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Thermal deformation in a thin circular plate due to a partially distributed heat supply

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Abstract. In this paper, we develop an integral transform to determine temperature distribution in a thin circular plate, subjected to a partially distributed and axisymmetric heat supply on the curved surface, and study the thermal deformation. The results, obtained in series form in terms of Bessel's functions, are illustrated numerically.

Keywords. Transient problem; thermal deformation; heat conduction problem.

1. Introduction

Thermal deflections of axisymmetrically heated circular plates in the case of fixed and simply supported edges have been considered by Boley & Weiner (1960). Roy Choudhuri (1973) discussed the normal deflection of a thin clamped circular plate due to ramp-type heating of a concentric circular region of the upper face and the lower face of the plate kept at zero temperature, with the circular edge being thermally insulated. Noda *et al* (1997) discussed the transient thermoelastoplastic bending problem making use of the strain increment theorem and determined the temperature field and thermoelastic deformation for the heating and cooling processes in thin circular plates, subjected to partially distributed and axisymmetric heat supply on the upper face. Deshmukh & Khobragade (2002) determined quasi-static thermal deflections in thin circular plates subjected to arbitrary initial temperatures on the lower face with the upper face at zero temperature and the fixed circular edge thermally insulated. Here, we extend the work of Noda *et al* (1997) and study the heat conduction problem to determine the temperature distribution and discuss the thermoelastic deformation in thin circular plates subjected to partially distributed and axisymmetric heat supply on the outer curved surface. The results are illustrated numerically. The results presented here will be more useful in engineering problems, particularly in machines subjected to heating and cooling.

2. Analysis

2.1 *Transient heat conduction problem*

We consider a thin circular plate of radius a and thickness h. The initial temperature of the plate is the same as the temperature of the surrounding medium, which is kept constant. From time $t = 0$ to $t = t_0$, the plate is subjected to a partially distributed and axisymmetric heat supply $Q_0 f(z)$ on the curved surface ($r = a$). After time t_0, the heat supply is removed and the plate is cooled by the surrounding medium. For heating processes, the heat conduction equation, the initial condition, and the boundary conditions are given, respectively, as

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t}, \tag{1}$$

$$T(r, z, t)|_{t=0} = 0, \tag{2}$$

$$T(r, z, t)|_{r=a} = -(Q_0/\lambda) f(z), \quad 0 \le t \le t_0, \tag{3}$$

$$T(r, z, t)|_{z=-h/2} = 0, \tag{4}$$

and $$T(r, z, t)|_{z=h/2} = 0, \tag{5}$$

where κ and λ are the thermal diffusivity and the thermal conductivity respectively.

On the other hand, for the cooling process, the temperature distribution $T'(r, z, t)$ satisfies the equation

$$\frac{\partial^2 T'}{\partial r^2} + \frac{1}{r}\frac{\partial T'}{\partial r} + \frac{\partial^2 T'}{\partial z^2} = \frac{1}{\kappa}\frac{\partial T'}{\partial t}, \tag{6}$$

with the initial and the boundary conditions given by

$$T'(r, z, t)|_{t=t_0} = T(r, z, t_0), \tag{7}$$

$$T'(r, z, t)|_{r=a} = 0, \quad t \ge t_0, \tag{8}$$

$$T'(r, z, t)|_{z=-h/2} = 0 \tag{9}$$

and $$T'(r, z, t)|_{z=h/2} = 0. \tag{10}$$

Equations (1) to (10) constitute the transient heat conduction problem for heating and cooling processes.

3. Solution of the problem

3.1 *Determination of the temperatures $T(r, z, t)$ and $T'(r, z, t)$*

On applying the finite Hankel and the finite Fourier transforms to (1) to (10) and then using their inversions, one obtains the expressions of the temperature distributions $T(r, z, t)$ and $T'(r, z, t)$ for the heating and the cooling processes respectively, as

$$T(r, z, t) = \frac{-2(2)^{1/2} Q_0}{a\lambda(h)^{1/2}} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\xi_n \overline{f}(\lambda_m) J_0(\xi_n r)\sin(\lambda_m z)\cdot[1 - e^{-\kappa(\xi_n^2+\lambda_m^2)t}]}{(\xi_n^2 + \lambda_m^2) J_1(\xi_n a)} \tag{11}$$

