Vibrations of stretched damped beams under non-ideal boundary conditions

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Abstract. A simply supported damped Euler–Bernoulli beam with immovable end conditions are considered. The concept of non-ideal boundary conditions is applied to the beam problem. In accordance, the boundaries are assumed to allow small deflections and moments. Approximate analytical solution of the problem is found using the method of multiple scales, a perturbation technique.

Keywords. Stretched beam vibrations; non-ideal boundary conditions; method of multiple time scales.

1. Introduction

Beams are frequently used as design models for vibration analysis. In such analysis, types of support conditions are important and have direct effect on the solutions and natural frequencies. Different types of supports such as simply supported, clamped, sliding, free etc. are defined and requirements associated with each of them are stated. Real systems are considered to satisfy those ideal boundary conditions. However, small deviations from ideal conditions in real systems indeed occur. For example, a beam connected at ends to rigid supports by pins is modelled using simply supported boundary conditions which require deflections and moments to be zero. However, the hole and pin assembly may have small gaps and/or friction which may introduce small deflections and/or moments at the ends. Similarly, a real built-in beam may have very small variations in deflection and/or slope. To represent such behaviour, non-ideal boundary condition concept has been proposed recently (Pakdemirli & Boyaci 2001, 2002, 2003).

Non-ideal boundary conditions are modelled using perturbations. Linear beam problems of different support conditions and an axially moving string problem have been treated (Pakdemirli & Boyaci 2002). Free vibration case for an undamped nonlinear beam problem with stretching has also been considered (Pakdemirli & Boyaci 2001). The damped forced case with a non-ideal simple support in an intermediate point is considered further (Pakdemirli & Boyaci 2003). Here in this work, the idea is extended to a damped forced nonlinear simple-simple beam vibration problem in which the nonlinearity is due to stretching effects. As non-ideality for a simple-simple support case, small variations in deflections and moments are

A list of symbols is given at the end of the paper.

allowed at both ends. Ideal and non-ideal frequencies as well as frequency response curves are contrasted. Combined effects of nonlinearity and non-ideal boundary conditions on the natural frequencies and mode shapes are examined using the method of multiple scales, a perturbation technique. While the stretching effect increases the frequencies, non-ideal boundary conditions may increase or decrease the frequencies.

2. Problem formulation and solution

The model considered is an Euler–Bernoulli beam with immovable end conditions causing nonlinear stretching effects. The dimensionless equation is

$$\ddot{w} + w^{iv} = \frac{1}{2}\varepsilon w'' \int_0^1 w'^2 dx - 2\varepsilon\mu\dot{w} + \hat{F} \cos \Omega t, \quad (1)$$

where w is the deflection, μ is the damping, t is the time, x is the spatial variable and F and Ω are the magnitude and frequency of the external excitation respectively. Dot denotes differentiation with respect to time t and prime denotes differentiation with respect to the spatial variable x . Review of the vast literature on beam vibrations is beyond the scope of this study. For some exact solutions of beams and comparison with approximate ones having different support conditions, see for example Pakdemirli & Nayfeh (1994), Ozkaya *et al* (1997), Low (1997), and Ozkaya & Pakdemirli (1999).

Here it is assumed that the beam is simply supported at both ends. However, the boundary conditions are not ideal and some slight variations occur in deflections and moments, hence

$$w(0, t) = \varepsilon\alpha_1(t), \quad w''(0, t) = \varepsilon\beta_1(t), \quad w(1, t) = \varepsilon\alpha_2(t), \quad w''(1, t) = \varepsilon\beta_2(t), \quad (2)$$

where ε is a small perturbation parameter denoting that the variations in deflections and moments are small.

