Two-dimensional static deformation of an anisotropic medium

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Abstract. The problem of two-dimensional static deformation of a monoclinic elastic medium has been studied using the eigenvalue method, following a Fourier transform. We have obtained expressions for displacements and stresses for the medium in the transformed domain. As an application of the above theory, the particular case of a normal line-load acting inside an orthotropic elastic half-space has been considered in detail and closed form expressions for the displacements and stresses are obtained. Further, the results for the displacements for a transversely isotropic as well as for an isotropic medium have also been derived in the closed form. The use of matrix notation is straightforward and avoids unwieldy mathematical expressions. To examine the effect of anisotropy, variations of dimensionless displacements for an orthotropic, transversely isotropic and isotropic elastic medium have been compared numerically and it is found that anisotropy affects the deformation significantly.

Keywords. Static deformation; anisotropic; orthotropic; monoclinic elastic medium.

1. Introduction

Maruyama (1966) obtained closed-form expressions for the displacement and stress fields in a homogeneous isotropic elastic half-space as a result of line-source. Dziewonski & Anderson (1981) have established that the upper part of the earth is anisotropic. Laminated composite anisotropic materials find a large number of engineering applications. Generalization of the solution to include anisotropy is, however, very difficult. For a transversely isotropic medium, some other relevant contribution are those of Small & Booker (1984) and Pan (1989).

Chou (1976) and Ting (1995) discussed antiplane strain deformation of an anisotropic medium. Garg *et al* (1996) obtained representations of seismic sources causing antiplane strain deformations of orthotropic media. The corresponding plane strain deformation of an

orthotropic elastic medium has been discussed by Garg *et al* (2003) using an eigenvalue approach. Orthotropic symmetry is exhibited by olivine and orthopyroxens, the principal rock forming minerals of the deep crust and upper mantle.

In the present paper, a novel analytical eigenvalue method is presented for a monoclinic solid. Fourier transformation of the equations of equilibrium for plane strain deformation of a monoclinic solid reduces them into a single linear homogeneous vector differential equation of second order, on which an eigenvalue method is applied to obtain a solution in the transformed domain. The form of the single governing equation for a monoclinic solid derived in this paper is a new contribution to the theory of monoclinic solids. The procedure developed in this paper is relatively simple and straightforward, avoids the cumbersome nature of the problem and is also convenient for numerical computation. As particular cases, normal line-loads acting inside orthotropic, transversely isotropic and perfectly isotropic elastic half-spaces have been considered in detail. The deformation at any point of the medium is useful to analyse the deformation field around mining tremors and drilling into the crust of the earth. It may also find application in various engineering problems, crystal physics and solid-earth geophysics regarding deformation of an anisotropic solid. In fact, the study of a single force acting on a monoclinic solid forms the basis for further investigations such as dipolar sources and faults in an anisotropic solid.

2. Basic equations and theory

The equations of equilibrium in the cartesian co-ordinate system (x_1, x_2, x_3) for zero body forces are

$$\tau_{ij,j} = 0, \quad (1)$$

where τ_{ij} ($i, j = 1, 2, 3$) are the components of the stress tensor.

The stress-strain relations for a monoclinic elastic medium, with x_1x_2 -plane as a plane of elastic symmetry are

$$\tau_{11} = d_{11}e_{11} + d_{12}e_{22} + d_{13}e_{33} + 2d_{16}e_{12}, \quad (2)$$

$$\tau_{22} = d_{12}e_{11} + d_{22}e_{22} + d_{23}e_{33} + 2d_{26}e_{12}, \quad (3)$$

$$\tau_{33} = d_{13}e_{11} + d_{23}e_{22} + d_{33}e_{33} + 2d_{36}e_{12}, \quad (4)$$

$$\tau_{23} = 2d_{44}e_{23} + 2d_{45}e_{13}, \quad (5)$$

$$\tau_{13} = 2d_{45}e_{23} + 2d_{55}e_{13}, \quad (6)$$

$$\tau_{12} = d_{16}e_{11} + d_{26}e_{22} + d_{36}e_{33} + 2d_{66}e_{12}, \quad (7)$$

where e_{ij} are the components of strain tensor and are related to displacement components (u_1, u_2, u_3) through the relations

$$e_{ij} = 1/2(u_{i,j} + u_{j,i}). \quad (8)$$

The two-suffix symmetric quantities d_{lk} ($l, k = 1, 2, \dots, 6$) are the elastic moduli for a monoclinic elastic medium. For an orthotropic solid with coordinate planes coinciding with the planes of symmetry, we have

$$d_{16} = d_{26} = d_{36} = d_{45} = 0, \quad (9)$$

and the number of independent elastic coefficients reduce to nine. A transversely isotropic medium is a particular case of orthotropy in which

$$d_{22} = d_{11}, d_{23} = d_{13}, d_{55} = d_{44}, d_{66} = 1/2(d_{11} - d_{12}), \quad (10)$$

and there are five independent elastic coefficients. Further, isotropy is also a particular case of an orthotropy in which

$$\begin{aligned} d_{11} &= d_{22} = d_{33} = \lambda + 2\mu, \\ d_{12} &= d_{13} = d_{23} = \lambda, \\ d_{44} &= d_{55} = d_{66} = \mu, \end{aligned} \quad (11)$$

where λ and μ are Lamé's constants. For convenience, we shall write

$$(x_1, x_2, x_3) = (x, y, z) \quad \text{and} \quad (u_1, u_2, u_3) = (u, v, w).$$

We now consider the plane-strain deformation, parallel to the xy -plane, in which the displacement components are independent of z and are of the type

$$u = u(x, y), v = v(x, y), w = 0. \quad (12)$$

The non-zero stresses for the plane-strain problem for a monoclinic medium are obtained from (2)–(8), (10) and (12), as

$$\tau_{11} = d_{11} \frac{\partial u}{\partial x} + d_{12} \frac{\partial v}{\partial y} + d_{16} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad (13)$$

$$\tau_{22} = d_{12} \frac{\partial u}{\partial x} + d_{22} \frac{\partial v}{\partial y} + d_{26} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad (14)$$

$$\tau_{33} = d_{13} \frac{\partial u}{\partial x} + d_{23} \frac{\partial v}{\partial y} + d_{36} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad (15)$$

$$\tau_{12} = d_{16} \frac{\partial u}{\partial x} + d_{26} \frac{\partial v}{\partial y} + d_{66} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right). \quad (16)$$

