

In time-averaging, therefore, $\varphi(t)$ may be written as

$$\varphi(t) = \omega(a)t + \theta. \tag{10}$$

With the above concept of periodic motion of the free vibration of undamped nonlinear systems about an equilibrium point with an averaged frequency given by (9), a stochastic averaging procedure is presented for obtaining the probability response of structure for a stochastic nonlinear SDOF system using FPK equation.

2.1 *Response of the system using stochastic averaging procedure*

Consider a nonlinear SDOF system subjected to both additive and multiplicative excitation. The system includes both stiffness and damping nonlinearities. The equation of motion can be expressed in the form

$$\ddot{x} + f(x) = \varepsilon g(x, \dot{x}) + \varepsilon^{1/2} \sum_{k=1}^{m} g_k(x, \dot{x})\eta_k(t), \tag{11}$$

in which ε is a very small quantity denoting that $g(x, \dot{x})$ and $g_k(x, \dot{x})\eta_k(t)$ are small; and $f(x)$ is the nonlinear function denoting the restoring action. For small values of ε, the motion of the system is nearly periodic and the response can be represented by expressions similar to (4a) and (5a), assuming a, φ, ψ and θ to be random processes. The solution of (11) in the form of (4a) and (5a) with random parameters can be regarded as a set of random Van Der Pol transformations from $x, \dot{x}$ to a and φ. Assuming $x(t), a(t), \varphi(t)$ as random process, $x(t)$ is written as

$$x(t) = a(t)\cos\varphi(t) + b, \tag{12}$$

$$\varphi(t) = \psi(t) + \theta(t), \tag{13}$$

$$\dot{x}(t) = -a(t)\beta(a, \varphi)\sin\varphi(t) \tag{14}$$

in which $\beta(a, \varphi)$ is given by 5(c). Differentiation of (12) with respect to t gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial}{\partial a}[a(t)\cos\varphi(t)]\frac{\partial a}{\partial t} + \frac{\partial b}{\partial a}\frac{\partial a}{\partial t} + \frac{\partial}{\partial \varphi}[a(t)\cos\varphi(t)]\frac{\partial \varphi}{\partial t}, \tag{15}$$

$$\dot{x} = \left[\cos\varphi(t) + \frac{\mathrm{d}b}{\mathrm{d}a}\right]\dot{a} - \dot{\varphi}a(t)\sin\varphi(t). \tag{16}$$

Equating (14) and (16), it is clear that

$$\dot{a}\left[\cos\phi + \frac{\mathrm{d}b}{\mathrm{d}a}\right] + a\sin\varphi\beta(a, \varphi) - \dot{\varphi}a\sin\varphi = 0, \tag{17a}$$

$$\dot{a}\,[\cos\phi + h] + a\sin\varphi\beta(a, \varphi) - \dot{\varphi}a\sin\varphi = 0, \tag{17b}$$

in which

$$h = \mathrm{d}b/\mathrm{d}a.$$

Substituting (12)–(14) in (11) and differentiating with respect to t, the following equations may be obtained.

$$\frac{\mathrm{d}}{\mathrm{d}t}[-a\beta(a,\varphi)\sin\varphi] + f(a\cos\varphi + b)$$
$$= \varepsilon g[a\cos\phi + b, -a\beta(a,\varphi)\sin\varphi] + \varepsilon^{1/2}\sum_{k=1}^{m} g_k[a\cos\varphi + b, -a\beta(a,\varphi)\sin\varphi]\eta_k, \tag{18}$$

$$-\dot{a}\beta(a,\varphi)\sin\varphi - a\dot{a}\frac{\partial}{\partial a}[\beta(a,\varphi)\sin\varphi] - \dot{\varphi}a\frac{\partial}{\partial\varphi}[\beta(a,\varphi)\sin\varphi] + f(a\cos\varphi + b)$$
$$= \varepsilon g[a\cos\phi + b, -a\beta(a,\varphi)\sin\varphi] + \sum_{k=1}^{m}\varepsilon^{1/2} g_k[a\cos\varphi + b, -a\beta(a,\varphi)\sin\varphi]\eta_k, \tag{19}$$

$$-\dot{a}\beta(a,\varphi)\sin\varphi - a\dot{a}\frac{\partial}{\partial a}\left\{\frac{2[V(a+b) - V(a\cos\varphi + b)]}{a^2}\right\}^{1/2}$$
$$-\dot{\varphi}\frac{\partial}{\partial\varphi}\{2[V(a+b) - V(a\cos\varphi + b)]\}^{1/2} + f(a\cos\varphi + b)$$
$$= \varepsilon g[a\cos\phi + b, -a\beta(a,\varphi)\sin\varphi] + \sum_{k=1}^{m}\varepsilon^{1/2} g_k[a\cos\varphi + b, -a\beta(a,\varphi)\sin\varphi]\eta_k, \tag{20}$$

$$-\dot{a}\beta(a,\varphi)\sin\varphi - \frac{a\dot{a}}{2}\left\{\frac{2[V(a+b) - V(a\cos\varphi + b)]}{a^2}\right\}^{-1/2}$$
$$\times\frac{\partial}{\partial a}\left\{\frac{2[V(a+b) - V(a\cos\varphi + b)]}{a^2}\right\} - \frac{\dot{\varphi}}{2}\{2[V(a+b) - V(a\cos\varphi + b)]\}^{-1/2}$$
$$\times\frac{\partial}{\partial\varphi}\{2[V(a+b) - V(a\cos\varphi + b)]\} + f(a\cos\varphi + b)$$
$$= \varepsilon g[a\cos\phi + b, -a\beta(a,\varphi)\sin\varphi] + \sum_{k}\varepsilon^{1/2} g_k[a\cos\varphi + b, -a\beta(a,\varphi)\sin\varphi]\eta_k, \tag{21}$$