and

$$T'(r,z,t) = \frac{-2(2)^{1/2}Q_0}{a\lambda(h)^{1/2}} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \times \frac{\xi_n \overline{f}(\lambda_m) J_0(\xi_n r)\sin(\lambda_m z)[1-e^{-\kappa(\xi_n^2+\lambda_m^2)t_0}][e^{-\kappa(\xi_n^2+\lambda_m^2)(t-t_0)}]}{(\xi_n^2+\lambda_m^2)J_1(\xi_n a)}, \tag{12}$$

where ξ_n is the nth positive root of the transcendental equation

$$J_0(\xi a) = 0, \tag{13}$$

λ_m is the mth positive root of the transcendental equation,

$$\sin(\lambda h/2) = 0, \tag{14}$$

$$\overline{f}(\lambda_m) = \int_{-h/2}^{h/2} f(z)\cdot K(\lambda_m, z)dz, \tag{15}$$

$K(\lambda_m, z)$ is the kernel for the finite Fourier transform defined by

$$K(\lambda_m, z) = ((2)^{1/2}/(h)^{1/2})\sin(\lambda_m z) \tag{16}$$

and $K_0(\xi_n, r)$ is the kernel for the finite Hankel transform defined by

$$K_0(\xi_n, r) = ((2)^{1/2}/a)\{J_0(\xi_n r)\}/\{J_0'(\xi_n a)\}. \tag{17}$$

Using the result (11) into (12), one obtains

$$T'(r,z,t) = T(r,z,t) - T(r,z,t-t_0). \tag{18}$$

Equation (18) represents the relation between the temperatures in the heating and cooling processes.

Further, equation (11) for the heating and the cooling processes together can be rewritten as

$$T(r,z,t) = \frac{-2(2)^{1/2}Q_0}{a\lambda(h)^{1/2}} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A(\xi_n,\lambda_m,t)\overline{f}(\lambda_m)J_0(\xi_n r)\sin(\lambda_m z), \tag{19}$$

where

$$\begin{aligned} A(\xi_n,\lambda_m,t) &= \{\xi_n[1-e^{-\kappa(\xi_n^2+\lambda_m^2)t}]\}/(\xi_n^2+\lambda_m^2)J_1(\xi_n a) \\ &= A_{mn}(\xi_n,\lambda_m,t), \quad (0 \le t \le t_0) \end{aligned} \tag{20}$$

and

$$\begin{aligned} A(\xi_n,\lambda_m,t) &= \{\xi_n[1-e^{-\kappa(\xi_n^2+\lambda_m^2)t_0}]\cdot e^{-\kappa(\xi_n^2+\lambda_m^2)(t-t_0)}\}/(\xi_n^2+\lambda_m^2)J_1(\xi_n a) \\ &= A_{mn}(\xi_n,\lambda_m,t_0)e^{-k(\xi_n^2+\lambda_m^2)(t-t_0)}, \quad (t \ge t_0). \end{aligned} \tag{21}$$

4. Thermoelastic problem

4.1 *Derivation of basic equations*

To derive the basic equations that govern the thermoelastic behaviour of the plate, assume that the plate is very thin. Therefore, the assumption that the plane perpendicular to the neutral plane ($z = 0$) before deformation, remains the one perpendicular to it after deformation. The differential equation satisfied by "the resultant force function" F is

$$\frac{\mathrm{d}}{\mathrm{d}r}[\nabla^2 F] = -\frac{\mathrm{d}N_T}{\mathrm{d}r}, \tag{22}$$

where ∇^2 denotes the two-dimensional and axisymmetric Laplace operator in cylindrical coordinate defined as

$$\nabla^2 = (\mathrm{d}^2/\mathrm{d}r^2) + (1/r)(\mathrm{d}/\mathrm{d}r). \tag{23}$$

The thermal resultant force N_T is defined as

$$N_T = \int_{-h/2}^{h/2} \alpha E T \mathrm{d}z, \tag{24}$$

α and E are the coefficients of the linear thermal expansion and Young's modulus respectively.