The equations of equilibrium, for a plane-strain deformation (parallel to the xy -plane) of a monoclinic elastic medium are

$$\begin{aligned} & \left[d_{11} \frac{\partial^2 u}{\partial x^2} + d_{66} \frac{\partial^2 u}{\partial y^2} + 2d_{16} \frac{\partial^2 u}{\partial x \partial y} \right] \\ & + \left[d_{16} \frac{\partial^2 v}{\partial x^2} + d_{26} \frac{\partial^2 v}{\partial y^2} + (d_{12} + d_{66}) \frac{\partial^2 v}{\partial x \partial y} \right] = 0, \end{aligned} \quad (17)$$

$$\begin{aligned} & \left[d_{16} \frac{\partial^2 u}{\partial x^2} + d_{26} \frac{\partial^2 u}{\partial y^2} + (d_{66} + d_{12}) \frac{\partial^2 u}{\partial x \partial y} \right] \\ & + \left[d_{66} \frac{\partial^2 v}{\partial x^2} + d_{22} \frac{\partial^2 v}{\partial y^2} + 2d_{26} \frac{\partial^2 v}{\partial x \partial y} \right] = 0, \end{aligned} \quad (18)$$

We define the following dimensionless quantities,

$$\alpha = \frac{x}{h}, \beta = \frac{y}{h}, U = \frac{u}{h}, V = \frac{v}{h}, \sigma_{11} = \frac{\tau_{11}}{d_{66}}, \sigma_{12} = \frac{\tau_{12}}{d_{66}}, \sigma_{22} = \frac{\tau_{22}}{d_{66}},$$

$$a = \frac{d_{11}}{d_{66}}, b = \frac{d_{22}}{d_{66}}, c = \frac{d_{12}}{d_{66}}, d = \frac{d_{26}}{d_{66}}, e = \frac{d_{16}}{d_{66}}, \quad (19)$$

where 'h' is a known fixed quantity and has the dimensions of length. Thus, the equations of equilibrium (17) and (18), in dimensionless form become

$$\left[a \frac{\partial^2 U}{\partial \alpha^2} + \frac{\partial^2 U}{\partial \beta^2} + 2e \frac{\partial^2 U}{\partial \alpha \partial \beta} \right] + \left[e \frac{\partial^2 V}{\partial \alpha^2} + d \frac{\partial^2 V}{\partial \beta^2} + (c+1) \frac{\partial^2 V}{\partial \alpha \partial \beta} \right] = 0, \quad (20)$$

$$\left[e \frac{\partial^2 U}{\partial \alpha^2} + d \frac{\partial^2 U}{\partial \beta^2} + (1+c) \frac{\partial^2 U}{\partial \alpha \partial \beta} \right] + \left[\frac{\partial^2 V}{\partial \alpha^2} + b \frac{\partial^2 V}{\partial \beta^2} + 2d \frac{\partial^2 V}{\partial \alpha \partial \beta} \right] = 0. \quad (21)$$

In (20) and (21), α is the dimensionless vertical distance, β is the dimensionless horizontal distance, U is the dimensionless vertical displacement, V is the dimensionless horizontal displacement, σ_{11} , σ_{12} , σ_{22} are the dimensionless stress components, and a, b, c, d, e are dimensionless elastic coefficients.

For an isotropic solid,

$$a = b = 1/(1 - \alpha^*), c = (1 - 2\alpha^*)/(\alpha^* - 1) = a - 2, d = 0, e = 0, \quad (22)$$

where

$$\alpha^* = (\lambda + \mu)/(\lambda + 2\mu), \quad (23)$$

in particular, for a Poissonian isotropic solid ($\lambda = \mu$), and $\alpha^* = 2/3$.

We define the Fourier transform $\bar{f}(\alpha, k)$ of a function $f(\alpha, \beta)$ by the relation

$$\bar{f}(\alpha, k) = F[f(\alpha, \beta)] = \int_{-\infty}^{+\infty} f(\alpha, \beta) e^{ik\beta} d\beta, \quad (24)$$

so that

$$f(\alpha, \beta) = (1/2\pi) \int_{-\infty}^{\infty} \bar{f}(\alpha, k) e^{-ik\beta} dk, \quad (25)$$

where k is the Fourier transform parameter.

Applying the Fourier transformation on equations of equilibrium (20) and (21), we find

$$\left[a \frac{d^2 \bar{U}}{d\alpha^2} - 2eik \frac{d\bar{U}}{d\alpha} - k^2 \bar{U} \right] + \left[e \frac{d^2 \bar{V}}{d\alpha^2} - ik(1+c) \frac{d\bar{V}}{d\alpha} - dk^2 \bar{V} \right] = 0, \quad (26)$$

$$\left[e \frac{d^2 \bar{U}}{d\alpha^2} - ik(1+c) \frac{d\bar{U}}{d\alpha} - k^2 d\bar{U} \right] + \left[\frac{d^2 \bar{V}}{d\alpha^2} - 2dik \frac{d\bar{V}}{d\alpha} - bk^2 \bar{V} \right] = 0. \quad (27)$$

The above two equations can be written as the following single homogeneous vector differential equation:

$$A^\circ \frac{d^2 N_0}{d\alpha^2} + B^\circ \frac{dN_0}{d\alpha} + C^\circ N_0 = 0, \quad (28)$$

where

$$\begin{aligned} A^\circ &= \begin{bmatrix} a & e \\ e & 1 \end{bmatrix}, B^\circ = \begin{bmatrix} -2\iota ke & -\iota k(1+c) \\ -\iota k(1+c) & -2\iota kd \end{bmatrix}, \\ C^\circ &= \begin{bmatrix} -k^2 & -k^2 d \\ -k^2 d & -bk^2 \end{bmatrix}, N_0 = \begin{bmatrix} \bar{U} \\ \bar{V} \end{bmatrix}. \end{aligned} \quad (29)$$

We note that matrices A° , B° and C° are all symmetric, matrix A° depends upon dimensionless elastic moduli while the matrices B° and C° depend upon parameter k also. The governing equation (28) for a monoclinic solid is a new contribution and eigenvalue method shall be applied to solve the problem. For this, we seek a solution of vector differential equation (28) of the type

$$N_0(\alpha, k) = E(k)e^{s\alpha}, \quad (30)$$

where s is a parameter and $E(k)$ is a matrix of the type 2×1 . Substitution of the value of N_0 from (30) into vector differential equation (28) gives the following characteristic equation:

$$a_0 s^4 + 4a_1 s^3 + 6a_2 s^2 + 4a_3 s + a_4 = 0, \quad (31)$$

where

$$\begin{aligned} a_0 &= a - e^2, \quad a_1 = -\iota(ad - ce)k/2, \\ a_2 &= \frac{-(ab - c^2 - 2c + 2ed)k^2}{6}, \\ a_3 &= \frac{2\iota(cd - be)k^3}{4}, \quad a_4 = (b - d^2)k^4. \end{aligned} \quad (32)$$