$$-\dot{a}\beta(a,\varphi)\sin\varphi - \frac{a\dot{a}}{2\beta(a,\varphi)\sin\varphi}\left\{\frac{2[f(a+b)(1+h) - f(a\cos\varphi + b)(\cos\varphi + h)]}{a^2}\right\}$$
$$-\frac{a\dot{a}}{2\beta(a,\varphi)\sin\varphi}\left\{-\frac{2\times 2[V(a+b) - V(a\cos\varphi + b)]}{a^3}\right\}$$
$$-\frac{\dot{\varphi}}{2}\{2[V(a+b) - V(a\cos\varphi + b)]\}^{-1/2}\times 2f(a\cos\varphi + b)a\sin\varphi + f(a\cos\varphi + b)$$
$$= \varepsilon g[a\cos\phi + b, -a\beta(a,\varphi)\sin\varphi] + \sum_{k}\varepsilon^{1/2} g_k[a\cos\varphi + b, -a\beta(a,\varphi)\sin\varphi]\eta_k, \tag{22}$$

$$- \dot{a}\beta(a,\varphi)\sin\varphi - \frac{\dot{a}}{a\beta(a,\varphi)\sin\varphi} \times [f(a+b)(1+h) - f(a\cos\varphi + b)(\cos\varphi + h)]$$

$$- \frac{\dot{a}\sin\varphi}{\beta(a,\varphi)}\{-\beta^2(a,\varphi)\} - \frac{\dot{\varphi} f(a\cos\varphi + b)}{\beta(a,\varphi)} + f(a\cos\varphi + b)$$

$$= \varepsilon g[a\cos\phi + b, -a\beta(a,\varphi)\sin\varphi] + \sum_k \varepsilon^{1/2} g_k[a\cos\varphi + b, -a\beta(a,\varphi)\sin\varphi]\eta_k, \quad (23)$$

$$- \frac{\dot{a}}{a\beta(a,\varphi)\sin\varphi}[f(a+b)(1+h) - f(a\cos\varphi + b)(\cos\varphi + h)]$$

$$- \frac{\dot{\varphi} f(a\cos\varphi + b)}{\beta(a,\varphi)} + f(a\cos\varphi + b)$$

$$= \varepsilon g[a\cos\phi + b, -a\beta(a,\varphi)\sin\varphi] + \sum_k \varepsilon^{1/2} g_k[a\cos\varphi + b, -a\beta(a,\varphi)\sin\varphi]\eta_k. \quad (24)$$

By solving (17) and (24), $\dot{a}$ and $\dot{\varphi}$ may be obtained as

$$\dot{a} = \frac{a}{f(a+b)(1+h)} \times \Bigg\{\varepsilon g[(a\cos\varphi + b), -a\beta(a,\varphi)\sin\varphi] + \varepsilon^{1/2}\sum_k g_k[(a\cos\varphi + b), -a\beta(a,\varphi)\sin\varphi]\eta_k\Bigg\} \times \beta(a,\varphi)\sin\varphi, \quad (25)$$

$$\dot{\varphi} = -\frac{1}{f(a+b)(1+h)} \times \Bigg\{\varepsilon g[(a\cos\varphi + b), -a\beta(a,\varphi)\sin\varphi] + \varepsilon^{1/2}\sum_k g_k[(a\cos\varphi + b), -a\beta(a,\varphi)\sin\varphi]\eta_k\Bigg\} \times \beta(a,\varphi)(\cos\varphi + h). \quad (26)$$

Equations (25) and (26) may also be written as

$$\dot{a} = \varepsilon q_1(a,\varphi) + \varepsilon^{1/2}\sum_{k=1}^{m} \sigma_{1k}(a,\varphi)\eta_k, \quad (27a)$$

$$\dot{\varphi} = \varepsilon q_2(a,\varphi) + \varepsilon^{1/2}\sum_{k=1}^{m} \sigma_{2k}(a,\varphi)\eta_k, \quad (27b)$$

in which,

$$q_1(a,\varphi) = -a\bar{f} g(a,\varphi)\beta(a,\varphi)\sin\varphi, \quad q_2(a,\varphi) = q_1(\cos\varphi + h)/\sin\varphi, \quad (28a,b)$$

$$\sigma_{1k}(a,\varphi) = -a\bar{f} g_k(a,\varphi)\beta(a,\varphi)\sin\varphi, \quad \sigma_{2k}(a,\varphi) = \sigma_{1k}(\cos\varphi + h)/\sin\varphi, \quad (28c,d)$$

where $\bar{f} = f^{-1}(a+b)/(1+h)$; $g(a,\varphi) = g[(a\cos\varphi + b), -a\beta(a,\varphi)\sin\varphi]$; $g_k(a,\varphi) = g_k[(a\cos\varphi + b), -a\beta(a,\varphi)\sin\varphi]$; $\beta(a,\varphi)$ is defined earlier. Since functions $q_1(a,\varphi)$,

$\sigma_{1k}(a, \varphi)$ etc. are periodic, they can be Fourier-synthesized using FFT for different assumed values of a. Thus, σ^r_{1kn}, σ^i_{2ln}, q_{10}etc are obtained from the real and imaginary parts of the FFT of $q_1(a, \varphi)$, $\sigma_{1k}(a, \varphi)$ etc. They are later used to obtain the drift and diffusion coefficients of Ito's equation.