An approximate solution of the form below is assumed

$$w(x, t) = w_0(x, T_0, T_1) + \varepsilon w_1(x, T_0, T_1) + \dots, \quad (3)$$

where T_0 is the usual fast time scale and $T_1 = \varepsilon t$ is the slow time scale in the method of multiple scales. Only primary resonance case is considered and hence the forcing term is ordered as $\hat{F} = \varepsilon F$. The time derivatives are defined as

$$d/dt = D_0 + \varepsilon D_1 + \dots, \quad d^2/dt^2 = D_0^2 + 2\varepsilon D_0 D_1 + \dots, \quad (4)$$

where

$$D_n = d/dT_n. \quad (5)$$

Substituting (3) and (4) into (1) and (2), separating at each order of ε , we get the following equations.

Order 1

$$\begin{aligned} D_0^2 w_0 + w_0^{iv} &= 0, \\ w_0(0, T_0, T_1) &= 0, \quad w_0''(0, T_0, T_1) = 0, \\ w_0(1, T_0, T_1) &= 0, \quad w_0''(1, T_0, T_1) = 0. \end{aligned} \quad (6)$$

Order ε

$$\begin{aligned} D_0^2 w_1 + w_1^{iv} &= -2\mu D_0 w_0 - 2D_0 D_1 w_0 + \frac{1}{2} w_0'' \int_0^1 w_0'^2 dx + F \cos \Omega T_0, \\ w_1(0, T_0, T_1) &= \alpha_1(T_0, T_1), \quad w_1''(0, T_0, T_1) = \beta_1(T_0, T_1), \\ w_1(1, T_0, T_1) &= \alpha_2(T_0, T_1), \quad w_1''(1, T_0, T_1) = \beta_2(T_0, T_1). \end{aligned} \quad (7)$$

The solution at the first order is

$$w_0 = (A(T_1)e^{i\omega T_0} + cc)Y(x), \quad (8)$$

where cc stands for complex conjugates of the preceding terms. Substituting (8) into (6) yields the boundary value problem,

$$Y^{iv} - \omega^2 Y = 0, \quad Y(0) = Y''(0) = Y(1) = Y''(1) = 0. \quad (9)$$

The solution is

$$Y(x) = \sqrt{2} \sin n\pi x, \quad \omega = n^2 \pi^2, \quad n = 1, 2, 3, \dots, \quad (10)$$

where $Y(x)$ is normalized such that $\int_0^1 Y^2 dx = 1$.

At order ε , one substitutes (8) into the right hand side of (7). The result is

$$\begin{aligned} D_0^2 w_1 + w_1^{iv} &= \left[-2i\omega D_1 A Y - 2i\mu\omega A Y + \frac{3}{2} A^2 \bar{A} Y'' \int_0^1 Y'^2 dx + \frac{1}{2} F e^{i\sigma T_1} \right] \\ &\times e^{i\omega T_0} + NST + cc, \end{aligned} \quad (11)$$

where NST stands for non-secular terms. It is also assumed that the external excitation frequency is close to one of the natural frequencies of the system; such that

$$\Omega = \omega + \varepsilon\sigma. \quad (12)$$

Here σ is a detuning parameter of order 1. A solution of the form below is assumed

$$w_1 = \varphi(x, T_1)e^{i\omega T_0} + W_1(x, T_0, T_1) + cc. \quad (13)$$

The first part of the solution is the one corresponding to secular terms and the second is the one corresponding to non-secular terms. Substituting (13) into (11) with boundary conditions yields

$$\begin{aligned} \varphi^{iv} - \omega^2 \varphi &= -2i\omega D_1 A Y - 2i\mu\omega A Y + \frac{3}{2} A^2 \bar{A} Y'' \int_0^1 Y'^2 dx + \frac{1}{2} F e^{i\sigma T_1}, \\ \varphi(0, T_1) &= \alpha_1 A(T_1), \quad \varphi''(0, T_1) = \beta_1 A(T_1), \quad \varphi(1, T_1) = \alpha_2 A(T_1), \\ \varphi''(1, T_1) &= \beta_2 A(T_1). \end{aligned} \quad (14)$$