To solve biquadratic equation (31), we make use of Descartes' method, in which the second term is removed by using the transformation

$$z^* = a_0 s + a_1, \quad (33)$$

so that (31) reduces to

$$(z^*)^4 + 6H(z^*)^2 + 4Gz^* + (a_0^2 I - 3H^2) = 0, \quad (34)$$

where

$$\begin{aligned} H &= a_0 a_2 - a_1^2, \quad G = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3, \\ I &= a_0 a_4 - 4a_1 a_3 + 3a_2^2. \end{aligned} \quad (35)$$

Following Descartes' method, the left hand side of (34) can be resolved into a product of two quadratic factors as

$$\begin{aligned} &(z^*)^4 + 6H(z^*)^2 + 4Gz^* + (a_0^2 I - 3H^2) \\ &= [(z^*)^2 + p_0 z^* + q][(z^*)^2 - p_0 z^* + q'], \end{aligned} \quad (36)$$

where

$$q + q' = p_0^2 + 6H, qq' = a_0^2 I - 3H^2, \quad (37)$$

$$p_0(q' - q) = 4G. \quad (38)$$

when $p_0 = 0$, the values of q and q' are determined from (37) directly, otherwise, eliminating q and q' from (37) and (38), we get

$$\xi^3 + 3b_1\xi^2 + 3b_2\xi + b_3 = 0, \quad (39)$$

where

$$\begin{aligned} \xi &= p_0^2, \quad b_1 = 4H, \\ b_2 &= 4(12H^2 - a_0^2 I)/3, \quad b_3 = -16G^2. \end{aligned} \quad (40)$$

Equation (39) is a cubic in ξ and hence, can be solved using Cardan's method and thus (39) is reduced to

$$\delta^3 + 3H_1\delta + G_1 = 0, \quad (41)$$

where

$$\delta = \xi + b_1, \quad H_1 = b_2 - b_1^2, \quad G_1 = b_3 - 3b_1b_2 + 2b_1^3. \quad (42)$$

The roots of the cubic equation (41) are

$$\delta = u_0 + v_0, \quad (43)$$

where u_0 and v_0 are given by

$$u_0^3 = \frac{-G_1 + \sqrt{G_1^2 + 4H_1^3}}{2}, \quad v_0^3 = \frac{-G_1 - \sqrt{G_1^2 + 4H_1^3}}{2}, \quad (44)$$

satisfying $u_0v_0 = -H_1$. Then p_0 is given by

$$p_0^2 = \delta - b_1, \quad (45)$$

and q and q' are then to be calculated from (37), (38) and (45). Equation (36) gives

$$\begin{aligned} z_1^* &= \frac{-p_0 + (p_0^2 - 4q)^{1/2}}{2}, \quad z_2^* = \frac{-p_0 - (p_0^2 - 4q)^{1/2}}{2}, \\ z_3^* &= \frac{p_0 + (p_0^2 - 4q')^{1/2}}{2}, \quad z_4^* = \frac{p_0 - (p_0^2 - 4q')^{1/2}}{2}, \end{aligned} \quad (46)$$

as the roots of the biquadratic equation (34). Hence, the eigenvalues of the problem are s_1, s_2, s_3, s_4 , where

$$s_L = \frac{z_L^* - a_1}{a_0}, \quad \text{for } L = 1, 2, 3, 4. \quad (47)$$

The eigenvectors for a monoclinic elastic medium are obtained by solving the matrix equation

$$[s^2 A^\circ + s B^\circ + C^\circ] E(k) = 0, \quad (48)$$

and the eigenvectors are found to be

$$X_L^T = [P_{1L}, 1], \text{ for } L = 1, 2, 3, 4, \quad (49)$$

where

$$P_{1L} = \frac{-[es_L^2 - \iota ks_L(1+c) - k^2 d]}{as_L^2 - 2\iota kes_L - k^2} = \frac{-[s_L^2 - 2\iota kds_L - bk^2]}{es_L^2 - \iota ks_L(1+c) - k^2 d} \quad (50)$$

and $(\dots)^T$ denote the transpose of the matrix $(\dots)$. Thus, a general solution of the vector-matrix differential equation (28) for a monoclinic elastic medium for non-repeated eigenvalues is of the form

$$N_0(\alpha, k) = \sum_{L=1}^4 (B_L X_L e^{s_L \alpha}), \quad (51)$$

where coefficients B_1, B_2, B_3 and B_4 are to be determined from prescribed boundary conditions. These coefficients may depend upon k . Equation (51) gives the following expressions for the dimensionless displacements in the transformed domain.

$$\begin{aligned} \bar{U}(\alpha, k) &= B_1 P_{11} e^{s_1 \alpha} + B_2 P_{12} e^{s_2 \alpha} + B_3 P_{13} e^{s_3 \alpha} + B_4 P_{14} e^{s_4 \alpha}, \\ \bar{V}(\alpha, k) &= B_1 e^{s_1 \alpha} + B_2 e^{s_2 \alpha} + B_3 e^{s_3 \alpha} + B_4 e^{s_4 \alpha}. \end{aligned} \quad (52)$$

Further, the expressions for dimensionless stresses for a plane strain deformation for a monoclinic solid are found to be

$$\begin{aligned} \bar{\sigma}_{11} &= [Q_{11} B_1 e^{s_1 \alpha} + Q_{12} B_2 e^{s_2 \alpha} + Q_{13} B_3 e^{s_3 \alpha} + Q_{14} B_4 e^{s_4 \alpha}], \\ \bar{\sigma}_{12} &= [R_{11} B_1 e^{s_1 \alpha} + R_{12} B_2 e^{s_2 \alpha} + R_{13} B_3 e^{s_3 \alpha} + R_{14} B_4 e^{s_4 \alpha}], \end{aligned} \quad (53)$$

where

$$\begin{aligned} Q_{1L} &= a P_{1L} s_L - \iota k c - \iota k e P_{1L} + s_L e, \\ R_{1L} &= e P_{1L} s_L - \iota k d - \iota k P_{1L} + s_L \text{ for } L = 1, 2, 3, 4. \end{aligned} \quad (54)$$