The effective band width of random excitation $\eta(t)$ in 27(a–b) depends on value of ε. If band width of $\eta(t)$ is ω, then effective band width becomes $\omega\varepsilon^{-1}$. So, as $\varepsilon \to 0$ effective band width tends to infinity and process $a(t)$ converges weakly to a diffusive Markov process. The Ito's equation of the limiting diffusion process represented by (27) is of the form (Lin & Cai 1995),

$$\mathrm{d}a = u(a)\mathrm{d}t + \sigma(a)\mathrm{d}B(t), \tag{29}$$

in which $u(a)$ and $\sigma(a)$ are the averaged drift and diffusion coefficients given by (Lin & Cai 1995)

$$u(a) = \varepsilon \left\langle q_1 + \int_{-\infty}^{\infty} \sum_{k=1}^{m} \sum_{l=1}^{m} \left(\left.\frac{\mathrm{d}\sigma_{1k}}{\mathrm{d}a}\right|_t \sigma_{1l}|_{t+\tau} + \left.\frac{\mathrm{d}\sigma_{1k}}{\mathrm{d}\theta}\right|_t \sigma_{2l}|_{t+\tau} \right) R_{kl}(\tau)\mathrm{d}\tau \right\rangle, \tag{30a}$$

$$\sigma^2(a) = \varepsilon \left\langle \int_{-\alpha}^{\alpha} \sum_{k=1}^{m} \sum_{l=1}^{m} \left(\sigma_{1k}|_t\ \sigma_{1l}|_{t+\tau} \right) R_{kl}(\tau)\mathrm{d}\tau \right\rangle_t . \tag{30b}$$

$\langle . \rangle$ represents time-averaging and $R_{kl}(\tau)$ is the cross-correlation function between η_k and η_l.

Using time-averaged relationship of $\varphi(t)$ given by (10) and substituting the Fourier series expansions of εq_i and $\varepsilon^{1/2}\sigma_{ik}$ in (30a–b), the following expressions for u and σ^2 for a particular value of a are derived:

$$\begin{aligned} u(a) = q_{10} &+ \sum_{k=1}^{m} \sum_{l=1}^{m} \pi \sigma'_{1ko} \sigma_{1lo} s_{kl}(o) \\ &+ \sum_{k=1}^{m} \sum_{l=1}^{m} \left[\frac{\pi}{2} \sum_{n=1}^{\infty} \left\{ (\sigma^r_{1kn})' \sigma^c_{1\,\mathrm{ln}} + (\sigma^i_{1kn})' \sigma^i_{1\,\mathrm{ln}} \right.\right. \\ &\left.\left. + n(\sigma^i_{1kn}\sigma^r_{2\,\mathrm{ln}} - \sigma^r_{1kn}\sigma^i_{2\,\mathrm{ln}}) \right\} S_{kl}(n\omega(a)) \right], \end{aligned} \tag{31}$$

$$\begin{aligned} \sigma^2(a) = &\sum_{k=1}^{m} \sum_{l=1}^{m} 2\pi \sigma_{1ko} \sigma_{1lo} s_{kl}(o) \\ &+ \sum_{k=1}^{m} \sum_{l=1}^{m} \left[\pi \sum_{n=1}^{\infty} (\sigma^r_{1kn}\sigma^r_{1\,\mathrm{ln}} + \sigma^i_{1kn}\sigma^i_{1\,\mathrm{ln}}) S_{kl}\ (n\omega(a)) \right], \end{aligned} \tag{32}$$

in which S_{kl} is the cross-power spectral density function between the process η_k and η_l; $(\sigma^r_{1l})'$ etc. is the derivative of σ^r_{1l} with respect to a.

The averaged FPK equation associated with Ito's equation (29) is of the standard form,

$$\frac{\sigma p}{\sigma t} = -\frac{\sigma}{\sigma r}[u(r)p] + \frac{1}{2}\frac{\sigma^2}{\sigma r^2}[\sigma^2(r)p], \tag{33}$$

in which $p = p(r, t|r_o, t_0)$is the transition probability density of displacement amplitude.

The initial condition of FPK equation is

$$p = \delta(r - r_o), \quad t = 0. \tag{34}$$

The two boundaries of the FPK equation are r=0 and ∞, if nonlinear restoring action exists. $r = 0$ is a regular boundary for non-zero external excitation, while $r = \infty$ is a singular boundary. For non-zero external excitation, the stationary solution of FPK equation under the assumption of zero probability flow at the two boundaries is of the form (Lin & Cai 1995),

$$p(a) = \frac{c}{\sigma^2(a)} \exp\left[\int_o^a \frac{2u(s)}{\sigma^2(s)} \mathrm{d}s\right], \tag{35}$$

where c is the normalization constant. For an assumed value of a, the numerical value of $p(a)$ can be obtained from (35) using the computed values of $u(a)$ and $\sigma^2(a)$ from (31)–(32). Note that the values of variables of (31)–(32) are obtained from the results of FFT. Once $p(a)$ is obtained, $p(x)$ is determined with following way.

The total energy of the system given by (2) can be written as $E = v(a+b)$, when amplitude of oscillation is maximum and a is the random process. The probability density function of total energy E can, therefore, be written as

$$p(E) = p(a)|\mathrm{d}a/\mathrm{d}E|. \tag{36}$$

Since

$$V(x) = \int f(x)\mathrm{d}x, \tag{37}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} V(x) = f(x). \tag{38}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}a} V(a+b) = f(a+b) \tag{39}$$

or

$$\frac{\mathrm{d}E}{\mathrm{d}a} = f(a+b) \tag{40}$$

or

$$\frac{\mathrm{d}a}{\mathrm{d}E} = \frac{1}{f(a+b)}. \tag{41}$$

Using (41), (36) can be written as,

$$p(E) = p(a)\left|\frac{\mathrm{d}a}{\mathrm{d}E}\right| = \left.\frac{p(a)}{f(a+b)}\right|_{a+b=v^{-1}(E)}, \tag{42}$$

where $a + b = V^{-1}(E)$ is the inverse function of the total energy $E = V(a+b)$. The joint probability density of displacement and velocity can be further obtained from $p(E)$ as follows:

$$p(x, \dot{x}) = \left.\frac{p(E)}{T(E)}\right|_{E=\dot{x}2/2+v(X)}, \tag{43}$$

where $T(E)$ is obtained from $T(a) = 2\pi/\omega(a)$ (with $\omega(a)$ given by (9)) by replacing a by E, in which E is given by $E = V(a+b)$. From the above equation, $p(x)$ is obtained as,

$$p(x) = \int_o^x p(x, \dot{x}) \mathrm{d}\dot{x}. \tag{44}$$

The sequence of operation to be followed for finding the $p(x)$ is explained with the help of the example problem.