For in-plane deformation, the boundary conditions are given by

$$F = \mathrm{d}F/\mathrm{d}r = 0, \text{ at } r = a. \tag{25}$$

The differential equation satisfied by "the deflection" $\omega(r, t)$ is

$$D(\mathrm{d}/\mathrm{d}r)[\nabla^2\omega] = [1/(1-\nu)](\mathrm{d}M_T/\mathrm{d}r), \tag{26}$$

where D is the flexural rigidity of the plate denoted by

$$D = Eh^3/[12(1-\nu^2)], \tag{27}$$

ν is Poisson's ratio and the thermal resultant moment M_T is defined as

$$M_T = \int_{-h/2}^{h/2} \alpha E T z \mathrm{d}z. \tag{28}$$

For out-of-plane deformation, the boundary conditions are given as

$$\omega = 0, \text{ at } r = 0, \tag{29}$$

and

$$D((\mathrm{d}^2\omega/\mathrm{d}r^2) + (\nu/r)(\mathrm{d}\omega/\mathrm{d}r)) = -M_T/(1-\nu), \text{ at } r = a. \tag{30}$$

Equations (22) to (30) constitute the formulation of the thermoelastic problem.

4.2 *Determination of the force component, $F(r,t)$*

Substituting the value of $T(r,z,t)$ from (19) into (24), we obtain

$$N_T = 0. \tag{31}$$

Solving (22) by using the value of N_T from (31) under the conditions (25), we obtain

$$F = F(r,t) = 0. \tag{32}$$

4.3 *Determination of the deflection, $\omega(r,t)$*

Substituting the value of $T(r,z,t)$ from (19) into (28), we obtain

$$M_T = \frac{2(2)^{1/2} Q_0 \alpha E(h)^{1/2}}{a\lambda} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{A(\xi_n, \lambda_m, t)\overline{f}(\lambda_m)\cos(\lambda_m h/2) J_0(\xi_n r)}{\lambda_m}. \tag{33}$$

From (32), we can solve (26) under the conditions (29) and (30). The deflection $\omega(r,t)$ is given as

$$\omega(r,t) = \frac{2(2)^{1/2} Q_0 \alpha E(h)^{1/2}}{D(1-\nu)a\lambda} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{A(\xi_n, \lambda_m, t)\overline{f}(\lambda_m)\cos(\lambda_m h/2)}{\xi_n \lambda_m} \times \left[J_0(\xi_n r) - \frac{r^2 \xi_n}{2(1+\nu)} \left\{ (1-\xi_n) J_0(\xi_n a) + \left(\frac{1-\nu}{a}\right) J_1(\xi_n a) \right\} - 1 \right]. \tag{34}$$

For the variation of angle with time, define the angle θ as

$$\theta = -\mathrm{d}\omega/\mathrm{d}r. \tag{35}$$

Substituting the value of $\omega(r,t)$ from (34) into (35), one obtains

$$\theta = \frac{2(2)^{1/2} Q_0 \alpha E(h)^{1/2}}{D(1-\nu^2)a^2\lambda} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left[\frac{A(\xi_n, \lambda_m, t)\overline{f}(\lambda_m)\cos(\lambda_m h/2)}{\lambda_m} \right] \times [a(1+\nu)J_1(\xi_n r) + r\{a(1-\xi_n)J_0(\xi_n a) + (1-\nu)J_1(\xi_n a)\}]. \tag{36}$$

5. Special case

Set

$$f(z) = (4z^2 - h^2)z. \tag{37}$$

Apply the finite Fourier transform to the equation (37), one obtains

$$\overline{f}(\lambda_m) = [24(2)^{1/2}(h)^{1/2}\cos(\lambda_m h/2)]/\lambda_m^3. \tag{38}$$

Putting the value of $\overline{f}(\lambda_m)$ from (38) into (19), (34) and (36), temperature, deflection and angle respectively, are obtained as

$$T(r,z,t) = \frac{-96Q_0}{a\lambda} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A(\xi_n,\lambda_m,t)\cdot\cos(\lambda_m h/2)\cdot J_0(\xi_n r)\cdot\sin(\lambda_m z)}{\lambda_m^3}, \tag{39}$$

$$\omega(r,t) = \frac{96Q_0\alpha Eh}{D(1-\nu)a\lambda} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A(\xi_n,\lambda_m,t)\cdot\cos(\lambda_m h/2)}{\xi_n\lambda_m^4} \times\left[J_0(\xi_n r) - \frac{r^2\xi_n}{2(1+\nu)}\left\{(1-\xi_n)J_0(\xi_n a) + \left(\frac{1-\nu}{a}\right)J_1(\xi_n a)\right\} - 1\right]. \tag{40}$$