3. Particular cases

3.1 Orthotropic symmetry

In this case

$$d = 0, e = 0, a_1 = 0, a_3 = 0, \quad (55)$$

and vector differential equation and characteristics equation for an orthotropic medium are obtained from (28), (29), (31), (32) and (55). We find

$$A \frac{d^2 N}{d\alpha^2} + B \frac{dN}{d\alpha} + C N = 0, \quad (56)$$

in which A, B, C are

$$A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\iota k(1+c) \\ -\iota k(1+c) & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} -k^2 & 0 \\ 0 & -bk^2 \end{pmatrix}, \quad N = \begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix}, \quad (57)$$

and

$$as^4 - (ab - c^2 - 2c)k^2s^2 + bk^4 = 0. \quad (58)$$

The characteristic equation (58) is a quadratic equation in s^2 and can be solved directly without using Descartes' method and gives the eigenvalues as

$$s^2 = m_1^2 k^2, m_2^2 k^2, \quad (59)$$

where

$$m_1^2 = \frac{A_0 + (A_0^2 - 4B_0)^{1/2}}{2}, \quad m_2^2 = \frac{A_0 - (A_0^2 - 4B_0)^{1/2}}{2},$$

$$A_0 = (ab - c^2 - 2c)/a = m_1^2 + m_2^2, \quad B_0 = b/a = m_1^2 m_2^2. \quad (60)$$

Under the assumption that $m_1 \neq m_2$, the eigenvalues are

$$s_1 = m_1 |k|, s_2 = m_2 |k|,$$

$$s_3 = -m_1 |k|, s_4 = -m_2 |k|, \quad (61)$$

with real parts of (m_1, m_2) as positive.

The eigenvectors for an orthotropic elastic medium are given by

$$X_L^T = [P_{2L}, 1], \quad X_{L+2}^T = [-P_{2L}, 1], \quad (62)$$

for $L = 1, 2$, in which

$$P_{21} = \frac{\iota m_1 |k|}{k} \left(\frac{1+c}{am_1^2 - 1} \right) = \frac{\iota k}{m_1 |k|} \left(\frac{b - m_1^2}{1+c} \right),$$

$$P_{22} = \frac{\iota m_2 |k|}{k} \left(\frac{1+c}{am_2^2 - 1} \right) = \frac{\iota k}{m_2 |k|} \left(\frac{b - m_2^2}{1+c} \right), \quad (63)$$

Thus, a solution of matrix equation (56) for the case of an orthotropic elastic medium is

$$N(\alpha, k) = \sum_{L=1}^2 [(C_L X_L e^{m_L |k| \alpha} + C_{L+2} X_{L+2} e^{-m_L |k| \alpha}], \quad (64)$$

where the coefficients C_1, C_2, C_3 and C_4 are to be determined from boundary conditions and they may depend upon k .

The transformed dimensionless displacements for an orthotropic elastic medium are

$$\begin{aligned}\bar{U}(\alpha, k) &= C_1 P_{21} e^{m_1 |k| \alpha} + C_2 P_{22} e^{m_2 |k| \alpha} - C_3 P_{21} e^{-m_1 |k| \alpha} - C_4 P_{22} e^{-m_2 |k| \alpha}, \\ \bar{V}(\alpha, k) &= C_1 e^{m_1 |k| \alpha} + C_2 e^{m_2 |k| \alpha} + C_3 e^{-m_1 |k| \alpha} + C_4 e^{-m_2 |k| \alpha},\end{aligned}\quad (65)$$

and stresses in the transformed domain are

$$\begin{aligned}\bar{\sigma}_{11}(\alpha, k) &= Q_{21} C_1 e^{m_1 |k| \alpha} + Q_{22} C_2 e^{m_2 |k| \alpha} + Q_{21} C_3 e^{-m_1 |k| \alpha} + Q_{22} C_4 e^{-m_2 |k| \alpha}, \\ \bar{\sigma}_{12}(\alpha, k) &= R_{21} C_1 e^{m_1 |k| \alpha} + R_{22} C_2 e^{m_2 |k| \alpha} - R_{21} C_3 e^{-m_1 |k| \alpha} - R_{22} C_4 e^{-m_2 |k| \alpha},\end{aligned}\quad (66)$$

where

$$\begin{aligned}Q_{2L} &= a P_{2L} m_L |k| - \iota c k, \\ R_{2L} &= m_L |k| - \iota P_{2L} k, \text{ for } L = 1, 2.\end{aligned}\quad (67)$$

3.2 Transversely isotropic symmetry

This case is a particular case of monoclinic symmetry as well as orthotropic symmetry and here the dimensionless elastic constants are

$$a = b = 2 \left(\frac{d_{11}}{d_{11} - d_{12}} \right), c = 2 \left(\frac{d_{12}}{d_{11} - d_{12}} \right) = a - 2, d = e = 0. \quad (68)$$

The characteristic equation for a transversely isotropic elastic medium becomes

$$s^4 - 2k^2 s^2 + k^4 = 0, \quad (69)$$

provided $d_{11} \neq 0$ and $d_{11} \neq d_{12}$ and it is observed that this characteristic equation is independent of its elastic moduli a, b, c . The solution of characteristic equation (69) gives repeated eigenvalues

$$s = s_1 = s_2 = -s_3 = -s_4 = |k|. \quad (70)$$

The equilibrium equations in the transformed domain for a transversely isotropic medium are equivalent to the first-order vector differential equation. Further treatment for this case is different from the treatment discussed earlier for a orthotropic/monoclinic medium with different eigenvalues.

Ross (1984) has given a procedure to tackle the problems with repeated eigenvalues provided the governing vector differential equation is of the first-order. For this situation, we define the process as given by Ross (1984) and the first order vector differential equation representing the equilibrium equations is

$$dN_1/d\alpha = A_1 N_1, \quad (71)$$

where

$$N_1 = \begin{bmatrix} \bar{U} \\ \bar{V} \\ \frac{d\bar{U}}{d\alpha} \\ \frac{d\bar{V}}{d\alpha} \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k^2}{a} & 0 & 0 & \frac{\iota k(a-1)}{a} \\ 0 & ak^2 & \iota k(a-1) & 0 \end{bmatrix} \quad (72)$$

Following the procedure as outlined by Ross (1984), the four independent eigen vectors are found to be

$$\begin{aligned}
 X_1 &= \begin{pmatrix} \iota|k| \\ k \\ \iota k^2 \\ k|k| \end{pmatrix}, \quad X_2 = \begin{bmatrix} \iota \left\{ |k|\alpha - \left(\frac{2a}{a-1} \right) \right\} \\ k \left(\alpha - \frac{1}{|k|} \right) \\ \iota|k| \left\{ |k|\alpha - \left(\frac{1+a}{a-1} \right) \right\} \\ k|k|\alpha \end{bmatrix}, \\
 X_3 &= \begin{pmatrix} -\iota|k| \\ k \\ \iota k^2 \\ -k|k| \end{pmatrix}, \quad X_4 = \begin{bmatrix} -\iota \left\{ |k|\alpha + \left(\frac{2a}{a-1} \right) \right\} \\ k \left(\alpha + \frac{1}{|k|} \right) \\ \iota|k| \left\{ |k|\alpha + \left(\frac{1+a}{a-1} \right) \right\} \\ -k|k|\alpha \end{bmatrix} \quad (73)
 \end{aligned}$$

Thus, a general solution of first order matrix differential equation (71) for a transversely isotropic medium (Ross 1984) is

$$N_1 = (D_1 X_1 + D_2 X_2) e^{|k|\alpha} + (D_3 X_3 + D_4 X_4) e^{-|k|\alpha}, \quad (74)$$

where D_1, D_2, D_3, D_4 are coefficients which may depend upon k and dimensionless elastic coefficient a .