In writing (15), the variations of deflections at the boundaries are considered to be of the same form as the time variations of the solutions and α_i and β_i are now constants. Since the

homogeneous problem has a non-trivial solution, the non-homogeneous problem (14) and (15) have a solution only if a solvability condition is satisfied (Nayfeh 1981). The solvability condition requires

$$2in^2\pi^2(D_1A + \mu A) + \frac{3}{2}n^4\pi^4A^2\bar{A} + \sqrt{2}n\pi KA - \frac{1}{2}fe^{i\sigma T_1} = 0, \quad (16)$$

where

$$K = (\beta_1 - \beta_2 \cos n\pi - n^2\pi^2\alpha_1 + n^2\pi^2\alpha_2 \cos n\pi) \text{ and } f = \int_0^1 FY dx. \quad (17)$$

Substituting the polar form

$$A(T_1) = \frac{1}{2}a(T_1)e^{i\theta(T_1)} \quad (18)$$

into (16), and separating real and imaginary parts, one obtains

$$\begin{aligned} n^2\pi^2 \frac{da}{dT_1} &= -n^2\pi^2\mu a + \frac{1}{2}f \sin \gamma, \\ n^2\pi^2 a \frac{d\gamma}{dT_1} &= n^2\pi^2\sigma a - \frac{3}{16}n^4\pi^4a^3 - \frac{\sqrt{2}}{2}n\pi K + \frac{1}{2}f \cos \gamma, \end{aligned} \quad (19)$$

where $\gamma = \sigma T_1 - \theta$.

In the steady state case, $a' = \gamma' = 0$ and solving for the detuning parameter yields

$$\sigma = \frac{3}{16}n^2\pi^2a^2 + \frac{\sqrt{2}}{2n\pi}K \pm \left(\frac{f^2}{4n^4\pi^4a^2} - \mu^2 \right)^{1/2}. \quad (20)$$

For free undamped vibrations, non-ideal nonlinear natural frequencies are

$$\omega_{ni} = \omega + \varepsilon \left(\frac{3}{16}\omega a^2 + \frac{\sqrt{2}}{2\sqrt{\omega}}K \right). \quad (21)$$

In the above relation, the first term in $O(\varepsilon)$ is due to the nonlinearity and the second term is due to the non-ideal boundary conditions. If only the non-ideality term of order ε is to be considered, it may increase or decrease the frequencies depending on the mode number n and amplitudes of end variations that are α_i due to deflections and β_i due to moments. Also it is apparent that deflection effects become dominant compared to the moment effects as the mode number increases. Different cases are summarized in table 1.

In figure 1a, fundamental nonlinear frequencies versus amplitudes are contrasted for the ideal and non-ideal cases, such that $\alpha_1 = \alpha_2 = 0.1$, $\beta_1 = \beta_2 = 10$, and $\varepsilon = 0.1$. In figure 1b, fundamental nonlinear frequencies versus amplitudes are contrasted for the ideal and non-ideal cases again, but this time $\alpha_1 = \alpha_2 = 2$, the other parameters remaining the same.

One can also obtain amplitude-excitation frequency relation from (12) and (20)

$$\Omega = \omega + \varepsilon \frac{3}{16}\omega a^2 + \varepsilon \frac{\sqrt{2}}{2\sqrt{\omega}}K \pm \left(\frac{\varepsilon^2 f^2}{4\omega^2 a^2} - \varepsilon^2 \mu^2 \right)^{1/2}. \quad (22)$$

Table 1. Effect of non-ideal boundary conditions on the frequencies.