and

$$\theta = \frac{96Q_0\alpha Eh}{D(1-\nu^2)a^2\lambda} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A(\xi_n,\lambda_m,t)\cdot\cos(\lambda_m h/2)}{\lambda_m^4} \times [a(1+\nu)J_1(\xi_n r) + r\{a(1-\xi_n)J_0(\xi_n a) + (1-\nu)J_1(\xi_n a)\}]. \tag{41}$$

6. Numerical results

The numerical calculations have been carried out for a steel (SN50C) plate with the properties $\kappa = 15{\cdot}9\times10^{-6}[\mathrm{m^2s^{-1}}]$, $\nu = 0{\cdot}281$, $\lambda = 59{\cdot}0[\mathrm{Wm^{-1}K^{-1}}]$. Setting

$$\alpha = \frac{-96Q_0}{a\lambda}, \beta = \frac{96Q_0\alpha Eh}{D(1-\nu)a\lambda}, \gamma = \frac{96Q_0\alpha Eh}{D(1-\nu^2)a^2\lambda}, \tag{42}$$

with the radius $a = 5$ and thickness $h = 1$ into (39), (40) and (41), one gets the expressions for temperature distribution, thermal deflection and angle respectively as,

$$\frac{T}{\alpha} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A(\xi_n,\lambda_m,t)\cdot\cos(\lambda_m/2)\cdot J_0(\xi_n r)\cdot\sin(\lambda_m z)}{\lambda_m^3} \tag{43}$$

$$\frac{\omega}{\beta} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A(\xi_n,\lambda_m,t)\cdot\cos(\lambda_m h/2)}{\xi_n\lambda_m^4} \times\left[J_0(\xi_n r) - \frac{r^2\xi_n}{2(1+\nu)}\left\{(1-\xi_n)J_0(5\xi_n) + \left(\frac{1-\nu}{5}\right)J_1(5\xi_n)\right\} - 1\right] \tag{44}$$

and

$$\frac{\theta}{\gamma} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A(\xi_n,\lambda_m,t)\cdot\cos^2(\lambda_m h/2)}{\lambda_m^4} \times [5(1+\nu)J_1(\xi_n r) + r\{5(1-\xi_m)J_0(5\xi_n) + (1-\nu)J_1(5\xi_n)\}] \tag{45}$$

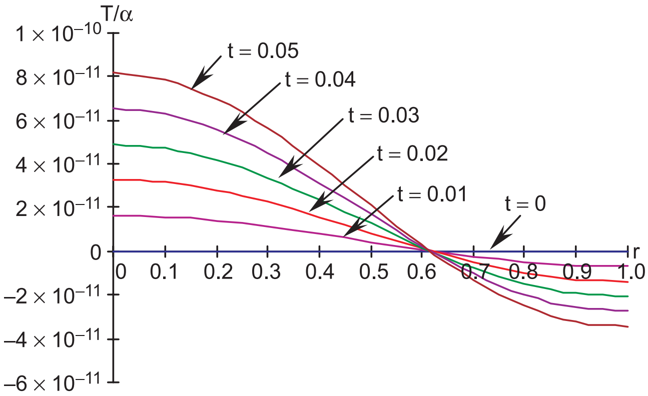

Figure 1. Temperature distribution T/α at $t = 0, 0{\cdot}01, \ldots, 0{\cdot}05$ for fixed $z = 0{\cdot}5$ $(0 \leq t \leq t_0)$.

where $\xi = 0{\cdot}4809, \xi_2 = 1{\cdot}1040, \xi_3 = 1{\cdot}7307, \xi_4 = 2{\cdot}3583, \xi_5 = 2{\cdot}9861, \xi_6 = 3{\cdot}6142$ are the positive roots of the transcendental equation $J_0(\xi a) = 0$ which takes the form $J_0(5\xi) = 0$ for $a = 5$ (Özisik 1968) and $\lambda_1 = 6{\cdot}28, \lambda_2 = 12{\cdot}56, \lambda_3 = 18{\cdot}84, \lambda_4 = 25{\cdot}12, \lambda_5 = 31{\cdot}40$ are positive roots of the transcendental equation $\sin(\lambda h/2) = 0$, which takes the form $\sin(\lambda/2) = 0$ for $h = 1$ (Özisik 1968). These values are used in (42), (43) and (44) to illustrate the temperature distribution, thermal deflection and angle of variation numerically in the heating as well as the cooling process with the help of a computer and variations, as shown in figures 1–6.