The displacements and stresses in the transformed domain for a transversely isotropic medium are found to be

$$\begin{aligned}
 \bar{U} &= \iota \left[\left\{ D_1 |k| + D_2 \left(|k|\alpha - \frac{2a}{a-1} \right) \right\} e^{|k|\alpha} \right. \\
 &\quad \left. - \left\{ D_3 |k| + D_4 \left(|k|\alpha + \frac{2a}{a-1} \right) \right\} e^{-|k|\alpha} \right] \\
 \bar{V} &= k \left[\left\{ D_1 + D_2 \left(\alpha - \frac{1}{|k|} \right) \right\} e^{|k|\alpha} + \left\{ D_3 + D_4 \left(\alpha + \frac{1}{|k|} \right) \right\} e^{-|k|\alpha} \right] \quad (75)
 \end{aligned}$$

$$\begin{aligned}
 \bar{\sigma}_{11} &= \iota \left[2k^2 D_1 e^{|k|\alpha} + \left\{ 2k^2 \alpha + |k| \left(\frac{2-4a}{a-1} \right) \right\} D_2 e^{|k|\alpha} \right. \\
 &\quad \left. + 2k^2 D_3 e^{-|k|\alpha} + \left\{ 2k^2 \alpha + |k| \left(\frac{4a-2}{a-1} \right) \right\} D_4 e^{-|k|\alpha} \right], \\
 \bar{\sigma}_{12} &= \left[\left\{ 2k|k| D_1 + D_2 \left(2k|k|\alpha - \frac{2ak}{a-1} \right) \right\} e^{|k|\alpha} \right. \\
 &\quad \left. - \left\{ 2k|k| D_3 + D_4 \left(2k|k|\alpha + \frac{2ak}{a-1} \right) \right\} e^{-|k|\alpha} \right]. \quad (76)
 \end{aligned}$$

4. An application: A normal line-load in an elastic half-space

We consider an elastic half-space with x -axis vertically downwards and the origin of the cartesian coordinate system (x, y, z) is taken at the boundary of the half-space. Let a normal line-load P_0 be acting vertically downwards on a line parallel to z -axis and passing through the point $(H, 0)$. The bounding surface $x = 0$ is assumed to be stress-free and we shall calculate the resulting stresses and displacements at any point of the half-space for the following three types of medium:

4.1 Deformation of an orthotropic elastic medium

Consider the elastic half-space consisting of region $\alpha < \alpha_0$ (medium I) and region $\alpha > \alpha_0$ (medium II). We find the displacements and stresses at any point of an orthotropic elastic half-space due to a normal line-load P_0 acting vertically downwards on a line parallel to z -axis (figure 1).

The boundary conditions are:

On the plane $x = 0$,

$$\tau_{11}^1(x = 0, y) = 0, \tau_{12}^1(x = 0, y) = 0. \quad (77)$$

On the plane $x = H$,

$$\begin{aligned} u^1(x = H, y) &= u^2(x = H, y), \\ v^1(x = H, y) &= v^2(x = H, y), \\ \tau_{12}^1(x = H, y) &= \tau_{12}^2(x = H, y), \\ \tau_{11}^2(x = H, y) - \tau_{11}^1(x = H, y) &= -P_0\delta(y), \end{aligned} \quad (78)$$

where $\delta(y)$ is the Dirac-delta function and $F[\delta(y)] = 1$.

Defining the dimensionless force

$$P = P_0/d_{66}. \quad (79)$$

Let the coefficients C_1, C_2, C_3, C_4 appearing in (65) and (66) in medium I be represented by $\bar{C}_1, \bar{C}_2, \bar{C}_3, \bar{C}_4$ and for medium II be represented by $C_1^+, C_2^+, C_3^+, C_4^+$.

Figure 1. A normal line-load P_0 .

The corresponding response in medium I is

$$\begin{aligned}
 \bar{U}^1(\alpha, k) &= \bar{C}_1 P_{21} e^{m_1 |k| \alpha} + \bar{C}_2 P_{22} e^{m_2 |k| \alpha} - \bar{C}_3 P_{21} e^{-m_1 |k| \alpha} - \bar{C}_4 P_{22} e^{-m_2 |k| \alpha}, \\
 \bar{V}^1(\alpha, k) &= \bar{C}_1 e^{m_1 |k| \alpha} + \bar{C}_2 e^{m_2 |k| \alpha} + \bar{C}_3 e^{-m_1 |k| \alpha} + \bar{C}_4 e^{-m_2 |k| \alpha}, \\
 \bar{\sigma}_{11}^1(\alpha, k) &= (a P_{21} m_1 |k| - \iota k c)(\bar{C}_1 e^{m_1 |k| \alpha} + \bar{C}_3 e^{-m_1 |k| \alpha}) \\
 &\quad + (a P_{22} m_2 |k| - \iota k c)(\bar{C}_2 e^{m_2 |k| \alpha} + \bar{C}_4 e^{-m_2 |k| \alpha}), \\
 \bar{\sigma}_{12}^1(\alpha, k) &= (m_1 |k| - \iota k P_{21})(\bar{C}_1 e^{m_1 |k| \alpha} - \bar{C}_3 e^{-m_1 |k| \alpha}) \\
 &\quad + (m_2 |k| - \iota k P_{22})(\bar{C}_2 e^{m_2 |k| \alpha} - \bar{C}_4 e^{-m_2 |k| \alpha}),
 \end{aligned} \tag{80}$$

and in medium II is given by,

$$\begin{aligned}
 \bar{U}^2(\alpha, k) &= -C_3^+ P_{21} e^{-m_1 |k| \alpha} - C_4^+ P_{22} e^{-m_2 |k| \alpha}, \\
 \bar{V}^2(\alpha, k) &= C_3^+ e^{-m_1 |k| \alpha} + C_4^+ e^{-m_2 |k| \alpha}, \\
 \bar{\sigma}_{11}^2(\alpha, k) &= (a P_{21} m_1 |k| - \iota k c)(C_3^+ e^{-m_1 |k| \alpha}) + (a P_{22} m_2 |k| - \iota k c)(C_4^+ e^{-m_2 |k| \alpha}), \\
 \bar{\sigma}_{12}^2(\alpha, k) &= (m_1 |k| - \iota k P_{21}) \bar{C}_3 e^{-m_1 |k| \alpha} (m_2 |k| - \iota k P_{22}) \bar{C}_4 e^{-m_2 |k| \alpha}.
 \end{aligned} \tag{81}$$