N	Frequency
Odd	Frequencies increase if $\beta_1 + \beta_2 > n^2\pi^2(\alpha_1 + \alpha_2)$ Frequencies decrease if $\beta_1 + \beta_2 < n^2\pi^2(\alpha_1 + \alpha_2)$ No change if $\beta_1 + \beta_2 = n^2\pi^2(\alpha_1 + \alpha_2)$
Even	Frequencies increase if $\beta_1 + n^2\pi^2\alpha_2 > \beta_2 + n^2\pi^2\alpha_1$ Frequencies decrease if $\beta_1 + n^2\pi^2\alpha_2 < \beta_2 + n^2\pi^2\alpha_1$ No change if $\beta_1 + n^2\pi^2\alpha_2 = \beta_2 + n^2\pi^2\alpha_1$

Figure 1a. Nonlinear frequencies versus amplitudes for ideal (dashed) and non-ideal (solid) cases for the first mode ($\alpha_1 = \alpha_2 = 0.1, \beta_1 = \beta_2 = 10, \varepsilon = 0.1$).

Figure 1b. Nonlinear frequencies versus amplitudes for ideal (dashed) and non-ideal (solid) cases for the first mode ($\alpha_1 = \alpha_2 = 2, \beta_1 = \beta_2 = 10, \varepsilon = 0.1$).

Figure 2a. Frequency response curves for ideal (dashed) and non-ideal (solid) cases for the first mode ($\alpha_1 = \alpha_2 = 0.1$, $\beta_1 = \beta_2 = 10$, $\varepsilon = 0.1$, $\mu = 0.5$, $f = 10$).

In figure 2a, frequency response graphs for the fundamental nonlinear frequencies are compared for the ideal and non-ideal cases for $\alpha_1 = \alpha_2 = 0.1$, $\beta_1 = \beta_2 = 10$, $\varepsilon = 0.1$, $\mu = 0.5$, $f = 10$. Also in figure 2b, frequency response graphs for the fundamental nonlinear frequencies are compared for the ideal and non-ideal cases, this time for $\alpha_1 = \alpha_2 = 2$, all the other parameters remaining the same.

The approximate beam deflection to the first order is as follows,

$$w = a \cos(\Omega t - \gamma) Y(x) + O(\varepsilon), \quad (23)$$

where $Y(x)$ is as given in (10).

Figure 2b. Frequency response curves for ideal (dashed) and non-ideal (solid) cases for the first mode ($\alpha_1 = \alpha_2 = 2$, $\beta_1 = \beta_2 = 10$, $\varepsilon = 0.1$, $\mu = 0.5$, $f = 10$).

3. Concluding remarks

Non-ideal boundary conditions are defined and formulated using perturbation theory. A sample problem of a stretched damped beam with external excitation was treated. An approximate analytical solution of the problem is presented using the method of multiple scales. It is shown that small variations of deflections and moments at the ends may affect the frequencies of the response. Depending on the mode numbers and amplitudes of variations, the frequencies may increase or decrease. Deviations from the ideal conditions lead to a drift in frequency-response curves which may be positive, negative or zero depending on the mode number and amplitudes of variations.

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List of symbols

a	real amplitude for the polar representation of the complex amplitude;
A	complex amplitude of the response function;
D_i	differentiation with respect to time scale T_i ;
f	excitation amplitude;
F	reordered amplitude of the external excitation;
$\hat{F}$	amplitude of the external excitation;
t	time variable;
T_0	fast time scale;
T_1	slow time scale of second order;
w	response function;
w_i	response function in order (i) of the perturbation;
W_1	part of the response function at the second order expressing non-secular terms;
x	spatial variable;
Y	spatial function of the response at the first order of perturbation;
α_1	amplitude of the deflection at the left non-ideal boundary;
α_2	amplitude of the deflection at the right non-ideal boundary;
β_1	amplitude of the moment at the left non-ideal boundary;
β_2	amplitude of the moment at the right non-ideal boundary;
γ	phase;
ε	a small perturbation parameter;
θ	phase function for the polar representation of complex amplitudes;
λ	a coefficient related to the natural frequency;
μ	damping coefficient;
σ	detuning parameter of order one;
ϕ	spatial function of the response at the second order of perturbation;
ω	one of the natural frequencies of the system with ideal boundary conditions;
ω_{ni}	one of the natural frequencies of the system with non-ideal boundary conditions;
Ω	frequency of the external excitation.

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