3. Application to Duffing–Van Der Pol oscillator

Consider a Duffing–Van Der Pol oscillator with both additive and multiplicative excitations. The equation of motion of the system is of the form,

$$\ddot{x} + (-\beta_1 + \beta_2 x^2)\dot{x} + \omega_s^2 x + x^3 = x f_1(t) + f_2(t), \tag{45}$$

where ω_s, β_1, β_2 are constant, $f_1(t)$ and $f_2(t)$ are stationary and ergodic process with zero mean and rational power spectral densities. The oscillator given by (45) when cast in the form of (11), provides

$$f(x) = \omega_s^2 x + x^3, \tag{46a}$$

$$g(x, \dot{x}) = -(-\beta_1 + \beta_2 x^2)\dot{x}, \tag{46b}$$

$$g_1(x, \dot{x})\eta_1 = x f_1(t), \tag{46c}$$

$$g_2(x, \dot{x})\eta_2 = 1 f_2(t), \tag{46d}$$

$$V(x) = \int_0^x f(u)\mathrm{d}u = \omega_s^2 \frac{x^2}{2} + \frac{x^4}{4}. \tag{46e}$$

For function $V(x)$ given by (46e), $E = V(a+b) = V(-a+b)$ gives b as zero and hence, h in (17b) also becomes zero. Substituting $V(x)$ in (5c), $\beta(a, \varphi)$ is obtained as,

$$\beta(a, \varphi) = \frac{1}{a \sin\varphi} \left[2 \left[\omega_s^2 \frac{a^2}{2} + \frac{a^4}{4} - \omega_s^2 \frac{a^2 \cos^2\varphi}{2} - \frac{a^4 \cos^4\varphi}{4} \right] \right]^{1/2}, \tag{47}$$

q_{1k}, σ_{2k} etc. of (28a–d) take the form,

$$q_1(a, \varphi) = -\frac{a^2}{f(a)}(-\beta_1 + \beta_2 a^2 \cos^2\varphi)\beta^2(a, \varphi)\sin^2\varphi, \tag{48a}$$

$$q_2(a, \varphi) = -\frac{a}{f(a)}(-\beta_1 + \beta_2 a^2 \cos^2\varphi)\beta^2(a, \varphi)\sin\varphi\cos\varphi, \tag{48b}$$

$$\sigma_{11}(a, \varphi) = -\frac{a^2}{f(a)}\beta(a, \varphi)\sin\varphi\cos\varphi, \tag{48c}$$

$$\sigma_{12}(a, \varphi) = -\frac{a}{f(a)}\beta(a, \varphi)\sin\varphi, \tag{48d}$$

$$\sigma_{21}(a,\varphi) = -\frac{a}{f(a)}\beta(a,\varphi)\cos^2\varphi, \tag{48e}$$

$$\sigma_{22}(a,\varphi) = -\frac{1}{f(a)}\beta(a,\varphi)\cos\varphi, \tag{48f}$$

in which $f(a) = \omega_s^3 a + a^3$.

The computation of $p(x)$ follows the steps as given below:

(1) Assume a set of values of a from zero to some desired maximum value at small interval (which dictates the accuracy);
(2) choose φ values at equal intervals between 0 to 2π (number of values may preferably be 2^n);
(3) for each value of a, obtain FFT of $q_1, \sigma_{11}, \sigma_{12}, \sigma_{21}$, and σ_{22} and obtain $T(a), \omega(a), q_{10}$; $\sigma^r_{11n}, \sigma^r_{12n}, \sigma^r_{21n}$, and σ^r_{22n}; $\sigma^i_{11n}, \sigma^i_{12n}, \sigma^i_{21n}$, and σ^i_{22n} for $n = 1$ to m (value of m to be selected depends upon the cut off frequency of the PSDF of excitation);
(4) once a set of values of σ^r_{11n} etc are obtained, $(\sigma^r_{11n})'$ is determined by numerical differentiation;
(5) $u(a)$ and $\sigma^2(a)$ values are obtain from (31)–(32);
(6) $p(a)$ is determined from (35) for each value of a; the integration in (35) is performed numerically;
(7) for each value of a, E is computed and $p(E)$ is obtained from (42);
(8) for different combinations of x and $\dot{x}$, E is obtained and $p(x,\dot{x})$ is computed from (43);
(9) $p(x)$ is obtained by numerical integration of $p(x,\dot{x})$.

4. Simulation analysis

Time history of $f_1 = f_2 = f$ (for this problem) is simulated from the given PSDF of $f(t)$ using the standard simulation procedure. In order to obtain a sufficiently smooth PDF of the response, sufficiently long time history of $f(t)$ is simulated with $\Delta t = 0{\cdot}03$ s and total duration, $T = 3000$ s. For the simulated time history of $f(t)$, the time history of $x(t)$ is obtained using Newmark's β method with iteration performed at each time step to consider the nonlinearities. For this purpose, all nonlinear terms of (11) are taken to the right hand side of the equation and treated as known, equal to those in the previous time step, in the beginning of the iteration. In subsequent iterations, they are updated till convergence is achieved. From the time history of $x(t)$, $p(x)$ is obtained using MATLAB.