Applying the boundary conditions from (77)–(78) and using the matrix method, we obtain the following values of coefficients.

$$\begin{aligned}
 \bar{C}_1 &= M_3 / D_5, \quad \bar{C}_2 = -M_3 / D_6, \\
 \bar{C}_3 &= \frac{-M_3}{M_1} \left(\frac{M_2}{D_5} - \frac{2D_2 D_4}{D_6} \right), \quad \bar{C}_4 = \frac{-M_3}{M_1} \left(\frac{M_2}{D_6} - \frac{2D_1 D_3}{D_5} \right), \\
 C_3^+ &= \frac{-M_3}{D_7} \left[1 + \frac{M_2}{M_1} \frac{D_7}{D_5} - \frac{2}{M_1} \frac{D_2 D_4 D_7}{D_6} \right], \\
 C_4^+ &= \frac{M_3}{D_8} \left[1 - \frac{M_2}{M_1} \frac{D_8}{D_6} + \frac{2}{M_1} \frac{D_1 D_3 D_8}{D_5} \right],
 \end{aligned} \tag{82}$$

where

$$\begin{aligned}
 D_1 &= a P_{21} m_1 |k| - \iota k c, & D_2 &= a P_{22} m_2 |k| - \iota k c, \\
 D_3 &= m_1 |k| - \iota k P_{21}, & D_4 &= m_2 |k| - \iota k P_{22}, \\
 D_5 &= e^{m_1 |k| \alpha_0}, & D_6 &= e^{m_2 |k| \alpha_0}, \\
 D_7 &= e^{-m_1 |k| \alpha_0}, & D_8 &= e^{-m_2 |k| \alpha_0}, \\
 M_1 &= D_1 D_4 - D_2 D_3, & M_2 &= D_1 D_4 + D_2 D_3, \\
 M_3 &= P/2(D_1 - D_2).
 \end{aligned} \tag{83}$$

Putting values of the various coefficients from (82) into displacements and stresses obtained in (80)–(81), we get the following expressions for displacements and stresses at any point of an orthotropic elastic half-space in the transformed domain:

$$\begin{aligned}
 \bar{U}(\alpha, k) &= \frac{P}{2T_1} \left[m_1 R_1 \frac{1}{|k|} e^{-m_1 |k| |\alpha - \alpha_0|} - m_2 R_2 \frac{1}{|k|} e^{-m_2 |k| |\alpha - \alpha_0|} \right. \\
 &\quad + \frac{m_1 R_1 T_3}{T_2} \frac{1}{|k|} e^{-m_1 |k| (\alpha + \alpha_0)} + \frac{m_2 R_2 T_3}{T_2} \frac{1}{|k|} e^{-m_2 |k| (\alpha + \alpha_0)} \\
 &\quad \left. + \frac{2m_1 R_1 T_4}{T_2} \frac{1}{|k|} e^{-|k| (m_1 \alpha + m_2 \alpha_0)} - \frac{2m_2 R_2 T_5}{T_2} \frac{1}{|k|} e^{-|k| (m_2 \alpha + m_1 \alpha_0)} \right] \\
 \bar{V}(\alpha, k) &= \frac{P}{2iT_1} \left[\pm \frac{1}{k} e^{-m_1 |k| |\alpha - \alpha_0|} \mp \frac{1}{k} e^{-m_2 |k| |\alpha - \alpha_0|} \right. \\
 &\quad - \frac{T_3}{T_2} \frac{1}{k} e^{-m_1 |k| (\alpha + \alpha_0)} - \frac{T_3}{T_2} \frac{1}{k} e^{-m_2 |k| (\alpha + \alpha_0)} \\
 &\quad \left. + \frac{2T_4}{T_2} \frac{1}{k} e^{-|k| (m_1 \alpha + m_2 \alpha_0)} + \frac{2T_5}{T_2} \frac{1}{k} e^{-|k| (m_1 \alpha_0 + m_2 \alpha)} \right], \\
 \bar{\sigma}_{11} &= \frac{P}{2T_1} \left[\pm (am_1^2 R_1 - c) e^{-m_1 |k| |\alpha - \alpha_0|} \mp (am_2^2 R_2 - c) e^{-m_2 |k| |\alpha - \alpha_0|} \right. \\
 &\quad - \frac{T_3}{T_2} (am_1^2 R_1 - c) e^{-m_1 |k| (\alpha + \alpha_0)} - \frac{T_3}{T_2} (am_2^2 R_2 - c) e^{-m_2 |k| (\alpha + \alpha_0)} \\
 &\quad + \frac{2T_4}{T_2} (am_1^2 R_1 - c) e^{-|k| (m_1 \alpha + m_2 \alpha_0)} \\
 &\quad \left. + \frac{2T_5}{T_2} (am_2^2 R_2 - c) e^{-|k| (m_2 \alpha + m_1 \alpha_0)} \right], \\
 \bar{\sigma}_{12} &= \frac{iP|k|}{T_1 k} \left[\frac{-m_1}{2} (1 + R_1) e^{-m_1 |k| |\alpha - \alpha_0|} + \frac{m_2}{2} (1 + R_2) e^{-m_2 |k| |\alpha - \alpha_0|} \right. \\
 &\quad - \frac{m_1 T_3}{2T_2} (1 + R_1) e^{-m_1 |k| (\alpha + \alpha_0)} - \frac{m_2 T_3}{2T_2} (1 + R_2) e^{-m_2 |k| (\alpha + \alpha_0)} \\
 &\quad + \frac{m_1 T_4}{T_2} (1 + R_1) e^{-|k| (m_1 \alpha + m_2 \alpha_0)} \\
 &\quad \left. + \frac{m_2 T_5}{T_2} (1 + R_2) e^{-|k| (m_2 \alpha + m_1 \alpha_0)} \right], \tag{84}
 \end{aligned}$$

where the upper sign is for the region $\alpha < \alpha_0$ and lower sign is for the region $\alpha > \alpha_0$, and $\alpha_0 = H/h$,

$$R_1 = \frac{1+c}{am_1^2-1}, R_2 = \frac{1+c}{am_2^2-1}, T_1 = \frac{aR_1(m_2^2-m_1^2)}{am_2^2-1},$$

$$T_2 = (am_1 m_2 + c)(m_1 R_1 - m_2 R_2) + (m_1 - m_2)am_1 m_2 R_1 R_2 + (m_1 - m_2)c,$$

$$T_3 = (am_1 m_2 - c)(m_1 R_1 + m_2 R_2) + (m_1 + m_2)am_1 m_2 R_1 R_2 - (m_1 + m_2)c,$$

$$\begin{aligned}
T_4 &= (am_2^2 - c)m_2R_2 + am_2^3R_2^2 - m_2c, \\
T_5 &= (am_1^2 - c)m_1R_1 + am_1^3R_1^2 - m_1c.
\end{aligned} \tag{85}$$