Figure 1 indicates that temperature increases in the heating process $(0 \leq t \leq t_0)$ while figure 2 shows that temperature decreases after removing the heat supply in the cooling process $(t \geq t_0)$. Figure 3 shown that deflection increases in the heating process $(0 \leq t \leq t_0)$ while figure 4 shows that deflection decreases after removing the heat supply in the cooling process $(t \geq t_0)$. In figure 5, the angle of variation increases during the heating process $(0 \leq t \leq t_0)$ while figure 6 shows that the angle of variation decreases after removing the heat supply in the cooling process $(t \geq t_0)$.

7. Conclusions

In this article, we extend the problem studied by Noda *et al* (1997) and study the heat conduction problem of a thin circular plate due to the partially distributed and axisymmetric heat

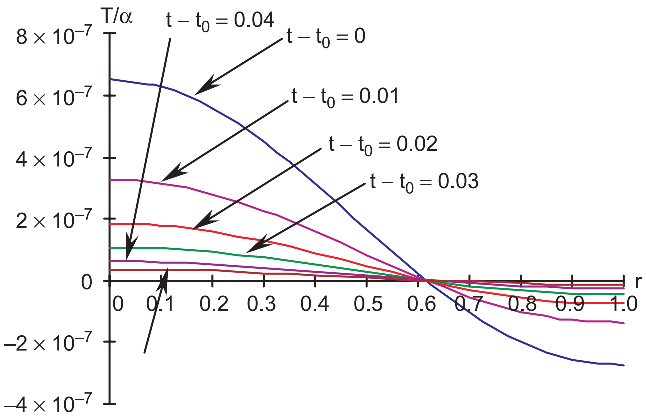

Figure 2. Distribution of temperature T/α at $t - t_0 = 0$, $0{\cdot}01, \ldots, 0{\cdot}05$ for fixed $z = 0{\cdot}5$ $(t \geq t_0)$.

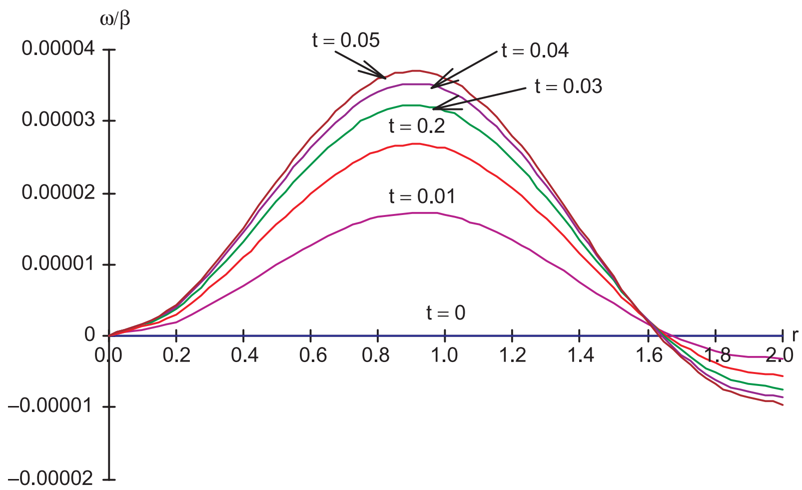

Figure 3. Distribution of deflection ω/β at $t = 0, 0{\cdot}01, \ldots, 0{\cdot}05 (0 \leq t \leq t_0)$.