The time history of $a(t)$ is obtained from that of $x(t)$ using the following transformation:

$$V(a) = \int_0^a f(x)\mathrm{d}x = \frac{1}{2}\omega_s^2 a^2 + \frac{1}{4}a^4. \tag{49}$$

The total energy E is given by

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_s^2 x^2 + \frac{1}{4}x^4 = V(a). \tag{50}$$

Equating (49) and (50), a quadratic equation in a^2 is obtained

$$\frac{1}{4}a^4 + \frac{1}{2}\omega_s^2 a^2 = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_s^2 x^2 + \frac{1}{4}x^4. \tag{51}$$

As time history of $x(t)$ and $\dot{x}(t)$ are known, the time history of $a(t)$ can be obtained by solving (51). Thus, for each time history of $x(t)$, a corresponding time history of $a(t)$ is obtained. $p(a)$ is obtained from the time history of $a(t)$ in the same way as $p(x)$ is obtained.

5. Numerical results

For the Duffing oscillator given by (45), the power spectral density function for both loadings is assumed to be the same and is taken to be of the form,

$$s_p(\omega) = D/[\pi(\omega^2 - \omega_p^2)^2 + 4\xi_p^2\omega_p^2\omega^2]. \tag{52}$$

By changing the values of ω_p and ξ_p, the shape of the PSDF can be changed from narrow to wide-band. By assigning suitable value to D, $\eta(t)/m$ can be made very small, i.e. ($\varepsilon \to 0$ in 11). For two combinations of $\omega_p$and ξ_p, the PSDFs are shown in figure 1, where combinations of D, $\omega_p$and ξ_p are also indicated. Note that the value of D is assumed such that the peak value of PSDFs are small (i.e $\eta(t)/m$ is small) and has the unit as shown in figure 1. The values of D for both PSDFs are given in the figure. It is seen from figure 1. that the PSDF 2 is comparatively broad-banded, while PSDF 1 is narrow-banded.

The values of β_1 and β_2 in (45) are assumed to be 0·01 and 0·02 respectively. The value of ω_p is varied depending upon the excitation used. For the PSDF represented by PSDF 2, ω_s is taken as 4 rad/s, while that for PSDF 1 is taken as 2·5 rad/s. Figures 2 to 4 show the probability density functions of responses obtained for PSDF 1. Also they compare PDFs of responses obtained from analysis and the simulation procedure. It is seen from figures 2 and 4 that PDFs of displacements both in transform domain and actual coordinates, i.e $p(a)$ and $p(x)$, compare well with those obtained from simulation studies. Figure 3 shows the joint probability density function $p(x, \dot{x})$ obtained analytically.

PDFs of responses for PSDF 2 are shown in figures 5 to 7. It is seen from the figures that the PDFs obtained from the analysis compare well with those obtained from simulation.

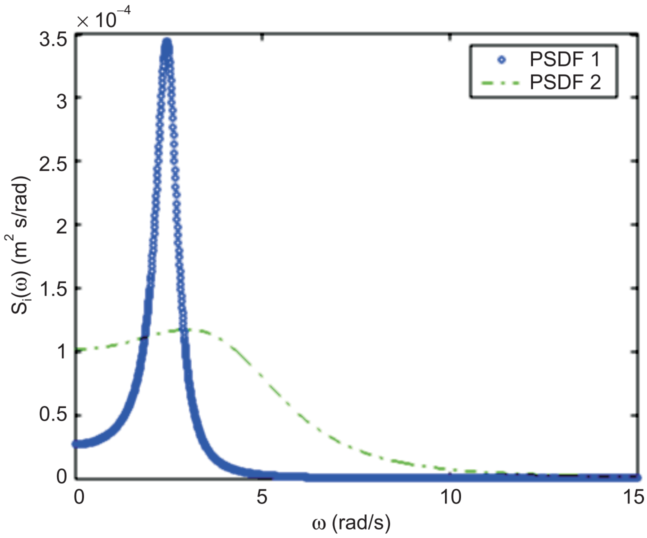

Figure 1. PSDFs of excitations defined by (52); PSDF 1: $\omega_p = 2{\cdot}5$ rad/s, $\xi_p = 0{\cdot}25$, $D = 0{\cdot}0032$; PSDF 2: $\omega_p = 5$ rad/s, $\xi_p = 1$, $D = 0{\cdot}2$.

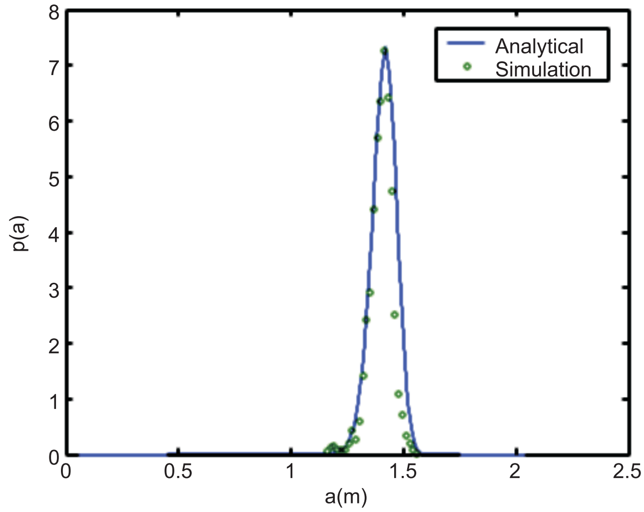

Figure 2. PDF for displacement amplitude (a) for PSDF 1; $\omega_s = 2{\cdot}5$ rad/s.

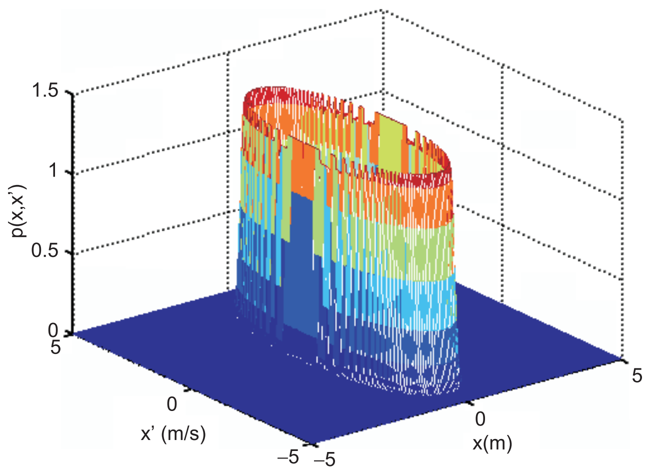

Figure 3. Joint PDF of displacement and velocity for PSDF 1; $\omega_s = 2{\cdot}5$ rad/s.