We note that quantities $R_1, R_2, T_1, T_2, T_3, T_4$ and T_5 are independent of k . Taking inversion and using appendix A, we get the following closed-form expressions for the displacements and stresses at any point of an orthotropic elastic half-space due to a normal line force acting at $(H, 0)$.

$$\begin{aligned}
U(\alpha, \beta) &= \frac{P}{4\pi T_1} \left[-m_1R_1 \log\{\beta^2 + m_1^2(\alpha - \alpha_0)^2\} \right. \\
&\quad + m_2R_2 \log\{\beta^2 + m_2^2(\alpha - \alpha_0)^2\} \\
&\quad - \frac{m_1R_1T_3}{T_2} \log\{\beta^2 + m_1^2(\alpha + \alpha_0)^2\} \\
&\quad - \frac{m_2R_2T_3}{T_2} \log\{\beta^2 + m_2^2(\alpha + \alpha_0)^2\} \\
&\quad + \frac{2m_1R_1T_4}{T_2} \log\{\beta^2 + (m_1\alpha + m_2\alpha_0)^2\} \\
&\quad \left. + \frac{2m_2R_2T_5}{T_2} \log\{\beta^2 + (m_2\alpha + m_1\alpha_0)^2\} \right], \\
V(\alpha, \beta) &= \frac{P}{2\pi T_1} \left[\mp \tan^{-1} \left(\frac{\beta}{m_1|\alpha - \alpha_0|} \right) \pm \tan^{-1} \left(\frac{\beta}{m_2|\alpha - \alpha_0|} \right) \right. \\
&\quad + \frac{T_3}{T_2} \tan^{-1} \left(\frac{\beta}{m_1(\alpha + \alpha_0)} \right) + \frac{T_3}{T_2} \tan^{-1} \left(\frac{\beta}{m_2(\alpha + \alpha_0)} \right) \\
&\quad \left. - \frac{2T_4}{T_2} \tan^{-1} \left(\frac{\beta}{|m_1\alpha + m_2\alpha_0|} \right) - \frac{2T_5}{T_2} \tan^{-1} \left(\frac{\beta}{|m_2\alpha + m_1\alpha_0|} \right) \right], \\
\sigma_{11}(\alpha, \beta) &= \frac{P}{2\pi T_1} \left[\pm(am_1^2R_1 - c) \left\{ \frac{m_1(\alpha_0 - \alpha)}{\beta^2 + m_1^2(\alpha_0 - \alpha)^2} \right\} \right. \\
&\quad \mp (am_2^2R_2 - c) \left\{ \frac{m_2|\alpha - \alpha_0|}{\beta^2 + m_2^2(\alpha - \alpha_0)^2} \right\} \\
&\quad - \frac{T_3}{T_2} (am_1^2R_1 - c) \left\{ \frac{m_1(\alpha + \alpha_0)}{\beta^2 + m_1^2(\alpha + \alpha_0)^2} \right\} \\
&\quad - \frac{T_3}{T_2} (am_2^2R_2 - c) \left\{ \frac{m_2(\alpha + \alpha_0)}{\beta^2 + m_2^2(\alpha + \alpha_0)^2} \right\} \\
&\quad + \frac{2T_4}{T_2} (am_1^2R_1 - c) \left\{ \frac{|m_1\alpha + m_2\alpha_0|}{\beta^2 + (m_1\alpha + m_2\alpha_0)^2} \right\} \\
&\quad \left. + \frac{2T_5}{T_2} (am_2^2R_2 - c) \left\{ \frac{|m_2\alpha + m_1\alpha_0|}{\beta^2 + (m_2\alpha + m_1\alpha_0)^2} \right\} \right],
\end{aligned}$$

$$\begin{aligned} \sigma_{12}(\alpha, \beta) = \frac{P}{2\pi T_1} & \left[-\frac{m_1(1+R_1)\beta}{\beta^2 + m_1^2(\alpha - \alpha_0)^2} + \frac{m_2(1+R_2)\beta}{\beta^2 + m_2^2(\alpha - \alpha_0)^2} \right. \\ & - \frac{m_1 T_3(1+R_1)\beta}{T_2\{\beta^2 + m_1^2(\alpha + \alpha_0)^2\}} - \frac{m_2 T_3(1+R_2)\beta}{T_2\{\beta^2 + m_2^2(\alpha + \alpha_0)^2\}} \\ & \left. + \frac{2m_1 T_4(1+R_1)\beta}{T_2\{\beta^2 + (m_1\alpha + m_2\alpha_0)^2\}} + \frac{2m_2 T_5(1+R_2)\beta}{T_2\{\beta^2 + (m_2\alpha + m_1\alpha_0)^2\}} \right]. \quad (86) \end{aligned}$$

4.2 Deformation of a transversely isotropic elastic half-space

Following the same procedure, as adopted for an orthotropic case, we obtain the following closed-form expressions for displacements at any point of a transversely isotropic elastic half-space due to a normal line-load acting at $(H, 0)$.

$$\begin{aligned} U(\alpha, \beta) = \frac{P}{4\pi} & \left[\frac{a-1}{a} \left\{ \frac{(\alpha_0 - \alpha)^2}{\beta^2 + (\alpha_0 - \alpha)^2} \right\} - \left(\frac{a+1}{2a} \right) \log\{\beta^2 + (\alpha_0 - \alpha)^2\} \right. \\ & + \left(\frac{a+1}{a} \right) \left\{ \frac{(\alpha_0 + \alpha)^2}{\beta^2 + (\alpha_0 + \alpha)^2} \right\} - \frac{1}{2a} \left(\frac{a^2 + 1}{a-1} \right) \log\{\beta^2 + (\alpha_0 + \alpha)^2\} \\ & \left. + 2 \left(\frac{a-1}{a} \right) \alpha_0 \alpha \left\{ \frac{(\alpha + \alpha_0)^2 - \beta^2}{((\alpha + \alpha_0)^2 + \beta^2)^2} \right\} \right], \\ V(\alpha, \beta) = \frac{P}{4\pi} & \left[- \left(\frac{a-1}{a} \right) (\alpha_0 - \alpha) \left\{ \frac{\beta}{\beta^2 + (\alpha_0 - \alpha)^2} \right\} \right. \\ & - \left(\frac{a+1}{a} \right) |\alpha_0 - \alpha| \left\{ \frac{\beta}{\beta^2 + (\alpha_0 + \alpha)^2} \right\} - \left(\frac{2}{a-1} \right) \tan^{-1} \left(\frac{\beta}{\alpha + \alpha_0} \right) \\ & \left. + 4\alpha\alpha_0 \left(\frac{a-1}{a} \right) \left\{ \frac{\beta(\alpha + \alpha_0)}{(\beta^2 + (\alpha_0 + \alpha)^2)^2} \right\} \right]. \quad (87) \end{aligned}$$