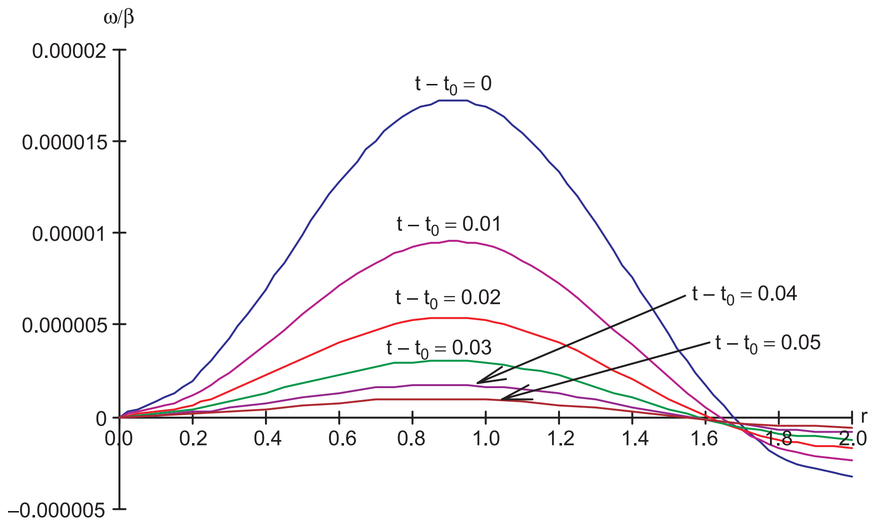

Figure 4. Distribution of deflection ω/β at $t - t_0 = 0, 0{\cdot}01, \ldots, 0{\cdot}05 (0 \geq t_0)$.

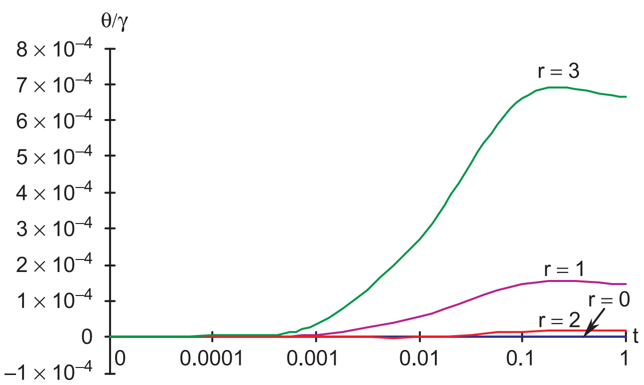

Figure 5. Variation of angle with time at $r = 0, 1, 2, 3 (0 \leq t \leq t_0)$.

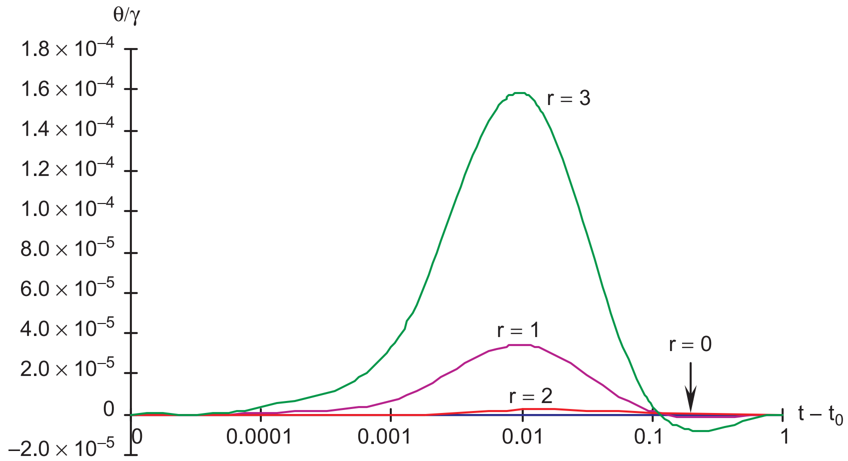

Figure 6. Variation of angle with time at $r = 0, 1, 2, 3 (t \geq t_0)$.

supply on the outer curved surface. We find the temperature field analytically by introducing the methods of the finite Fourier and the finite Hankel transforms and determine the thermal deformation by using the temperature field for the heating and cooling processes. Considering steel, we obtain expressions for the temperature distribution, thermal deflection and angle of variation in the heating as well as cooling media, which are illustrated numerically and shown in figures 1–6.

The force component $F(r, t)$ reduces to zero due to the absence of plane strain on the circular region of the plate. This type of problem is mainly applicable in the engineering area, particularly in machines subjected to heating and cooling. Any particular case of special interest can be described by assigning suitable values to the parameters and functions in (42), (43) and (44).

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