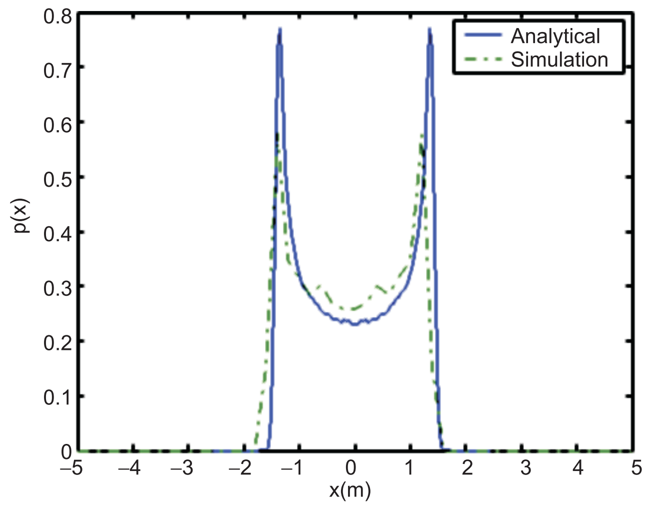

Figure 4. PDF of displacement (x) for PSDF 1; $\omega_s = 2{\cdot}5$ rad/s.

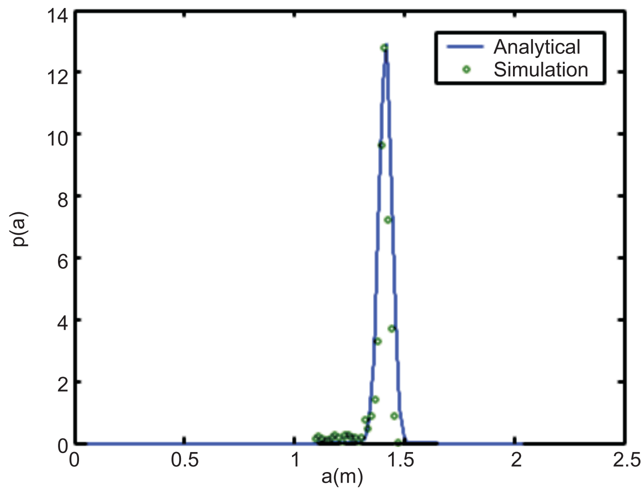

Figure 5. PDF for displacement amplitude (a) for PSDF 2; ω_s = 4 rad/s.

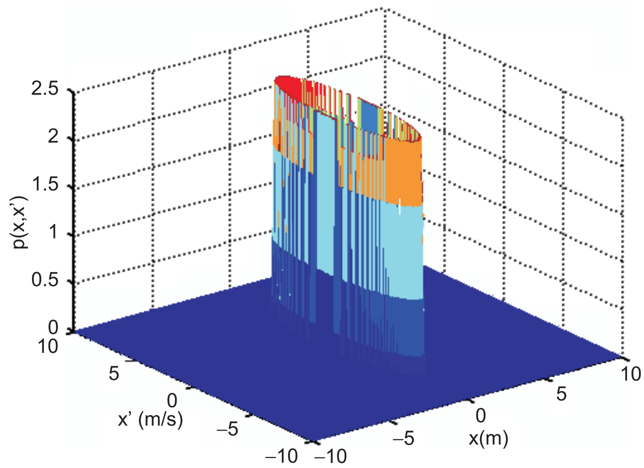

Figure 6. Joint PDF of displacement and velocity for PSDF 2; $\omega_s = 4$ rad/s.

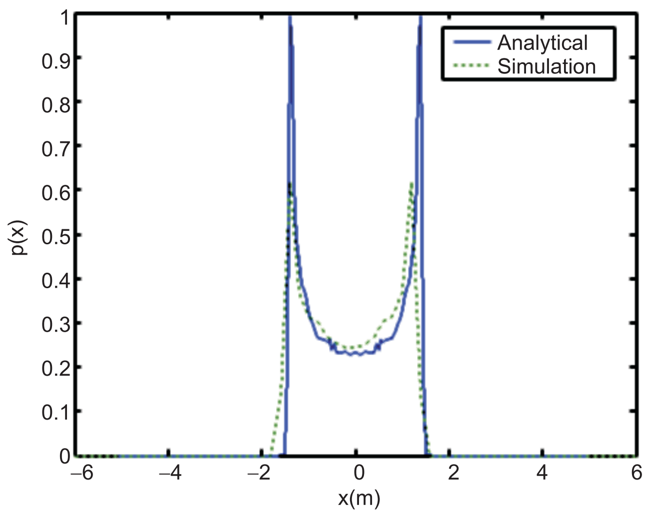

Figure 7. PDF of displacement (x) for PSDF 2; $\omega_s = 4$ rad/s.

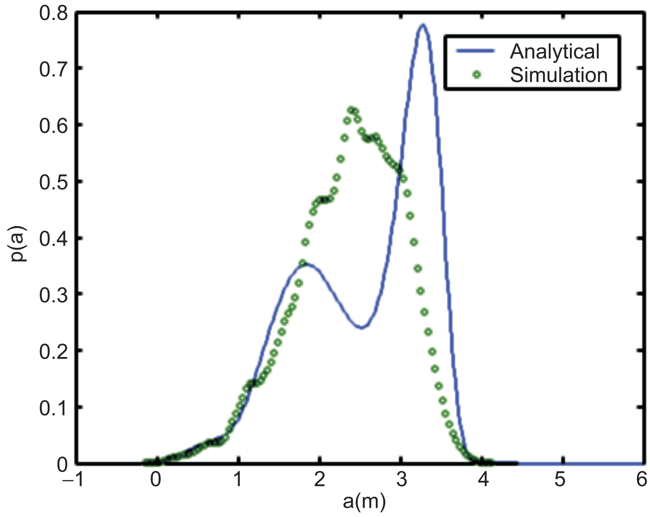

Figure 8. PDF for displacement amplitude (a) for PSDF 1 with $D = 9{\cdot}6$; $\omega_s = 1$ rad/s.