4.3 Deformation of an isotropic elastic half-space

For an isotropic case $a = 1/(1 - \alpha^*)$, using this in (87), we obtain the following closed-form expressions for the displacements at any point of an isotropic elastic half-space due to a normal line-load acting at $(H, 0)$.

$$\begin{aligned} U(\alpha, \beta) = \frac{P}{4\pi} & \left[\frac{\alpha^*(\alpha_0 - \alpha)^2}{\beta^2 + (\alpha_0 - \alpha)^2} - \frac{(2 - \alpha^*)}{2} \log\{\beta^2 + (\alpha_0 - \alpha)^2\} \right. \\ & + \frac{(2 - \alpha^*)(\alpha_0 + \alpha)^2}{\beta^2 + (\alpha_0 + \alpha)^2} - \frac{1}{2} \left(\frac{2 - 2\alpha^* + \alpha^{*2}}{\alpha^*} \right) \log\{\beta^2 + (\alpha_0 + \alpha)^2\} \\ & \left. + 2\alpha^*\alpha_0\alpha \left\{ \frac{(\alpha + \alpha_0)^2 - \beta^2}{((\alpha + \alpha_0)^2 + \beta^2)^2} \right\} \right], \\ V(\alpha, \beta) = \frac{P}{4\pi} & \left[\frac{-\alpha^*\beta(\alpha_0 - \alpha)}{\beta^2 + (\alpha_0 - \alpha)^2} - \frac{(2 - \alpha^*)(\alpha_0 - \alpha)\beta}{\beta^2 + (\alpha_0 + \alpha)^2} \right. \\ & \left. - 2 \left(\frac{1 - \alpha^*}{\alpha^*} \right) \tan^{-1} \left(\frac{\beta}{\alpha + \alpha_0} \right) + \frac{4\alpha_0\alpha\alpha^*\beta(\alpha + \alpha_0)}{\{\beta^2 + (\alpha + \alpha_0)^2\}^2} \right]. \quad (88) \end{aligned}$$

The displacements for an isotropic elastic half-space obtained in (88) coincide with the corresponding results obtained by Maruyama (1966).

5. Numerical results and discussion

We wish to examine the effect of anisotropy of the elastic half-space. For numerical computations, we use the values of elastic constants given by Love (1944) for topaz material (orthotropic), which are

$$a = 2.15789, \quad b = 2.67669, \quad c = 0.96240.$$

To examine the effect of anisotropy of elastic medium, various distribution curves in figures 2 to 5 are plotted for orthotropic, transversely isotropic and isotropic elastic medium. For a transversely isotropic medium, the following relations

$$b = a, \quad c = a - 2$$

are taken into consideration. For an isotropic elastic medium we consider $\alpha^* = 0.6624$. In these figures, the normal and tangential dimensionless displacements at the surface level $\alpha = 0$ and subsurface level $\alpha = 0.5$ have been shown for $\alpha_0 = 1.0$. Figures 2 and 4 show the variation of dimensionless normal displacement with dimensionless horizontal distance for orthotropic, transversely isotropic and isotropic elastic medium. In each case it is found that as we move away from the point of application, the difference between the dimensionless displacements corresponding to orthotropic, transversely and isotropic elastic medium increases. Also it is found that in magnitude the difference between the normal displacements corresponding to

Figure 2. Variation of dimensionless normal displacement for orthotropic, transversely isotropic and isotropic elastic media.

Figure 3. Variation of dimensionless tangential displacement for orthotropic, transversely isotropic and isotropic elastic media.

Figure 4. Variation of dimensionless normal displacement for orthotropic, transversely isotropic and isotropic elastic media.

Figure 5. Variation of dimensionless tangential displacement distribution for orthotropic, transversely isotropic and isotropic elastic media.

orthotropic and isotropic is greater than that of the difference between transversely isotropic and isotropic. Figures 3 and 5 show the variation of dimensionless tangential displacement with dimensionless horizontal distance for orthotropic, transversely isotropic and isotropic elastic medium. As we move away from the point of application the difference between the dimensionless tangential displacement corresponding to orthotropic and isotropic elastic medium increases. While the difference between the dimensionless tangential displacements corresponding to isotropic and transversely isotropic medium decreases.

From these figures, it is concluded that the anisotropy is affecting the deformation substantially.

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Appendix ($x > 0$) A

$$(1) \int_{-\infty}^{\infty} e^{-t|ky|} dk = 2\pi \delta(y)$$

$$(2) \int_{-\infty}^{\infty} (|k|)^{-1} e^{-|k|x} e^{-t|ky|} dk = -\log(y^2 + x^2)$$

$$\begin{aligned}
(3) \quad & \int_{-\infty}^{\infty} k^{-1} e^{-|k|x} e^{-iky} dk = -2\iota \tan^{-1} \left(\frac{y}{x} \right) \\
(4) \quad & \int_{-\infty}^{\infty} e^{-|k|x} e^{-iky} dk = \frac{2z}{y^2 + z^2} \\
(5) \quad & \int_{-\infty}^{\infty} \frac{k}{|k|} e^{-|k|x} e^{-iky} dk = \frac{-2\iota y}{y^2 + x^2} \\
(6) \quad & \int_{-\infty}^{\infty} k e^{-|k|x} e^{-iky} dk = \frac{-4\iota yx}{(y^2 + x^2)^2} \\
(7) \quad & \int_{-\infty}^{\infty} |k| e^{-|k|x} e^{-iky} dk = \frac{2(z^2 - y^2)}{(z^2 + y^2)^2} \\
(8) \quad & \int_{-\infty}^{\infty} k^2 e^{-|k|x} e^{-iky} dk = \frac{4x(x^2 - 3y^2)}{(x^2 + y^2)^3} \\
(9) \quad & \int_{-\infty}^{\infty} k|k| e^{-|k|x} e^{-iky} dk = \frac{-4\iota y(3z^2 - y^2)}{(z^2 + y^2)^3}
\end{aligned}$$

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