Therefore, the proposed analytical method provides good estimates of PSDFs of responses both for narrow-band (PSDF 1) and wide-band (PSDF 2) excitations so long as $\eta(t)/m$ is small (or $\varepsilon \rightarrow 0$).

The value of D for the expressions of PSDFs given in (52) is now changed such that the ordinates of the PSDF in figure 1 are increased 3000 times. Thus $\eta(t)/m$ no longer remains very small and, therefore, the condition of $\varepsilon \rightarrow 0$ is not satisfied. For these PSDFs, the PDFs of responses are obtained both analytically and using simulation procedure. Responses are obtained for two cases. In the first case, ω_s is taken as 1 rad/s for both types of PSDFs, while for the second case, the ω_s is selected as 5 rad/s for PSDF 1 and $\omega_s = 12$ rad/s for PSDF 2.

Results for the first case are shown in figures 8 to 11. It is seen from the figures that PDFs obtained from the analytical methods do not compare well with those obtained from

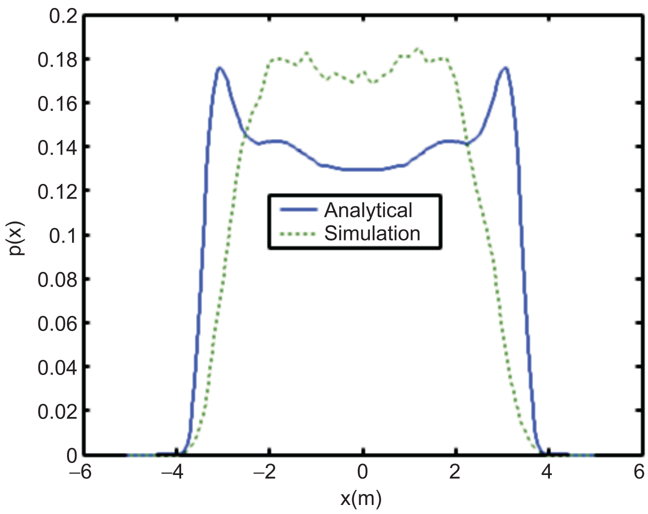

Figure 9. PDF of displacement (x) for PSDF 1 with $D = 9{\cdot}6$; $\omega_s = 1$ rad/s.

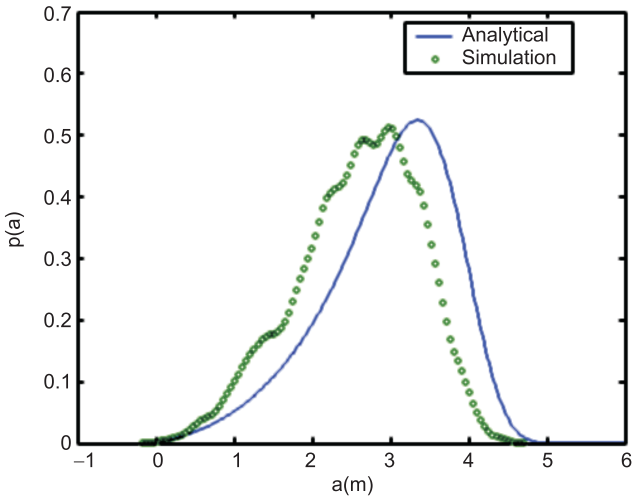

Figure 10. PDF for displacement amplitude (a) for PSDF 2 with $D = 600$; $\omega_s = 1$ rad/s.

simulation results. The reason for this is that the ordinates of $S(n\varpi)$, (52), have large values for $n = 2, 3$ etc. As a result, the excitation divided by the mass does not remain small.

For the second case, the results are shown in figures 12 to 15. It is seen from the figures that the results from the simulation compare well with those of analytical results even though the PSDFs correspond to large excitation (PSDFs ordinates in figure 1 are multiplied by 3000). This is the case because the value of ω_s for the oscillators are selected such that the ordinates of $S(n\varpi)$ (53), for $n = 2, 3$ are at the tail end of the spectrum. Hence, the excitation divided by mass becomes small.

6. Conclusions

A probability domain response analysis of nonlinear SDOF system is presented under both multiplicative and additive stochastic excitation. The method uses random Van Der Pol

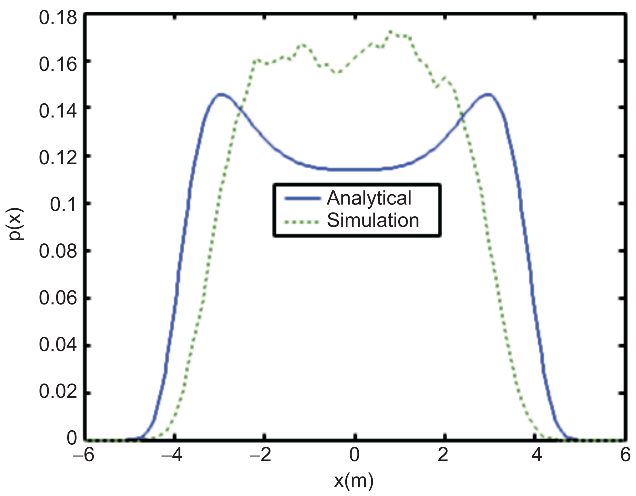

Figure 11. PDF of displacement (x) for PSDF 2 with $D = 600$; $\omega_s = 1$ rad/s.

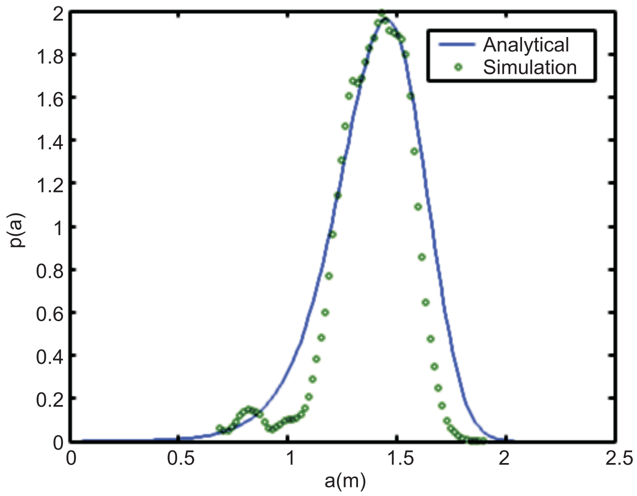

Figure 12. PDF for displacement amplitude (a) for PSDF 1 with $D = 9{\cdot}6$; $\omega_s = 5$ rad/s.

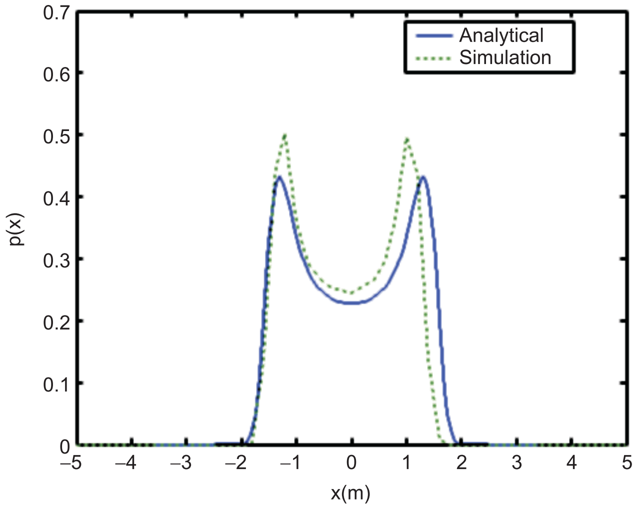

Figure 13. PDF of displacement (x) for PSDF 1 with $D = 9{\cdot}6$; $\omega_s = 5$ rad/s.

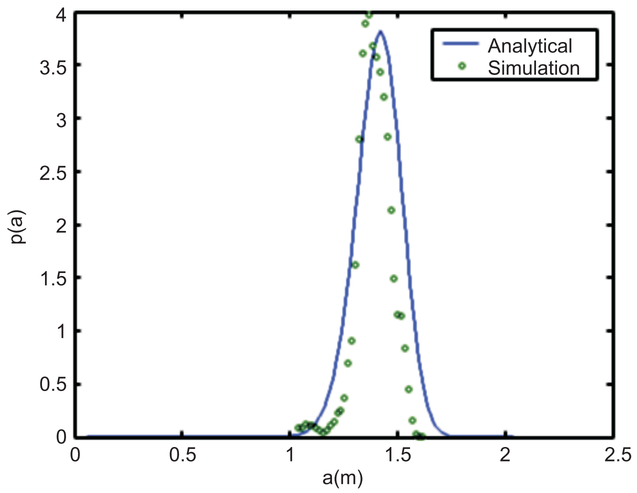

Figure 14. PDF for displacement amplitude (a) for PSDF 2 with $D = 600$; $\omega_s = 12$ rad/s.

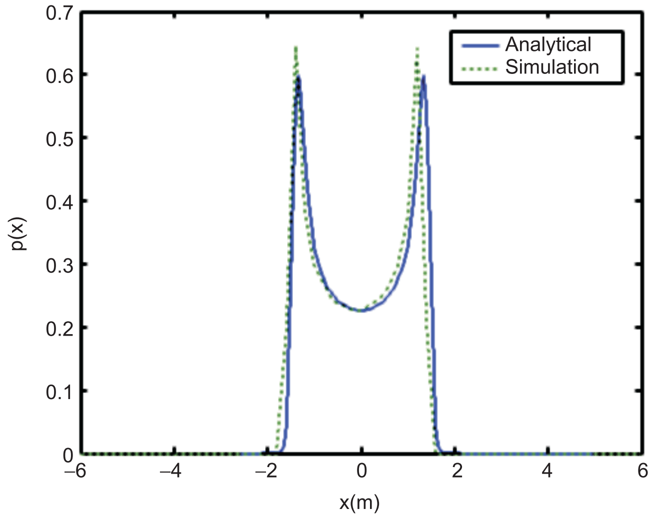

Figure 15. PDF of displacement (x) for PSDF 2 with $D = 600$; $\omega_s = 12$ rad/s.

transformation along with stochastic averaging procedure to determine averaged drift and diffusion coefficients of Ito's equation. The PSDFs of responses are obtained from the solution of FPK equation. The validity of the method is established by comparing the analytical results with those of simulation study. As a numerical example, a Duffing oscillator driven by both narrow- and wide-band excitations is considered. The results of the study show the following.

(I) For small normalized excitation (normalized with respect to the mass of system), the responses obtained from the analysis compare well with those obtained from simulation for both narrow and wide-band excitations.

(II) For normalized excitations which are not small, the results obtained from the simulation studies do not compare well with those obtained from analysis; varying degrees of error are observed depending upon the nature of excitation. Thus, the method is not applicable for such conditions.

(III) However, for oscillator (linear) frequency lying in the tail end of PSDFs of large normalized excitations, the method provides good results for both narrow- and wide-band spectrums.

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